

ATTRACTION SETS IN ATTAINABILITY PROBLEMS WITH ASYMPTOTIC-TYPE CONSTRAINTS

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Abstract: In control theory, the problem of constructing and investigating attainability domains is very important. However, under perturbations of constraints, this problem lacks stability. It is useful to single out the case when the constraints are relaxed. In this case, greater opportunities arise in terms of attainability, and often a useful effect can be observed even under slight relaxation of the constraints. This situation is analogous to the duality gap in convex programming. Very often, it is not possible to specify in advance how much relaxation of the constraints will occur. Therefore, attention is focused on the limit of the attainability domains under unrestricted tightening of the relaxed conditions. As a result, a certain attainability problem with asymptotic-type constraints arises. This problem formulation can be significantly generalized. Namely, we do not consider any unperturbed conditions at all and instead pose asymptotic-type constraints directly by means of a nonempty family of sets in the space of ordinary controls. Moreover, not only the case of control problems can be considered. In this general formulation, an analogue of the limit of attainability domains naturally appears as the relaxed conditions are infinitely tightened. For asymptotic constraints of this kind, we introduce solutions which are, at the conceptual level, similar to the approximate solutions of J. Warga, but we use filters or directedness, and not just sequences of ordinary solutions (controls). We investigate the most general attainability problem, in which asymptotic-type constraints can be generated by any nonempty family of sets in the ordinary solution space. It is shown, however, that the most practically interesting case is realized by filters, and the role of ultrafilters is noted as well. The action of constraints is associated with sets and elements of attraction. Furthermore, some properties of the family of all attraction sets are investigated.

Keywords: Attraction set, Constraints, Filter, Topology, Ultrafilter.

1. Introduction

We consider attainability problems in topological spaces with asymptotic-type constraints. These asymptotic-type constraints may arise when standard constraints (such as inequalities in mathematical programming, phase constraints, or boundary conditions in control theory) are relaxed, but they can also be posed from the outset. In all cases, we deal with a nonempty family of sets in the space of ordinary (implementable) solutions. Thus, our concrete solutions must be essentially asymptotic; here we focus on the approximate solutions in the sense of Warga (see [17, Ch. III]), allowing, however, for nonsequential variants (i.e., directed sets or filters). In addition, for the family generating asymptotic-type constraints, we require that the solution direction eventually takes values in each set of this family (a similar requirement is imposed when using filters and, in particular, ultrafilters).

In addition, we have a certain target operator with values in a topological space. Using the solution direction, we obtain a directed set of its values (when using a filter, the filter base is realized). We consider those points in the topological space that are realized as generalized limits of such directed sets of values. The set of these generalized limits is called the attraction set for the given asymptotic-type constraints. Thus, for every nonempty family of sets in the space of

ordinary solutions, the corresponding attraction set in the fixed topological space is defined. By varying these families, we obtain a family of attraction sets. The latter family is the main subject of our research. We strive to develop a kind of “calculus” of attraction sets. Filters and ultrafilters will play an important role in this construction.

We note that, for the investigation of extremal problems with weakened constraints, extension constructions are used very widely (see [17, Ch. III–V]). This approach motivated the development of the theory of generalized solutions (controls); in this connection, we would like to especially mention the monographs [9, 11, 17, 18]. In [11, 12], the fundamental alternative theorem was established; this theorem defined the current state of differential game theory. In the construction of the proof, the idea of observing phase constraints in the form of sections of the stable bridge of N.N. Krasovskii was employed. We also note the wide application of generalized controls in solving the performance problem; see [9].

For control problems involving impulses, N.N. Krasovskii suggested (see [13]) using the apparatus of generalized functions to represent (generalized) controls. This approach served as the basis for the development of impulse control theory (see [7, 10, 13, 15, 16, 19] and others). In [2, 3, 6], for abstract control problems with impulse-type and momentary-type constraints, and with discontinuous dependencies among the conditions, extension constructions in the class of finitely additive measures were proposed. Finally, we note the approach of [4, Ch. 8], which is connected with the use of ultrafilters as generalized elements in attainability problems with asymptotic-type constraints. The present article continues the investigations of [4, Ch. 8].

Now, we note essential differences between the present investigation and the constructions in the author’s earlier works. Namely, here, not a single attraction set is considered, but rather the space of such objects is explored. In particular, we study the transformations of attraction sets when the asymptotic-type constraints are varied. Cases where attraction sets are generated by filters forming asymptotic-type constraints are particularly highlighted. The role of ultrafilters in the above-mentioned transformations is clarified. Namely, each ultrafilter on the set of ordinary solutions is associated with an element of attraction. As a consequence, an attraction operator is defined; by means of this operator, a new representation for attraction sets generated by filters is established.

2. General notions and definitions

We use standard set-theoretical notation, including quantifiers ($\forall$, $\exists$), logical connectives ($\&$, $\vee$, $\implies$, $\iff$, and others), and special symbols: def (by definition), $\triangleq$ (equality by definition), and $\exists!$ (there exists a unique element). We assume that a family is a set whose elements are themselves sets. We also adopt the axiom of choice.

If a and b are objects, then by $\{a; b\}$ we denote the set such that $a \in \{a; b\}$, $b \in \{a; b\}$, and for any $z \in \{a; b\}$, $(z = a) \vee (z = b)$ holds; that is, $\{a; b\}$ is the unordered pair of these objects. For any object x , the set $\{x\} \triangleq \{x; x\}$ is the singleton corresponding to x . Sets are objects; therefore, for any objects x and y , the expression $(x, y) \triangleq \{\{x\}; \{x; y\}\}$ defines the ordered pair with first element x and second element y (see [14, Ch. II, Sect. 3]). If h is an ordered pair, then $\text{pr}_1(h)$ and $\text{pr}_2(h)$ denote the first and second elements of h , respectively; by virtue of the equality $h = (\text{pr}_1(h), \text{pr}_2(h))$, these elements are uniquely defined.

If H is a set, then $\mathcal{P}(H)$ denotes the family of all subsets of H , and $\mathcal{P}'(H) \triangleq \mathcal{P}(H) \setminus \{\emptyset\}$. Moreover, let $\text{Fin}(H)$ denote the family of all finite sets in $\mathcal{P}'(H)$, that is, the family of all nonempty finite subsets of H (any family can be used as H).

Functions. If A and B are nonempty sets, then B^A denotes the set of all functions from A

to B ; for $g \in B^A$ (that is, for $g : A \rightarrow B$) and $a \in A$, the element $g(a) \in B$ is the value of g at the point a . If A and B are nonempty sets, $f \in B^A$, and $C \in \mathcal{P}(A)$, then [14, Ch. II, Sect. 7]

$$f^1(C) \triangleq \{f(x) : x \in C\} \in \mathcal{P}(B)$$

is the image of the set C under the action of f ; if $D \in \mathcal{P}(B)$, then, as usual, $f^{-1}(D)$ denotes the preimage of the set D under f . For a nonempty family $\mathcal{M}$, we introduce the family

$$(\text{Cen})[\mathcal{M}] \triangleq \left\{ \mathcal{Z} \in \mathcal{P}'(\mathcal{M}) \mid \bigcap_{Z \in \mathcal{K}} Z \neq \emptyset \quad \forall \mathcal{K} \in \text{Fin}(\mathcal{Z}) \right\} \in \mathcal{P}(\mathcal{P}'(\mathcal{M}))$$

of all nonempty centered subfamilies of $\mathcal{M}$. As usual, $\mathbb{R}$ is the real line, $\mathbb{N} \triangleq \{1; 2; \dots\} \in \mathcal{P}'(\mathbb{R})$, and $\overline{1, n} \triangleq \{k \in \mathbb{N} \mid k \leq n\}$ under $n \in \mathbb{N}$. We suppose that the elements of $\mathbb{N}$ (the natural numbers) are not sets. Taking this into account, for every nonempty set H and $n \in \mathbb{N}$, we use the notation H^n instead of $H^{\overline{1, n}}$ for the set of all functions from $\overline{1, n}$ to H (these functions are called tuples). Of course, any nonempty family can be used as H . In denoting functions, we often use the index form (families with indices, see [17, Sect. 1.1]).

For every family $\mathcal{H}$ and set T , we define

$$([\mathcal{H}](T) \triangleq \{H \in \mathcal{H} \mid T \subset H\} \in \mathcal{P}(\mathcal{H})) \& (\mathcal{H}|_T \triangleq \{H \cap T : H \in \mathcal{H}\} \in \mathcal{P}(\mathcal{P}(T))).$$

If $\mathbb{M}$ is a set and $\mathcal{M} \in \mathcal{P}'(\mathcal{P}(\mathbb{M}))$, then

$$\mathbf{C}_{\mathbb{M}}[\mathcal{M}] \triangleq \{\mathbb{M} \setminus M : M \in \mathcal{M}\} \in \mathcal{P}'(\mathcal{P}(\mathbb{M}))$$

is the family of subsets of $\mathbb{M}$ dual to $\mathcal{M}$.

Special families. Fix a set $\mathbf{I}$ throughout this section. We consider families from $\mathcal{P}'(\mathcal{P}(\mathbf{I}))$, that is, nonempty families of subsets of $\mathbf{I}$. In particular,

$$\pi[\mathbf{I}] \triangleq \{\mathcal{I} \in \mathcal{P}'(\mathcal{P}(\mathbf{I})) \mid (\emptyset \in \mathcal{I}) \& (\mathbf{I} \in \mathcal{I}) \& (A \cap B \in \mathcal{I} \quad \forall A \in \mathcal{I} \quad \forall B \in \mathcal{I})\} \quad (2.1)$$

is the family of all π -systems of subsets of $\mathbf{I}$ containing the “zero” $\emptyset$ and the “unit” $\mathbf{I}$. Define

$$(\text{LAT})_0[\mathbf{I}] \triangleq \{\mathcal{I} \in \pi[\mathbf{I}] \mid A \cup B \in \mathcal{I} \quad \forall A \in \mathcal{I} \quad \forall B \in \mathcal{I}\}$$

as the family of all lattices of subsets of $\mathbf{I}$ containing the “zero” and “unit”. Next,

$$\tilde{\pi}^0[\mathbf{I}] \triangleq \{\mathcal{I} \in \pi[\mathbf{I}] \mid \forall I \in \mathcal{I} \quad \forall x \in \mathbf{I} \setminus I \quad \exists J \in \mathcal{I} : (x \in J) \& (J \cap I = \emptyset)\} \quad (2.2)$$

is the family of all separable π -systems of (2.1). We also use the family

$$(\text{top})[\mathbf{I}] \triangleq \left\{ \tau \in \pi[\mathbf{I}] \mid \bigcup_{G \in \mathcal{G}} G \in \tau \quad \forall \mathcal{G} \in \mathcal{P}(\tau) \right\} = \left\{ \tau \in (\text{LAT})_0[\mathbf{I}] \mid \bigcup_{G \in \mathcal{G}} G \in \tau \quad \forall \mathcal{G} \in \mathcal{P}(\tau) \right\}$$

of all topologies on the set $\mathbf{I}$. If $\tau \in (\text{top})[\mathbf{I}]$, then $(\mathbf{I}, \tau)$ is a topological space with unit $\mathbf{I}$, and $\mathbf{C}_{\mathbf{I}}[\tau] \in (\text{LAT})_0[\mathbf{I}]$ is the family of all closed in $(\mathbf{I}, \tau)$ subsets of $\mathbf{I}$. Define

$$(\mathbf{c} - \text{top})[\mathbf{I}] \triangleq \left\{ \tau \in (\text{top})[\mathbf{I}] \mid \bigcap_{F \in \mathcal{F}} F \neq \emptyset \quad \forall \mathcal{F} \in (\text{Cen})[\mathbf{C}_{\mathbf{I}}[\tau]] \right\} \quad (2.3)$$

as the family of all compact topologies on $\mathbf{I}$. If $\tau \in (\mathbf{c} - \text{top})[\mathbf{I}]$, then $(\mathbf{I}, \tau)$ is a compact topological space. For $\tau \in (\text{top})[\mathbf{I}]$ and $x \in \mathbf{I}$, let $N_\tau^0(x) \triangleq \{G \in \tau \mid x \in G\}$ and

$$N_\tau(x) \triangleq \{H \in \mathcal{P}(\mathbf{I}) \mid \exists G \in N_\tau^0(x) : G \subset H\} \quad (2.4)$$

be the family of all neighborhoods of the point x in the topological space $(\mathbf{I}, \tau)$. Define

$$\begin{aligned} (\text{top})_0[\mathbf{I}] &\triangleq \{\tau \in (\text{top})[\mathbf{I}] \mid \forall y \in \mathbf{I} \forall z \in \mathbf{I} \setminus \{y\} \exists G_1 \in N_\tau^0(y) \exists G_2 \in N_\tau^0(z) : G_1 \cap G_2 = \emptyset\} \\ &= \{\tau \in (\text{top})[\mathbf{I}] \mid \forall y \in \mathbf{I} \forall z \in \mathbf{I} \setminus \{y\} \exists H_1 \in N_\tau(y) \exists H_2 \in N_\tau(z) : H_1 \cap H_2 = \emptyset\} \end{aligned}$$

as the family of all topologies that make $\mathbf{I}$ a T_2 -space. Let

$$(\mathbf{c} - \text{top})_0[\mathbf{I}] \triangleq (\mathbf{c} - \text{top})[\mathbf{I}] \cap (\text{top})_0[\mathbf{I}];$$

if $\tau \in (\mathbf{c} - \text{top})_0[\mathbf{I}]$, then the topological space $(\mathbf{I}, \tau)$ is called a compactum.

If $\tau \in (\text{top})[\mathbf{I}]$ and $A \in \mathcal{P}(\mathbf{I})$, then $[\mathbf{C}_\mathbf{I}[\tau]](A) \in \mathcal{P}'(\mathbf{C}_\mathbf{I}[\tau])$ and

$$\text{cl}(A, \tau) \triangleq \bigcap_{F \in [\mathbf{C}_\mathbf{I}[\tau]](A)} F$$

is the closure of A in the topological space $(\mathbf{I}, \tau)$.

3. Some topological constructions

If (X, τ) is a topological space and $Y \in \mathcal{P}(X)$, then $\tau|_Y \in (\text{top})[Y]$; the resulting topological space $(Y, \tau|_Y)$ is called a subspace of (X, τ) . For every topological space (X, τ) , define

$$(\tau - \text{comp})[X] \triangleq \{K \in \mathcal{P}(X) \mid \tau|_K \in (\mathbf{c} - \text{top})[K]\}$$

as the family of all compact (in (X, τ)) subsets of X . Throughout this (brief) section, we fix topological spaces (U, τ_1) and (V, τ_2) with $U \neq \emptyset$ and $V \neq \emptyset$; that is, $\tau_1 \in (\text{top})[U]$ and $\tau_2 \in (\text{top})[V]$. Define

$$C(U, \tau_1, V, \tau_2) \triangleq \{f \in V^U \mid f^{-1}(G) \in \tau_1 \ \forall G \in \tau_2\}, \quad (3.1)$$

$$\begin{aligned} C_{\text{cl}}(U, \tau_1, V, \tau_2) &\triangleq \{f \in C(U, \tau_1, V, \tau_2) \mid f^1(F) \in \mathbf{C}_V[\tau_2] \ \forall F \in \mathbf{C}_U[\tau_1]\} \\ &= \{f \in V^U \mid f^1(\text{cl}(A, \tau_1)) = \text{cl}(f^1(A), \tau_2) \ \forall A \in \mathcal{P}(U)\}. \end{aligned} \quad (3.2)$$

Note the following important special case:

$$((\tau_1 \in (\mathbf{c} - \text{top})[U]) \& (\tau_2 \in (\text{top})_0[V])) \implies (C(U, \tau_1, V, \tau_2) = C_{\text{cl}}(U, \tau_1, V, \tau_2)). \quad (3.3)$$

In (3.1), the set of all continuous functions from (U, τ_1) to (V, τ_2) is defined; (3.2) is the set of all closed (i.e., continuous and closed) functions between these spaces. By (3.3), every continuous function from a compact topological space to a T_2 -space is closed. Of course, every constant function is continuous.

If $f \in V^U$ and $\mathcal{H} \in \mathcal{P}'(\mathcal{P}(U))$, then the family

$$f^1[\mathcal{H}] \triangleq \{f^1(H) : H \in \mathcal{H}\} \in \mathcal{P}'(\mathcal{P}(V)) \quad (3.4)$$

is called the “image” of the initial nonempty family $\mathcal{H}$. If $\mathbb{H} \in \mathcal{P}(U)$ and $\mathcal{H} = \{\mathbb{H}\}$, then

$$f^1[\mathcal{H}] = f^1[\{\mathbb{H}\}] = \{f^1(\mathbb{H})\}.$$

The following important property holds:

$$f^1(K) \in (\tau_2 - \text{comp})[V] \quad \forall f \in C(U, \tau_1, V, \tau_2) \quad \forall K \in (\tau_1 - \text{comp})[U];$$

see [8, 3.1.10]. That is, the continuous image of a compact set is compact.

4. Directed families, filters, and filter bases

In this section, we fix a nonempty set J .

In what follows, this set may be realized in various ways. In essence, J serves as a parameter with specific realizations to be considered as needed. We consider various subfamilies of $\mathcal{P}(J)$. In particular,

$$\beta[J] \triangleq \{\mathcal{J} \in \mathcal{P}'(\mathcal{P}(J)) \mid \forall J_1 \in \mathcal{J} \forall J_2 \in \mathcal{J} \exists J_3 \in \mathcal{J} : J_3 \subset J_1 \cap J_2\} \quad (4.1)$$

is the family of all nonempty directed subfamilies of $\mathcal{P}(J)$. In addition,

$$\{\cap\}_\#(\tilde{\mathcal{J}}) \triangleq \left\{ \bigcap_{\Sigma \in \mathcal{K}} \Sigma : \mathcal{K} \in \text{Fin}(\tilde{\mathcal{J}}) \right\} \in \beta[J] \quad \forall \tilde{\mathcal{J}} \in \mathcal{P}'(\mathcal{P}(J)). \quad (4.2)$$

Now, we introduce filter bases; namely, we consider the family

$$\beta_0[J] \triangleq \{\mathcal{B} \in \beta[J] \mid \emptyset \notin \mathcal{B}\} = \{\mathcal{B} \in \mathcal{P}'(\mathcal{P}(J)) \mid \forall B_1 \in \mathcal{B} \forall B_2 \in \mathcal{B} \exists B_3 \in \mathcal{B} : B_3 \subset B_1 \cap B_2\} \quad (4.3)$$

of all filter bases on the set J . Moreover, note that (see [1, Ch. I])

$$\mathfrak{F}[J] \triangleq \{\mathcal{F} \in \mathcal{P}'(\mathcal{P}(J)) \mid (A \cap B \in \mathcal{F} \quad \forall A \in \mathcal{F} \quad \forall B \in \mathcal{F}) \& ([\mathcal{P}(J)](F) \subset \mathcal{F} \quad \forall F \in \mathcal{F})\} \quad (4.4)$$

is the nonempty family (indeed, $\{J\} \in \mathfrak{F}[J]$) of all filters on J . In addition,

$$(J - \mathbf{f})[\mathcal{B}] \triangleq \{F \in \mathcal{P}[J] \mid \exists B \in \mathcal{B} : B \subset F\} \in \mathfrak{F}[J] \quad \forall \mathcal{B} \in \beta_0[J]. \quad (4.5)$$

Thus (see (4.5)), filter bases from (4.3) generate filters of the family (4.4) via the simple rule (4.5). In connection with (2.4), note that for all $\tau \in (\text{top})[J]$, for all $x \in J$,

$$N_\tau^0(x) \in \beta_0[J] : \quad N_\tau[x] = (J - \mathbf{f})[N_\tau^0(x)] \in \mathfrak{F}[J]. \quad (4.6)$$

Using (4.6), recall the well-known convergence notion [1, Ch. I]: for all $\tau \in (\text{top})[J]$, $\mathcal{B} \in \beta_0[J]$, and $x \in J$,

$$(\mathcal{B} \xrightarrow{\tau} x) \stackrel{\text{def}}{\iff} (N_\tau(x) \subset (J - \mathbf{f})[\mathcal{B}]). \quad (4.7)$$

Using the inclusion $\mathfrak{F}[J] \subset \beta_0[J]$ and the evident property

$$(J - \mathbf{f})[\mathcal{F}] = \mathcal{F} \quad \forall \mathcal{F} \in \mathfrak{F}[J],$$

from (4.7), we obtain the following natural corollary for filters: for all $\tau \in (\text{top})[J]$, $\mathcal{F} \in \mathfrak{F}[J]$, and $x \in J$,

$$(\mathcal{F} \xrightarrow{\tau} x) \iff (N_\tau(x) \subset \mathcal{F}). \quad (4.8)$$

We use ultrafilters, i.e., maximal filters; then,

$$\mathfrak{F}_u[J] \triangleq \{\mathcal{U} \in \mathfrak{F}[J] \mid \forall \mathcal{F} \in \mathfrak{F}[J] (\mathcal{U} \subset \mathcal{F} \implies (\mathcal{U} = \mathcal{F}))\} \quad (4.9)$$

is the nonempty family of all ultrafilters on J . As the simplest example of an ultrafilter, for $x \in J$, we set

$$(J - \text{ult})[x] \triangleq \{F \in \mathcal{P}(J) \mid x \in F\} \in \mathfrak{F}_u[J] \quad (4.10)$$

((4.10) is the trivial ultrafilter associated with x). Clearly, (4.10) realizes an embedding of the set J into the family (4.9):

$$(J - \text{ult})[\cdot] \triangleq ((J - \text{ult})[x])_{x \in J} \in \mathfrak{F}_u[J]^J. \quad (4.11)$$

Along with (4.11), define the mapping $\mathbf{S}_J \in \mathcal{P}(\mathfrak{F}_{\mathbf{u}}[J])^{\mathcal{P}(J)}$ as follows:

$$\mathbf{S}_J(A) \triangleq \{\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[J] \mid A \in \mathcal{U}\} \quad \forall A \in \mathcal{P}(J). \quad (4.12)$$

By [4, Sect. 8.2], we define the Stone topology

$$\tau_{\mathfrak{H}}[J] \triangleq \{G \in \mathcal{P}(\mathfrak{F}_{\mathbf{u}}[J]) \mid \forall \mathcal{U} \in G \exists U \in \mathcal{U} : \mathbf{S}_J(U) \subset G\} \in (\mathbf{c} - \text{top})_0[\mathfrak{F}_{\mathbf{u}}[J]]; \quad (4.13)$$

thus, we obtain a zero-dimensional compactum

$$(\mathfrak{F}_{\mathbf{u}}[J], \tau_{\mathfrak{H}}[J]). \quad (4.14)$$

5. Attraction sets in topological spaces

In this section, we fix a nonempty set E , whose elements are called usual solutions. We keep in mind that each element $e \in E$ admits immediate realization. We also fix a nonempty set X and a topology $\tilde{\tau} \in (\text{top})[X]$; thus, $(X, \tilde{\tau})$, $X \neq \emptyset$, is a topological space. Finally, we fix $f \in X^E$ as a target operator. Recall that $f^1(\Sigma) = \{f(x) : x \in \Sigma\}$ for $\Sigma \in \mathcal{P}(E)$. Then,

$$(\text{AS})[E; X; \tilde{\tau}; f; \mathcal{E}] \triangleq \bigcap_{\Sigma \in \mathcal{E}} \text{cl}(f^1(\Sigma), \tilde{\tau}) \in \mathbf{C}_X[\tilde{\tau}] \quad \forall \mathcal{E} \in \beta[E]; \quad (5.1)$$

where this definition is considered as a preliminary one. If $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, then (see (4.2)) we consider the following attraction set:

$$(\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{E}] \triangleq (\text{AS})[E; X; \tilde{\tau}; f; \{\cap\}_{\#}(\mathcal{E})] \in \mathbf{C}_X[\tilde{\tau}]. \quad (5.2)$$

In connection with (5.1) and (5.2), we note (see [4, (8.3.10), Propositions 8.3.1 and 8.4.1]) that a series of equivalent representations for attraction sets can be obtained. Now, recall that

$$(\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{E}] = (\text{AS})[E; X; \tilde{\tau}; f; \mathcal{E}] \quad \forall \mathcal{E} \in \beta[E] \quad (5.3)$$

(see [4, Proposition 8.4.1]).

Now, let us consider the simplest example of an attraction set (5.1), (5.3). Here, we present the construction in a meaningful way, using a scalar controlled system:

$$\dot{x}(t) = u(t), \quad t \in [0, 1[, \quad (5.4)$$

with zero initial state: $x(0) = 0$. In (5.4), we allow nonnegative controls u of the following type: u is any piecewise constant, right-continuous, real-valued function on $[0, 1[$ satisfying

$$\int_0^1 u(t) dt \leq 1. \quad (5.5)$$

Let $\mathbb{U}$ denote the set of all such functions (see (5.5)). Then,

$$\mathbf{x}_u(t) \triangleq \int_0^t u(\tau) d\tau \in [0, \infty[\quad \forall u \in \mathbb{U} \quad \forall t \in [0, 1].$$

In this example, we identify E with $\mathbb{U}$. For $u \in \mathbb{U}$, consider the following phase constraints:

$$\mathbf{x}_u(t) = 0 \quad \forall t \in [0, 1]. \quad (5.6)$$

Define the set

$$\mathbb{G} \triangleq \{\mathbf{x}_u(1) : u \in \mathbb{U}, \mathbf{x}_u(t) = 0 \ \forall t \in [0, 1[\}$$

as the reachability domain under these phase constraints. It is clear that $\mathbb{G} = \{0\}$. Now, let

$$\mathbf{B}_\theta \triangleq \{u \in \mathbb{U} \mid \mathbf{x}_u(t) = 0 \ \forall t \in [0, \theta[\} \quad \forall \theta \in [0, 1[$$

and define

$$\mathfrak{B} \triangleq \{\mathbf{B}_\tau : \tau \in [0, 1[\}.$$

Clearly, the intersection of all sets in $\mathfrak{B}$ is the set of all $u \in \mathbb{U}$ satisfying (5.6), which coincides with $\{\mathbf{O}\}$, where $\mathbf{O} \in \mathbb{U}$ and $\mathbf{O}(t) = 0$ for all $t \in [0, 1[$. Moreover, we have $\mathfrak{B} \in \beta_0[\mathbb{U}]$, so we may use (5.1) with $\mathcal{E} = \mathfrak{B}$. Define $\tilde{h} \in \mathbb{R}^{\mathbb{U}}$ by

$$\tilde{h}(u) \triangleq \int_0^1 u(t) dt \quad \forall u \in \mathbb{U}.$$

That is, in (5.1), we set $f = \tilde{h}$. Then,

$$\tilde{h}^1(\mathbf{B}_\theta) = \{\tilde{h}(u) : u \in \mathbf{B}_\theta\} = [0, 1]$$

for each $\theta \in [0, 1[$. Indeed, for $\theta \in [0, 1[$ we can construct a function $\tilde{u}_\theta \in \mathbf{B}_\theta$, defined by

$$(\tilde{u}_\theta(\xi) \triangleq 0 \quad \forall \xi \in [0, \theta[\}) \& \left(\tilde{u}_\theta(\xi) \triangleq \frac{1}{1-\theta} \quad \forall \xi \in [\theta, 1[\right),$$

so that $\mathbf{x}_{\tilde{u}_\theta}(1) = 1$. Moreover, $\tilde{h}^1(\mathbf{B}_\theta)$ is a convex set. Therefore, in this example, the attraction set (5.1) coincides with $[0, 1]$, while $[0, 1] \neq \overline{\mathbb{G}}$, where the overline denotes the closure in $\mathbb{R}$ with respect to the usual $|\cdot|$ -topology. Thus, there is a jump when (5.6) is weakened. Therefore, in this example, (5.1) is more interesting from a practical point of view.

Of course, we can use filter bases and filters as $\mathcal{E}$ in (5.1)–(5.3); in addition, $\mathfrak{F}[E] \subset \beta_0[E]$. In this connection, we note the following easily verifiable property:

$$(\text{AS})[E; X; \tilde{\tau}; f; \mathcal{B}] = (\text{AS})[E; X; \tilde{\tau}; f; (E - \mathbf{f})[\mathcal{B}]] \quad \forall \mathcal{B} \in \beta_0[E]. \quad (5.7)$$

Recall that for any $\mathcal{B} \in \beta_0[E]$, the property $f^1[\mathcal{B}] \in \beta_0[X]$ holds and

$$((E - \mathbf{f})[\mathcal{B}] \in \mathfrak{F}_{\mathbf{u}}[E]) \implies ((X - \mathbf{f})[f^1[\mathcal{B}]] \in \mathfrak{F}_{\mathbf{u}}[X]) \quad (5.8)$$

(see [4, Proposition 8.2.1; 1, Ch. I]). Using (5.8), we obtain the following representation of the attraction set (see [4, Propositions 8.3.1, 8.4.1, and 8.4.2]): for any $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$

$$(\text{as})[E; X; \tilde{\tau}; f; \mathcal{E}] = \{x \in X \mid \exists \mathcal{U} \in [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{E}) : f^1[\mathcal{U}] \xrightarrow{\tilde{\tau}} x\}; \quad (5.9)$$

so, by (5.9), ultrafilters can be used as analogs of Warga's approximate solutions (see [17, Ch. III]). Moreover, for any $\Sigma \in \mathcal{P}(E)$, we have the inclusion $\{\Sigma\} \in \beta[E]$, and by (5.1),

$$(\text{AS})[E; X; \tilde{\tau}; f; \{\Sigma\}] = \text{cl}(f^1(\Sigma), \tilde{\tau}). \quad (5.10)$$

In connection with (5.9) and (5.10), we note the equivalent representation [4, (8.3.10)] realized in the directed class. Now, we introduce two families of attraction sets. Set

$$(\tilde{\tau} - \text{AS})[f] \triangleq \{(\text{as})[E; X; \tilde{\tau}; f; \mathcal{E}] : \mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))\} \in \mathcal{P}'(\mathbf{C}_X[\tilde{\tau}]) \quad (5.11)$$

as the family of all attraction sets under fixed X , $\tilde{\tau}$, and f (recall that X is uniquely specified by $\tilde{\tau}$). Moreover,

$$\mathfrak{F}_u[E] \subset \mathfrak{F}[E] \subset \beta_0[E] \subset \beta[E]. \quad (5.12)$$

Then, by (5.1), (5.3), and (5.12), for any $\mathcal{F} \in \mathfrak{F}[E]$,

$$(\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{F}] = (\mathbf{AS})[E; X; \tilde{\tau}; f; \mathcal{F}] = \bigcap_{F \in \mathcal{F}} \text{cl}(f^1(F), \tilde{\tau}) \in \mathbf{C}_X[\tilde{\tau}]; \quad (5.13)$$

where, of course, ultrafilters can be used as $\mathcal{F}$. Using (5.13), we set

$$\begin{aligned} ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] &\stackrel{\Delta}{=} \{(\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{F}] : \mathcal{F} \in \mathfrak{F}[E]\} \\ &= \{(\mathbf{AS})[E; X; \tilde{\tau}; f; \mathcal{F}] : \mathcal{F} \in \mathfrak{F}[E]\} \in \mathcal{P}'(\mathbf{C}_X[\tilde{\tau}]). \end{aligned} \quad (5.14)$$

Clearly,

$$((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] \subset (\tilde{\tau} - \mathbf{AS})[f].$$

Proposition 1. *The following equality holds:*

$$(\tilde{\tau} - \mathbf{AS})[f] = ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] \cup \{\emptyset\}. \quad (5.15)$$

P r o o f. Let $M \in (\tilde{\tau} - \mathbf{AS})[f]$. Using (5.11), we choose $\mathcal{M} \in \mathcal{P}'\mathcal{P}(E)$ such that

$$M = (\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{M}].$$

Then, by (4.2), for

$$\mu \stackrel{\Delta}{=} \{\cap\}_{\#}(\mathcal{M}) \in \beta[E]$$

we obtain (see (5.2))

$$M = (\mathbf{AS})[E; X; \tilde{\tau}; f; \mu]. \quad (5.16)$$

In addition, by (4.1) and (4.3), either $\mu \in \beta_0[E]$ or $\emptyset \in \mu$. We consider both cases separately.

Let $\mu \in \beta_0[E]$. Then, by (4.5), define $\mathfrak{M} \stackrel{\Delta}{=} (E - \mathfrak{f})[\mu] \in \mathfrak{F}[E]$. Therefore, by (5.1), (5.7), and (5.14),

$$M = (\mathbf{AS})[E; X; \tilde{\tau}; f; \mathfrak{M}] \in ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f].$$

Hence,

$$(\mu \in \beta_0[E]) \implies (M \in ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f]).$$

If $\emptyset \in \mu$, then by (5.1), $(\mathbf{AS})[E; X; \tilde{\tau}; f; \mu] = \emptyset$, and by (5.16), $M = \emptyset$. Thus, $(\emptyset \in \mu) \implies (M = \emptyset)$. Consequently,

$$M \in ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] \cup \{\emptyset\}.$$

Therefore,

$$(\tilde{\tau} - \mathbf{AS})[f] \subset ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] \cup \{\emptyset\}. \quad (5.17)$$

Note that $\emptyset \in \mathcal{P}(E)$ and $\{\emptyset\} \in \beta[E]$. Then, by (5.1) and (5.3),

$$(\mathbf{as})[E; X; \tilde{\tau}; f; \{\emptyset\}] = (\mathbf{AS})[E; X; \tilde{\tau}; f; \{\emptyset\}] = \emptyset \in (\tilde{\tau} - \mathbf{AS})[f].$$

Therefore, $\{\emptyset\} \subset (\tilde{\tau} - \mathbf{AS})[f]$, and hence,

$$((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] \cup \{\emptyset\} \subset (\tilde{\tau} - \mathbf{AS})[f].$$

Using (5.17), we obtain the required equality (5.15).

Let us recall the example of [4, Sect. 8.9]. In this example, attainability problems are presented in which the attraction set coincides with $\emptyset$. The asymptotic-type constraints are specified by filter bases. By using (5.7), we can interpret this example as an attainability problem where the asymptotic-type constraints are generated by a filter. Thus, in general, the families appearing in (5.15)—specifically, those on the right-hand side—need not be disjoint. In the following sections, we will introduce a natural condition that excludes this possibility.

6. Attainability problem with precompact target operator and some representations for attraction sets

In what follows, we fix a nonempty topological space (Y, τ) , $Y \neq \emptyset$, as the main object. Thus, $\tau \in (\text{top})[Y]$. Let

$$\mathbb{F}_c^0[E; Y; \tau] \triangleq \{f \in Y^E \mid f^1(E) \in (\tau - \text{comp})^0[Y]\}, \quad (6.1)$$

where $(\tau - \text{comp})^0[Y] \triangleq \{H \in \mathcal{P}(Y) \mid \exists K \in (\tau - \text{comp})[Y] : H \subset K\}$. We call functions from (6.1) precompact functions. It is easy to check that if $\tau \in (\text{top})_0[Y]$, $h \in \mathbb{F}_c^0[E; Y; \tau]$, and $\mathcal{B} \in \beta_0[E]$, then

$$(\text{AS})[E; Y; \tau; h; \mathcal{B}] \in (\tau - \text{comp})[Y] \setminus \{\emptyset\}. \quad (6.2)$$

In this connection, recall that $\beta_0[E] \subset (\text{Cen})[\mathcal{P}(E)]$ (see (4.3) and [3, (3.3.16)]). From (6.2), it follows that if $\tau \in (\text{top})_0[Y]$, $h \in \mathbb{F}_c^0[E; Y; \tau]$, and $\mathcal{F} \in \mathfrak{F}[E]$, then

$$(\text{AS})[E; Y; \tau; h; \mathcal{F}] \in (\tau - \text{comp})[Y] \setminus \{\emptyset\}. \quad (6.3)$$

Recall that for any topological space $(K, \mathbf{t})$, $K \neq \emptyset$, with $\mathbf{t} \in (\mathbf{c} - \text{top})[K]$ (i.e., any nonempty compact topological space $(K, \mathbf{t})$), $m \in K^E$, $\tau \in (\text{top})_0[Y]$, and $g \in C(K, \mathbf{t}, Y, \tau)$,

$$g \circ m \in \mathbb{F}_c^0[E; Y; \tau]$$

(where, $\circ$ denotes composition). Furthermore; we have the following useful property (see [3, Proposition 5.2.1]):

$$(\text{AS})[E; Y; \tau; g \circ m; \mathcal{E}] = g^1((\text{AS})[E; K; \mathbf{t}; m; \mathcal{E}]) \quad \forall \mathcal{E} \in \beta[E]. \quad (6.4)$$

We note that (6.4) allows a number of generalizations (for example, see [6, Propositions 3.4.10 and 3.4.11], [5]). Of course, in (6.4), the compactum (4.14) can be taken as $(K, \mathbf{t})$. Moreover, recall that by [4, Proposition 8.3.1],

$$(\text{as})[E; Y; \tau; f; \mathcal{E}] = \{y \in Y \mid \exists \mathcal{U} \in [\mathfrak{F}_u[E]](\mathcal{E}) : f^1[\mathcal{U}] \xrightarrow{\tau} y\} \quad \forall f \in Y^E \quad \forall \mathcal{E} \in \mathcal{P}'(\mathcal{P}(E));$$

see also [4, (8.3.10)], where a representation of the attraction set in the directedness class is given.

7. Filters and attainability sets, 1

Recall some properties noted in [4, Ch. 9] and [1, Ch. I]. To this end, we assume that

$$\mathcal{E}_1 \{\cap\} \mathcal{E}_2 \triangleq \{\text{pr}_1(z) \cap \text{pr}_2(z) : z \in \mathcal{E}_1 \times \mathcal{E}_2\} \quad \forall \mathcal{E}_1 \in \mathcal{P}'(\mathcal{P}(E)) \quad \forall \mathcal{E}_2 \in \mathcal{P}'(\mathcal{P}(E)); \quad (7.1)$$

see [4, (9.3.6)]. We can use (7.1) for filters; furthermore, by [4, Proposition 9.3.1], for all $\mathcal{F}_1 \in \mathfrak{F}[E]$, $\mathcal{F}_2 \in \mathfrak{F}[E]$, and $\mathcal{F}_3 \in \mathfrak{F}[E]$,

$$\begin{aligned} ((\mathcal{F}_1 \subset \mathcal{F}_3) \& (\mathcal{F}_2 \subset \mathcal{F}_3) \& (\forall \mathcal{F} \in \mathfrak{F}[E] ((\mathcal{F}_1 \subset \mathcal{F}) \& (\mathcal{F}_2 \subset \mathcal{F})) \implies (\mathcal{F}_3 \subset \mathcal{F}))) \\ \implies (\mathcal{F}_3 = \mathcal{F}_1 \{\cap\} \mathcal{F}_2). \end{aligned} \quad (7.2)$$

In connection with (7.2), we also recall the constructions of [1, Ch. I, § 6]. The following obvious corollary holds: in (7.2), the specified representation of the supremum for $\{\mathcal{F}_1; \mathcal{F}_2\}$ applies if this supremum exists. We also have the following consequence (see [4, Corollary 9.3.1]): for any $\mathcal{F}_1 \in \mathfrak{F}[E]$ and $\mathcal{F}_2 \in \mathfrak{F}[E]$,

$$(\exists \mathcal{F} \in \mathfrak{F}[E] : (\mathcal{F}_1 \subset \mathcal{F}) \& (\mathcal{F}_2 \subset \mathcal{F})) \iff (\mathcal{F}_1 \{\cap\} \mathcal{F}_2 \in \mathfrak{F}[E]). \quad (7.3)$$

From (7.3), we obtain the following equivalence:

$$(A \cap B \neq \emptyset \quad \forall A \in \mathcal{F}_1 \quad \forall B \in \mathcal{F}_2) \iff (\mathcal{F}_1 \{\cap\} \mathcal{F}_2 \in \mathfrak{F}[E]). \quad (7.4)$$

(In connection with (7.4), we only note that $\mathcal{F}_1 \{\cap\} \mathcal{F}_2 \in \beta_0[E]$ under the property $A \cap B \neq \emptyset$ for all $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Here, (4.5) and (7.3) should be used.) Note that in the general case, for $\mathcal{F}_1 \in \mathfrak{F}[E]$ and $\mathcal{F}_2 \in \mathfrak{F}[E]$,

$$\mathcal{F}_1 \{\cap\} \mathcal{F}_2 \in \beta[E] : (\mathcal{F}_1 \subset \mathcal{F}_1 \{\cap\} \mathcal{F}_2) \& (\mathcal{F}_2 \subset \mathcal{F}_1 \{\cap\} \mathcal{F}_2); \quad (7.5)$$

moreover, in this case, we obtain the following equality:

$$[\mathfrak{F}_u[E]](\mathcal{F}_1 \cup \mathcal{F}_2) = [\mathfrak{F}_u[E]](\mathcal{F}_1 \{\cap\} \mathcal{F}_2), \quad (7.6)$$

where the following natural representation holds for $\mathcal{F}_1 \{\cap\} \mathcal{F}_2$:

$$\mathcal{F}_1 \{\cap\} \mathcal{F}_2 = \{\cap\}_\#(\mathcal{F}_1 \cup \mathcal{F}_2). \quad (7.7)$$

From (7.7), by induction, we obtain: for any $n \in \mathbb{N}$ and $(\mathcal{F}_i)_{i \in \overline{1, n}} \in \mathfrak{F}[E]^n$, for the families

$$\left(\bigcup_{i=1}^n \mathcal{F}_i \in \mathcal{P}'(\mathcal{P}'(E)) \right) \& \left(\{\cap\}_{i=1}^n(\mathcal{F}_i) \triangleq \left\{ \bigcap_{i=1}^n F_i : (F_i)_{i \in \overline{1, n}} \in \prod_{i=1}^n \mathcal{F}_i \right\} \in \beta[E] \right), \quad (7.8)$$

the following equality holds:

$$\{\cap\}_{i=1}^n(\mathcal{F}_i) = \{\cap\}_\# \left(\bigcup_{i=1}^n \mathcal{F}_i \right). \quad (7.9)$$

(The verification of (7.9) follows straightforwardly from the definitions). As a consequence, if $\tau \in (\text{top})[Y]$, $h \in Y^E$, $n \in \mathbb{N}$, and $(\mathcal{F}_i)_{i \in \overline{1, n}} \in \mathfrak{F}[E]^n$, we have

$$(\text{as}) \left[E; X; \tau; h; \bigcup_{i=1}^n \mathcal{F}_i \right] = (\text{AS}) \left[E; Y; \tau; h; \{\cap\}_{i=1}^n(\mathcal{F}_i) \right]. \quad (7.10)$$

Now, consider the case of an arbitrary family of filters. That is, fix a nonempty set T and $(\mathcal{F}_t)_{t \in T} \in \mathfrak{F}[E]^T$; consider the family

$$\bigcup_{t \in T} \mathcal{F}_t \in \mathcal{P}'(\mathcal{P}'(E)). \quad (7.11)$$

To study the attraction set corresponding to asymptotic-type constraints generated by (7.11), we introduce the family

$$\{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t) \triangleq \bigcup_{K \in \text{Fin}(T)} \left\{ \bigcap_{t \in K} F_t : (F_t)_{t \in K} \in \prod_{t \in K} \mathcal{F}_t \right\} \in \mathcal{P}'(\mathcal{P}(E)). \quad (7.12)$$

Proposition 2. *If $A \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t)$ and $B \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t)$, then*

$$A \cap B \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t).$$

The proof follows directly from (4.4) and (7.12); in this argument, standard properties of finite sets are used. From (4.1), (7.12), and Proposition 2, we obtain

$$\{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t) \in \beta[E]. \quad (7.13)$$

Proposition 3. *The following equivalence is valid:*

$$\left(\bigcap_{t \in K} F_t \neq \emptyset \quad \forall K \in \text{Fin}(T) \quad \forall (F_t)_{t \in K} \in \prod_{t \in K} \mathcal{F}_t \right) \iff (\{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t) \in \mathfrak{F}[E]). \quad (7.14)$$

P r o o f. Let

$$\bigcap_{t \in K} F_t \neq \emptyset \quad \forall K \in \text{Fin}(T) \quad \forall (F_t)_{t \in K} \in \prod_{t \in K} \mathcal{F}_t. \quad (7.15)$$

Then, by (7.12) and (7.15),

$$\emptyset \notin \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t).$$

Therefore,

$$\{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t) \in \mathcal{P}'(\mathcal{P}'(E)) : A \cap B \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t) \quad \forall A \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t) \quad \forall B \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t). \quad (7.16)$$

Let $\Phi \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t)$. Using (7.12), we choose $\mathbb{K} \in \text{Fin}(T)$ and

$$(\Phi_t)_{t \in \mathbb{K}} \in \prod_{t \in \mathbb{K}} \mathcal{F}_t$$

such that

$$\Phi = \bigcap_{t \in \mathbb{K}} \Phi_t. \quad (7.17)$$

Let $\mathbb{H} \in [\mathcal{P}(E)](\Phi)$. Then $\mathbb{H} \in \mathcal{P}(E)$ and $\Phi \subset \mathbb{H}$. By (4.4), we obtain

$$\tilde{\Phi}_t \triangleq \Phi_t \cup \mathbb{H} \in \mathcal{F}_t \quad (7.18)$$

for all $t \in \mathbb{K}$ (indeed, $\Phi_t \in \mathcal{F}_t$ and $\tilde{\Phi}_t \in [\mathcal{P}(E)](\Phi_t)$). From (7.18), we have

$$(\tilde{\Phi}_t)_{t \in \mathbb{K}} \in \prod_{t \in \mathbb{K}} \mathcal{F}_t : \quad \tilde{\Phi} \triangleq \bigcap_{t \in \mathbb{K}} \tilde{\Phi}_t \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t). \quad (7.19)$$

By (7.18) and (7.19), $\mathbb{H} \subset \tilde{\Phi}$. Let $x_* \in \tilde{\Phi}$. Then, by (7.19), we have

$$x_* \in \tilde{\Phi}_t \quad \forall t \in \mathbb{K}. \quad (7.20)$$

In addition, $(x_* \notin \mathbb{H}) \vee (x_* \in \mathbb{H})$. Suppose $x_* \notin \mathbb{H}$. Then, by (7.20), $x_* \in \tilde{\Phi}_t \setminus \mathbb{H}$ for all $t \in \mathbb{K}$, so by (7.18),

$$x_* \in \Phi_t \quad \forall t \in \mathbb{K}.$$

Therefore, $x_* \in \Phi$ (see (7.17)), and consequently $x_* \in \mathbb{H}$, which contradicts the assumption. Therefore, the property $x_* \notin \mathbb{H}$ is impossible, and so $x_* \in \mathbb{H}$. Since the choice of x_* was arbitrary, the inclusion $\tilde{\Phi} \subset \mathbb{H}$ is established. As a consequence (see (7.19)),

$$\mathbb{H} = \tilde{\Phi} \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t).$$

Thus, we obtain the following important property:

$$[\mathcal{P}(E)](\Phi) \subset \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t).$$

Since Φ was arbitrary, by (4.4) and (7.16), the inclusion

$$\{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t) \in \mathfrak{F}[E]$$

holds (under condition (7.15)). Thus, the implication

$$\left(\bigcap_{t \in K} F_t \neq \emptyset \quad \forall K \in \text{Fin}(T) \quad \forall (F_t)_{t \in K} \in \prod_{t \in K} \mathcal{F}_t \right) \implies (\{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t) \in \mathfrak{F}[E])$$

is valid. From (4.4) and (7.12), the converse implication follows directly. Accordingly, (7.14) is established.

Proposition 4. *The following equality holds:*

$$\{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t) = \{\cap\}_{\#} \left(\bigcup_{t \in T} \mathcal{F}_t \right). \quad (7.21)$$

P r o o f. Thus, we have two nonempty families. Let $\mathbb{P} \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t)$. Then, by (7.12), for some $\mathbb{K} \in \text{Fin}(T)$ and $(P_t)_{t \in \mathbb{K}} \in \prod_{t \in \mathbb{K}} \mathcal{F}_t$, the equality

$$\mathbb{P} = \bigcap_{t \in \mathbb{K}} P_t \quad (7.22)$$

holds. Set

$$\mathbf{P} \triangleq \{P_t : t \in \mathbb{K}\} \in \text{Fin} \left(\bigcup_{t \in T} \mathcal{F}_t \right).$$

Then, $\mathbb{P}$ is the intersection of all sets of $\mathbf{P}$, and by (4.2) and (7.22),

$$\mathbb{P} \in \{\cap\}_{\#} \left(\bigcup_{t \in T} \mathcal{F}_t \right).$$

Therefore, we obtain the inclusion

$$\{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t) \subset \{\cap\}_{\#} \left(\bigcup_{t \in T} \mathcal{F}_t \right). \quad (7.23)$$

Now, choose any set

$$Q \in \{\cap\}_{\#} \left(\bigcup_{t \in T} \mathcal{F}_t \right).$$

Then, for some $r \in \mathbb{N}$ and tuple

$$(Q_l)_{l \in \overline{1, r}} \in \left(\bigcup_{t \in T} \mathcal{F}_t \right)^r, \quad (7.24)$$

we have

$$Q = \bigcap_{l=1}^r Q_l. \quad (7.25)$$

From (7.24), we have the following obvious property:

$$\mathbb{T}_l \triangleq \{t \in T \mid Q_l \in \mathcal{F}_t\} \in \mathcal{P}'(T) \quad \forall l \in \overline{1, r}. \quad (7.26)$$

Hence,

$$(\mathbb{T}_l)_{l \in \overline{1, r}} \in \mathcal{P}'(T)^r : \prod_{l=1}^r \mathbb{T}_l = \{(t_l)_{l \in \overline{1, r}} \in T^r \mid t_s \in \mathbb{T}_s \quad \forall s \in \overline{1, r}\} \in \mathcal{P}'(T^r); \quad (7.27)$$

Using (7.27), fix any tuple

$$(\theta_l)_{l \in \overline{1, r}} \in \prod_{l=1}^r \mathbb{T}_l. \quad (7.28)$$

Then, by (7.27) and (7.28), $(\theta_l)_{l \in \overline{1, r}} \in T^r$ and, as a result,

$$\Theta \triangleq \{\theta_l : l \in \overline{1, r}\} \in \text{Fin}(T). \quad (7.29)$$

From (7.26) and (7.28), for each $l \in \overline{1, r}$, we have $Q_l \in \mathcal{F}_{\theta_l}$. For $t \in \Theta$, set

$$\mathcal{L}_t \triangleq \{l \in \overline{1, r} \mid \theta_l = t\} \in \mathcal{P}'(\overline{1, r}). \quad (7.30)$$

We note that, by (4.4) and reasoning by induction, the following property is established:

$$\bigcap_{i=1}^m F_i \in \mathcal{F} \quad \forall \mathcal{F} \in \mathfrak{F}[E] \quad \forall m \in \mathbb{N} \quad \forall (F_i)_{i \in \overline{1, m}} \in \mathcal{F}^m. \quad (7.31)$$

Using (7.30) and (7.31), for each $t \in \Theta$, set

$$\mathbb{Q}_t \triangleq \bigcap_{l \in \mathcal{L}_t} Q_l \in \mathcal{F}_t. \quad (7.32)$$

Thus,

$$(\mathbb{Q}_t)_{t \in \Theta} \in \prod_{t \in \Theta} \mathcal{F}_t.$$

From (7.12) and (7.29),

$$\mathbf{Q} \triangleq \bigcap_{t \in \Theta} \mathbb{Q}_t \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t). \quad (7.33)$$

Consider two sets: Q and $\mathbf{Q}$. Let $y_* \in Q$. Then $y_* \in Q_l$ for all $l \in \overline{1, r}$. By (7.30) and (7.32),

$$y_* \in \mathbb{Q}_t \quad \forall t \in \Theta,$$

so, by (7.33), $y_* \in \mathbf{Q}$. Thus,

$$Q \subset \mathbf{Q}. \quad (7.34)$$

Let $y^* \in \mathbf{Q}$. Then, for $y^* \in E$, we have

$$y^* \in \mathbb{Q}_t \quad \forall t \in \Theta. \quad (7.35)$$

Now, let $\nu \in \overline{1, r}$. Then $\mathbb{T}_\nu = \{t \in T \mid Q_\nu \in \mathcal{F}_t\}$ (see (7.26)). By (7.28), $\theta_\nu \in \mathbb{T}_\nu$, so

$$Q_\nu \in \mathcal{F}_{\theta_\nu},$$

where $\theta_\nu \in \Theta$ by (7.29). From (7.35), $y^* \in \mathbb{Q}_{\theta_\nu}$. Therefore, by (7.32),

$$y^* \in Q_l \quad \forall l \in \mathcal{L}_{\theta_\nu}. \quad (7.36)$$

By (7.30), $\nu \in \mathcal{L}_{\theta_\nu}$, so by (7.36), $y^* \in Q_\nu$. Since ν was arbitrary, it follows that

$$y^* \in Q_l \quad \forall l \in \overline{1, r}.$$

By (7.25), $y^* \in Q$. Thus, $\mathbf{Q} \subset Q$. Using (7.34), we obtain the equality $Q = \mathbf{Q}$ and, by (7.33), $Q \in \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t)$. Therefore, the inclusion

$$\{\cap\}_{\#} \left(\bigcup_{t \in T} \mathcal{F}_t \right) \subset \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t)$$

holds. Using (7.23), we obtain the required equality (7.21).

Corollary 1. *If $h \in Y^E$, then*

$$(\mathbf{as})\left[E; Y; \tau; h; \bigcup_{t \in T} \mathcal{F}_t\right] = (\mathbf{AS})\left[E; Y; \tau; h; \{\cap\}_{t \in T}^{(\#)}(\mathcal{F}_t)\right].$$

The corresponding proof uses (5.1), (7.13), and Proposition 4. In Corollary 1, an essential generalization is obtained in comparison with (7.10). By Proposition 3 and (5.14), we have

$$\begin{aligned} & \left(\bigcap_{t \in K} F_t \neq \emptyset \quad \forall K \in \text{Fin}(T) \quad \forall (F_t)_{t \in K} \in \prod_{t \in K} \mathcal{F}_t \right) \\ \implies & \left((\mathbf{as})\left[E; Y; \tau; h; \bigcup_{t \in T} \mathcal{F}_t\right] \in ((\tau, \mathfrak{F}) - \mathbf{AS})[h] \quad \forall h \in Y^E \right). \end{aligned}$$

8. Filters and attainability sets, 2

In what follows, we suppose that $\tau \in (\text{top})_0[Y]$. Thus, we consider the T_2 -space (Y, τ) , $Y \neq \emptyset$. Moreover, we fix a precompact function $\mathbf{h} \in \mathbb{F}_c^0[E; Y; \tau]$. By (6.2), we have the following important property:

$$(\mathbf{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}] \in (\tau - \text{comp})[Y] \setminus \{\emptyset\} \quad \forall \mathcal{F} \in \mathfrak{F}[E]. \quad (8.1)$$

Returning to Proposition 1, we note that, by (5.14) and (8.1),

$$((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}] \subset (\tau - \mathbf{AS})[\mathbf{h}] \setminus \{\emptyset\}$$

and, as, a consequence (see Proposition 1),

$$((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}] = (\tau - \mathbf{AS})[\mathbf{h}] \setminus \{\emptyset\}. \quad (8.2)$$

Thus, in our case, the attraction set is nonempty if and only if it can be generated by a filter. Therefore, in this case, we avoid pathologies such as those in the example of [4, Sect. 8.9]. It is useful to note that both the precompactness condition for $\mathbf{h}$ and the T_2 -separability of (Y, τ) are typical in control problems. Thus, (8.2) holds for an important class of practical problems.

Now, we note that in (8.1) we can take ultrafilters as $\mathcal{F}$; that is:

$$(\mathbf{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] \in (\tau - \text{comp})[Y] \setminus \{\emptyset\} \quad \forall \mathcal{U} \in \mathfrak{F}_u[E]. \quad (8.3)$$

We will now consider certain constructions related to (8.3). For this, we first introduce some auxiliary statements regarding filter convergence. If $\mathcal{F} \in \mathfrak{F}[Y]$, we define the sets

$$\left((\tau - \text{LIM})[\mathcal{F}] \triangleq \{y \in Y \mid \mathcal{F} \xrightarrow{\tau} y\} \in \mathcal{P}(Y) \right) \& \left((\tau - \text{CL})[\mathcal{F}] \triangleq \bigcap_{F \in \mathcal{F}} \text{cl}(F, \tau) \in \mathcal{P}(Y) \right) \quad (8.4)$$

which satisfy

$$(\tau - \text{LIM})[\mathcal{F}] \subset (\tau - \text{CL})[\mathcal{F}]$$

(see [4, (8.3.37)]), and

$$((\tau - \text{LIM})[\mathcal{F}] = \emptyset) \vee (\exists y \in (\tau - \text{LIM})[\mathcal{F}] : (\tau - \text{CL})[\mathcal{F}] = \{y\}) \quad (8.5)$$

(see [4, Proposition 8.3.3]). Moreover, by [4, Proposition 8.3.2], we always have

$$(\tau - \text{LIM})[\mathcal{U}] = (\tau - \text{CL})[\mathcal{U}] \quad \forall \mathcal{U} \in \mathfrak{F}_u[E]. \quad (8.6)$$

We will use statements (8.4)–(8.6) in the investigation of the properties of the attraction set (8.3). In more general form, these statements are presented in [4, Sect. 8.3].

Proposition 5. *If $\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]$, then $\exists! \mathbf{y} \in Y : (\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] = \{\mathbf{y}\}$.*

P r o o f. Fix $\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]$. Then, in particular, $\mathcal{U} \in \beta_0[E]$ and $\mathbf{h}^1[\mathcal{U}] \in \beta_0[Y]$. By [4, Proposition 8.2.1],

$$(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]] \in \mathfrak{F}_{\mathbf{u}}[Y]. \quad (8.7)$$

By (5.1), (5.7), and (8.4), we have

$$(\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] = (\tau - \text{CL})[(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]]]. \quad (8.8)$$

Using (8.6) and (8.7), we obtain the following equality:

$$(\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] = (\tau - \text{LIM})[(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]]].$$

From (8.3), we have

$$(\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] \neq \emptyset.$$

Therefore, by (8.5), (8.7), and (8.8),

$$(\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] = \{\mathbf{y}\}, \quad (8.9)$$

where

$$\mathbf{y} \in (\tau - \text{LIM})[(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]]].$$

From (8.4), it follows that $\mathbf{y} \in Y$. The element $\mathbf{y} \in Y$ satisfying (8.9) is, of course, unique.

In connection with Proposition 5, we recall (5.9). Using this proposition, we introduce the operator

$$\Psi[E; Y; \tau; \mathbf{h}] \in Y^{\mathfrak{F}_{\mathbf{u}}[E]} \quad (8.10)$$

by the following natural rule: for any $\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]$, the value $\Psi[E; Y; \tau; \mathbf{h}](\mathcal{U}) \in Y$ is defined by the equality

$$(\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] = \{\Psi[E; Y; \tau; \mathbf{h}](\mathcal{U})\}; \quad (8.11)$$

we call $\Psi[E; Y; \tau; \mathbf{h}](\mathcal{U})$ the attraction element corresponding to the ultrafilter $\mathcal{U}$.

Proposition 6. *If $\mathcal{F} \in \mathfrak{F}[E]$, then*

$$(\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}] = \Psi[E; Y; \tau; \mathbf{h}]^1([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})). \quad (8.12)$$

P r o o f. Fix $\mathcal{F} \in \mathfrak{F}[E]$. Let $y_0 \in \Psi[E; Y; \tau; \mathbf{h}]^1([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}))$. Then $y_0 \in Y$, and for some ultrafilter $\mathcal{U}_0 \in [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})$, the equality $y_0 = \Psi[E; Y; \tau; \mathbf{h}](\mathcal{U}_0)$ holds. Using (8.11), we have

$$(\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}_0] = \{y_0\}. \quad (8.13)$$

By (5.1), we have the inclusion

$$(\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}_0] \subset (\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}]$$

(since, by the choice of $\mathcal{U}_0$, we have $\mathcal{F} \subset \mathcal{U}_0$). Then, by (8.13), $y_0 \in (\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}]$. Since y_0 was arbitrary, the inclusion

$$\Psi[E; Y; \tau; \mathbf{h}]^1([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})) \subset (\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}] \quad (8.14)$$

is established. Let $y_* \in (\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}]$. Now, we use (5.9) and [4, (8.2.6) and Proposition 8.3.1]. Then, for some $\mathcal{U}_* \in [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})$,

$$\mathbf{h}^1[\mathcal{U}_*] \xrightarrow{\tau} y_*.$$

As a consequence, by (4.7) and (4.8) we obtain

$$(Y - \mathbf{f})[\mathbf{h}^1[\mathcal{U}_*]] \xrightarrow{\tau} y_*.$$

Therefore,

$$y_* \in (\tau - \text{LIM})[(Y - \mathbf{f})[\mathbf{h}^1[\mathcal{U}_*]]],$$

where

$$(Y - \mathbf{f})[\mathbf{h}^1[\mathcal{U}_*]] \in \mathfrak{F}_{\mathbf{u}}[Y]$$

(see (5.8)). By (8.6),

$$y_* \in (\tau - \text{CL})[(Y - \mathbf{f})[\mathbf{h}^1[\mathcal{U}_*]]].$$

Using (3.4), (4.5), and (8.4), we obtain the following chain of equalities:

$$\begin{aligned} (\tau - \text{CL})[(Y - \mathbf{f})[\mathbf{h}^1[\mathcal{U}_*]]] &= \bigcap_{\Sigma \in (Y - \mathbf{f})[\mathbf{h}^1[\mathcal{U}_*]]} \text{cl}(\Sigma, \tau) \\ &= \bigcap_{\Sigma \in \mathbf{h}^1[\mathcal{U}_*]} \text{cl}(\Sigma, \tau) = \bigcap_{U \in \mathcal{U}_*} \text{cl}(\mathbf{h}^1(U), \tau) = (\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}_*]. \end{aligned}$$

Thus,

$$y_* \in (\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}_*].$$

By (8.11),

$$y_* = \Psi[E; Y; \tau; \mathbf{h}](\mathcal{U}_*).$$

Since $\mathcal{U}_* \in [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})$, we conclude (by the definition of the image) that

$$y_* \in \Psi[E; Y; \tau; \mathbf{h}]^1([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})).$$

Therefore, the inclusion

$$(\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}] \subset \Psi[E; Y; \tau; \mathbf{h}]^1([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}))$$

is established. Using (8.14), we obtain the required equality (8.12).

So, the attraction set for asymptotic-type constraints generated by a filter is exhausted by the attraction elements corresponding to ultrafilters that majorize this filter. We note the following obvious property of the attraction element for trivial ultrafilters:

$$\Psi[E; Y; \tau; \mathbf{h}]((E - \text{ult})[x]) = \mathbf{h}(x) \quad \forall x \in E. \quad (8.15)$$

Next, we state two simple facts regarding the nonemptiness of the attraction set. For $\mathcal{E} \in \beta[E]$,

$$((\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{E}] \neq \emptyset) \iff (\mathcal{E} \in \beta_0[E]).$$

Moreover, for $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, the following equivalence holds:

$$((\text{as})[E; Y; \tau; \mathbf{h}; \mathcal{E}] \neq \emptyset) \iff (\mathcal{E} \in (\text{Cen})[E]).$$

Proposition 7. *The following equality is valid:*

$$((\tau, \mathfrak{F}) - \text{AS})[\mathbf{h}] = (\tau - \text{AS})[\mathbf{h}] \setminus \{\emptyset\}. \quad (8.16)$$

P r o o f. By Proposition 1, we obtain

$$(\tau - \mathbf{AS})[\mathbf{h}] \setminus \{\emptyset\} \subset ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}]. \quad (8.17)$$

On the other hand, from (5.14) and (6.3), we have (in our case)

$$((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}] \subset (\tau - \mathbf{AS})[\mathbf{h}] \setminus \{\emptyset\}.$$

Using (8.17), we obtain (8.16).

Recall Proposition 6. Now, we will use some properties of ultrafilters. We have that for all $\mathcal{F}_1 \in \mathfrak{F}[E]$, $\mathcal{F}_2 \in \mathfrak{F}[E]$, and $\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]$,

$$(\mathcal{F}_1 \cap \mathcal{F}_2 \subset \mathcal{U}) \implies ((\mathcal{F}_1 \subset \mathcal{U}) \vee (\mathcal{F}_2 \subset \mathcal{U})); \quad (8.18)$$

here we use [4, Proposition 9.4.3 and (1.5.1)]; in addition, $\mathcal{F}_1 \cap \mathcal{F}_2 \in \mathfrak{F}[E]$. Given $\mathcal{E}_1 \in \mathcal{P}'(\mathcal{P}(E))$ and $\mathcal{E}_2 \in \mathcal{P}'(\mathcal{P}(E))$, we define

$$\mathcal{E}_1 \{\cup\} \mathcal{E}_2 \triangleq \{\text{pr}_1(z) \cup \text{pr}_2(z) : z \in \mathcal{E}_1 \times \mathcal{E}_2\} \in \mathcal{P}'(\mathcal{P}(E)).$$

If $\mathcal{F}_1 \in \mathfrak{F}[E]$ and $\mathcal{F}_2 \in \mathfrak{F}[E]$, the following obvious equality holds:

$$\mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}_1 \{\cup\} \mathcal{F}_2 \in \mathfrak{F}[E]. \quad (8.19)$$

From (8.18) and (8.19), we obtain the following chain of equalities:

$$[\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1 \cap \mathcal{F}_2) = [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1 \{\cup\} \mathcal{F}_2) = [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1) \cup [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_2). \quad (8.20)$$

Now, recall Proposition 6. Then, by (8.19) and (8.20),

$$\begin{aligned} (\mathbf{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}_1 \cap \mathcal{F}_2] &= \Psi[E; Y; \tau; \mathbf{h}]^1([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1 \cap \mathcal{F}_2)) \\ &= \Psi[E; Y; \tau; \mathbf{h}]^1([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1)) \cup \Psi[E; Y; \tau; \mathbf{h}]^1([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_2)) \\ &= (\mathbf{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}_1] \cup (\mathbf{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}_2] \quad \forall \mathcal{F}_1 \in \mathfrak{F}[E] \quad \forall \mathcal{F}_2 \in \mathfrak{F}[E]. \end{aligned} \quad (8.21)$$

From (4.4), it follows that for $m \in \mathbb{N}$ and $(\mathcal{F}_i)_{i \in \overline{1, m}} \in \mathfrak{F}[E]^m$,

$$\bigcap_{i=1}^m \mathcal{F}_i \in \mathfrak{F}[E]. \quad (8.22)$$

In connection with (8.22), we introduce

$$\{\cup\}_{i=1}^m (\mathcal{E}_i) \triangleq \left\{ \bigcup_{i=1}^m \Sigma_i : (\Sigma_i)_{i \in \overline{1, m}} \in \prod_{i=1}^m \mathcal{E}_i \right\} \quad \forall m \in \mathbb{N} \quad \forall (\mathcal{E}_i)_{i \in \overline{1, m}} \in \mathcal{P}'(\mathcal{P}(E))^m.$$

It is easy to see that for $m \in \mathbb{N}$ and $(\mathcal{F}_i)_{i \in \overline{1, m}} \in \mathfrak{F}[E]^m$,

$$\bigcap_{i=1}^m \mathcal{F}_i = \{\cup\}_{i=1}^m (\mathcal{F}_i). \quad (8.23)$$

Remark 1. In fact, (8.22) and (8.23) can be generalized as follows: if T is a nonempty set and $(\mathcal{F}_t)_{t \in T} \in \mathfrak{F}[E]^T$, then

$$\bigcap_{t \in T} \mathcal{F}_t = \left\{ \bigcup_{t \in T} F_t : (F_t)_{t \in T} \in \prod_{t \in T} \mathcal{F}_t \right\} \in \mathfrak{F}[E].$$

By (8.21) and reasoning by induction, the following general statement is established.

Proposition 8. *If $n \in \mathbb{N}$ and $(\mathcal{F}_i)_{i \in \overline{1, n}} \in \mathfrak{F}[E]^n$, then*

$$\bigcup_{i=1}^n (\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}_i] = (\text{AS})\left[E; Y; \tau; \mathbf{h}; \bigcap_{i=1}^n \mathcal{F}_i\right].$$

Corollary 2. *If $n \in \mathbb{N}$ and $(\mathcal{B}_i)_{i \in \overline{1, n}} \in \beta_0[E]^n$, then*

$$\bigcup_{i=1}^n (\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{B}_i] = (\text{AS})\left[E; Y; \tau; \mathbf{h}; \bigcap_{i=1}^n (E - \mathbf{f})[\mathcal{B}_i]\right].$$

The corresponding proof follows immediately from (5.7) and Proposition 8. As a consequence, from (5.14), we obtain

$$\bigcup_{i=1}^n (\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{B}_i] \in ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}] \quad \forall n \in \mathbb{N} \quad \forall (\mathcal{B}_i)_{i \in \overline{1, n}} \in \beta_0[E]^n.$$

9. Some topological properties

Now, we consider the question of the continuity property of the operator $\Psi[E; Y; \tau; \mathbf{h}]$ and some its consequences. Since $\mathcal{P}(E) \in \tilde{\pi}^0[E]$, by [4, (1.5.8), (2.4.4)] we use

$$\mathbb{F}_{\text{lim}}[E; Y; \mathcal{P}(E); \tau] \triangleq \{g \in Y^E \mid \forall \mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E] \exists y \in Y : g^1[\mathcal{U}] \xrightarrow{\tau} y\} \in \mathcal{P}'(Y^E) \quad (9.1)$$

for which

$$\mathbb{F}_{\mathbf{c}}^0[E; Y; \tau] \subset \mathbb{F}_{\text{lim}}[E; Y; \mathcal{P}(E); \tau]$$

(the corresponding proof is obvious). Thus, $\mathbf{h} \in \mathbb{F}_{\text{lim}}[E; Y; \mathcal{P}(E); \tau]$, and (see [4, p. 58]) we define

$$\varphi_{\text{lim}}[E; Y; \mathcal{P}(E); \tau; \mathbf{h}] \in Y^{\mathfrak{F}_{\mathbf{u}}[E]};$$

moreover, in our case, the following equality holds:

$$\varphi_{\text{lim}}[E; Y; \mathcal{P}(E); \tau; \mathbf{h}] = \Psi[E; Y; \tau; \mathbf{h}]. \quad (9.2)$$

Remark 2. In connection with (9.2), we note (8.6). Indeed, let $\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]$. Then, $\mathcal{U} \in \beta[E]$, and by (3.4), (5.1), and (8.6),

$$\begin{aligned} (\text{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] &= \bigcap_{U \in \mathcal{U}} \text{cl}(\mathbf{h}^1(U), \tau) = \bigcap_{\Sigma \in \mathbf{h}^1[\mathcal{U}]} \text{cl}(\Sigma, \tau) \\ \bigcap_{\Sigma \in (Y - \mathbf{f})[\mathbf{h}^1[\mathcal{U}]]} \text{cl}(\Sigma, \tau) &= (\tau - \text{CL})[(Y - \mathbf{f})[\mathbf{h}^1[\mathcal{U}]]] = (\tau - \text{LIM})[(Y - \mathbf{f})[\mathbf{h}^1[\mathcal{U}]]], \end{aligned} \quad (9.3)$$

where

$$(Y - \mathbf{f})[\mathbf{h}^1[\mathcal{U}]] \in \mathfrak{F}_{\mathbf{u}}[Y]$$

(see (5.8)). As a consequence, by (8.4), (8.11), and (9.3),

$$(Y - \mathbf{f})[\mathbf{h}^1[\mathcal{U}]] \xrightarrow{\tau} \Psi[E; Y; \tau; \mathbf{h}](\mathcal{U}). \quad (9.4)$$

From (4.7), (4.8), and (9.4), we obtain the following convergence:

$$\mathbf{h}^1[\mathcal{U}] \xrightarrow{\tau} \Psi[E; Y; \tau; \mathbf{h}](\mathcal{U}).$$

Now, by (9.1) and [4, (1.5.8),(2.4.5),(2.4.6)], the obvious equality holds:

$$\varphi_{\lim}[E; Y; \mathcal{P}(E); \tau; \mathbf{h}](\mathcal{U}) = \Psi[E; Y; \tau; \mathbf{h}](\mathcal{U}).$$

Since the choice of $\mathcal{U}$ was arbitrary, equality (9.2) is established.

Until the end of this section, we suppose that (Y, τ) is a regular topological space; that is, (Y, τ) is both a T_1 -space and a T_3 -space. Furthermore, the separability property holds in our case, that is, $\tau \in (\text{top})_0[Y]$. From (9.2) and [4, Proposition 2.4.2], we obtain the following statement.

Proposition 9. *The mapping $\Psi[E; Y; \tau; \mathbf{h}]$ has the continuity property:*

$$\Psi[E; Y; \tau; \mathbf{h}] \in C(\mathfrak{F}_{\mathbf{u}}[E], \tau_{\mathbf{h}}[E], Y, \tau).$$

Using (3.3) and (4.13), we obtain

$$\Psi[E; Y; \tau; \mathbf{h}] \in C_{\text{cl}}(\mathfrak{F}_{\mathbf{u}}[E], \tau_{\mathbf{h}}[E], Y, \tau);$$

therefore, by (3.2),

$$\Psi[E; Y; \tau; \mathbf{h}]^1(\text{cl}(A, \tau_{\mathbf{h}}[E])) = \text{cl}(\Psi[E; Y; \tau; \mathbf{h}]^1(A), \tau) \quad \forall A \in \mathcal{P}(\mathfrak{F}_{\mathbf{u}}[E]). \quad (9.5)$$

Now, we use (9.5) and the natural variant of [4, (9.7.18)]:

$$\mathfrak{F}_{\mathbf{u}}[E] = \text{cl}(\{(E - \text{ult})[x] : x \in E\}, \tau_{\mathbf{h}}[E]); \quad (9.6)$$

in connection with (9.6), we also recall [4, (1.5.8), (1.5.9), (8.2.4)]. Using (8.15), (9.5), and (9.6), we obtain

$$\begin{aligned} \Psi[E; Y; \tau; \mathbf{h}]^1(\mathfrak{F}_{\mathbf{u}}[E]) &= \text{cl}(\{\Psi[E; Y; \tau; \mathbf{h}]((E - \text{ult})[x]) : x \in E\}, \tau) \\ &= \text{cl}(\{\mathbf{h}(x) : x \in E\}, \tau) = \text{cl}(\mathbf{h}^1(E), \tau). \end{aligned} \quad (9.7)$$

Proposition 10. *Nonempty finite subsets of $\text{cl}(\mathbf{h}^1(E), \tau)$ are attraction sets generated by filters:*

$$\text{Fin}(\text{cl}(\mathbf{h}^1(E), \tau)) \subset ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}]. \quad (9.8)$$

P r o o f. We use (9.7) to verify (9.8). Let

$$V \in \text{Fin}(\text{cl}(\mathbf{h}^1(E), \tau)). \quad (9.9)$$

Then for some $n \in \mathbb{N}$ and some tuple $(v_i)_{i \in \overline{1, n}} \in V^n$, we have

$$V = \{v_i : i \in \overline{1, n}\}.$$

In particular, $(v_i)_{i \in \overline{1, n}} \in Y^n$. Moreover, by (9.9),

$$v_j \in \text{cl}(\mathbf{h}^1(E), \tau) \quad \forall j \in \overline{1, n}. \quad (9.10)$$

By (9.7) and (9.10), we obtain

$$\mathfrak{V}_j \triangleq \{\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E] \mid v_j = \Psi[E; Y; \tau; \mathbf{h}](\mathcal{U})\} \in \mathcal{P}'(\mathfrak{F}_{\mathbf{u}}[E]) \quad \forall j \in \overline{1, n}.$$

It follows that

$$\prod_{i=1}^n \mathfrak{V}_i = \{(\mathcal{U}_i)_{i \in \overline{1, n}} \in \mathfrak{F}_{\mathbf{u}}[E]^n \mid \mathcal{U}_j \in \mathfrak{V}_j \quad \forall j \in \overline{1, n}\} \in \mathcal{P}'(\mathfrak{F}_{\mathbf{u}}[E]^n). \quad (9.11)$$

Since the set (9.11) is nonempty, we can choose

$$(\mathcal{V}_i)_{i \in \overline{1, n}} \in \prod_{i=1}^n \mathfrak{V}_i. \quad (9.12)$$

From (9.11) and (9.12), for each $j \in \overline{1, n}$, the ultrafilter $\mathcal{V}_j \in \mathfrak{F}_u[E]$ satisfies the equality

$$v_j = \Psi[E; Y; \tau; \mathbf{h}](\mathcal{V}_j).$$

Of course, $\mathcal{V}_j \in \mathfrak{F}[E]$ for all $j \in \overline{1, n}$. Therefore, by (8.22),

$$\bigcap_{i=1}^n \mathcal{V}_i \in \mathfrak{F}[E].$$

Then, by (5.14),

$$(\mathbf{AS})[E; Y; \tau; \mathbf{h}; \bigcap_{i=1}^n \mathcal{V}_i] \in ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}].$$

Using Proposition 8, we obtain

$$\bigcup_{i=1}^n (\mathbf{AS})[E; Y; \tau; \mathbf{h}; \mathcal{V}_i] \in ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}]. \quad (9.13)$$

By (8.11) and (9.12),

$$(\mathbf{AS})[E; Y; \tau; \mathbf{h}; \mathcal{V}_j] = \{v_j\}$$

for all $j \in \overline{1, n}$. Thus, by (9.13),

$$\bigcup_{i=1}^n \{v_i\} \in ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}],$$

where the union $\{v_i\}$, $i \in \overline{1, n}$, coincides with V . As a consequence,

$$V \in ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}].$$

Since the choice of V in (9.9) was arbitrary, (9.8) holds.

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