URAL MATHEMATICAL JOURNAL, Vol. 11, No. 2, 2025, pp. 144–157

DOI: 10.15826/umj.2025.2.010

EQUILIBRIUM TRAJECTORIES FOR CONTROL SYSTEMS WITH HETEROGENEOUS DYNAMICS[1]

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Abstract: The paper considers the construction of equilibrium in bimatrix games with heterogeneous dynamics of players' interaction. Heterogeneity of dynamics is connected with difference in maximal rates of the participants. In such a formulation, the switching curves of players' controls are represented by fractional rational functions and are constructed on the basis of N.N. Krasovskii's guaranteed strategies using elements of L.S. Pontryagin's maximum principle. Equilibrium trajectories are generated within the framework of the concept of the dynamic Nash equilibrium introduced by A.F. Kleimenov and are obtained by pasting together the characteristics of the Hamilton-Jacobi equations expressed as exponential functions. The sensitivity analysis is carried out for the shapes of control switching curves with respect to the proportions of players' maximal rates. The comparative analysis is implemented for the values of players' payoffs calculated on equilibrium trajectories of the dynamic game.

Keywords: Dynamic bimatrix games, Heterogeneous dynamics, Average integral payoffs, Characteristics of Hamilton–Jacobi equations, Equilibrium trajectories.

1. Introduction

The paper is devoted to analysis of a dynamic bimatrix game with heterogeneous dynamics of interaction between players. Heterogeneity of the dynamics means that the participants of the game have different maximum rates for changing their behavior.

The considered model analyzes a non-zero-sum game that evolves over an infinite time interval [3, 6, 7, 11]. The quality of game trajectories for the players is described by average integral payoff functionals (see, for example, [1, 13]). The construction of optimal strategies of players is connected with the concept of switching curves for players' controls, which are built on the basis of the guaranteed approach of N.N. Krasovskii [5, 8, 9] using elements of the maximum principle of L.S. Pontryagin [12]. In the paper, it is shown that for game with the heterogeneous dynamics the control switching curves are represented by fractional rational functions.

Equilibrium trajectories are generated within the framework of the concept of the dynamic Nash equilibrium proposed by A.F. Kleimenov [4] and are obtained as a result of pasting together the characteristics of the Hamilton–Jacobi equations, expressed as exponential functions. This approach, as well as the concept of dynamic stable cooperative solutions [2, 10], is directed on shifting game trajectories from the static Nash equilibrium toward the Pareto maximum points.

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[1] The research of the second author, Alexander M. Tarasyev, is supported by the Russian Science Foundation (project No. 24-21-00424, https://rscf.ru/project/24-21-00424/).

The construction of equilibrium trajectories is based on positional strategies generated by control switching curves. The structure of the switching curves depends significantly on the proportions of the maximal rates for players' velocities. In this regard, the sensitivity analysis is carried out for the switching curves depending on the change of these proportions.

The obtained results for constructing positional strategies and equilibrium trajectories are used in the analysis of the test model for the considered bimatrix games [6, 7]. In this analysis, the values of the functionals of players' payoffs are calculated and compared on the equilibrium trajectories. It is shown that the obtained equilibrium solutions dominate the static Nash equilibrium [14].

2. Heterogeneous dynamics of evolutionary game

We consider the following dynamics of players' interaction

$$\begin{cases} \dot{z_1}(t) = -\alpha z_1(t) + \alpha u(t), \quad z_1(t_0) = z_1^0, \quad \alpha > 0, \\ \dot{z_2}(t) = -\beta z_2(t) + \beta v(t), \quad z_2(t_0) = z_2^0, \quad \beta > 0. \end{cases} \tag{2.1}$$

Here parameter z_1, $0 \le z \le 1$, represents the probability of choosing the first strategy by the first player (respectively, $(1 - z_1)$ is the probability that he holds to the second strategy). Parameter z_2, $0 \le z_2 \le 1$, means the probability of choosing the first strategy by the second player (respectively, $(1 - z_2)$ is the probability that he holds to the second strategy). Control parameters u and v satisfy the conditions $0 \le u \le 1$, $0 \le v \le 0$ and can be interpreted as signals that recommend players to change their strategies. For example, the value $u = 0$ $(v = 0)$ corresponds to the signal: "change the first strategy to the second". The value $u = 1$ $(v = 1)$ corresponds to the signal: "change the second strategy to the first". The value $u = z_1$ $(v = z_2)$ corresponds to the signal: "keep the previous strategy". In particular, in the game simulating the investment process, the control signals u and v determine the movement of funds from one project to another. Parameters α and β stand for the maximal rates of changing players' behavior, so the parameter β/α denotes the proportion of these maximal rates.

The unit square, $(z_1, z_2) \in [0, 1] \times [0, 1]$, of the game is a strongly invariant set due to the specific properties of the dynamics (2.1). Consequently, every trajectory of the dynamics (2.1), initiated inside this square, stays in it on the infinite time horizon.

3. "Local" payoff functionals

Matrices A and B reflect payoffs of the first and second player, respectively:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Terminal payoff functionals of players in period t, where $t \in [t_0; +\infty)$, are defined as the mathematical expectation of payoffs given by corresponding matrices A and B in a bimatrix game, and can be interpreted as "local" interests of players

$$g_A(z_1(T), z_2(T)) = C_A z_1(T) z_2(T) - \varphi_1 z_1(T) - \varphi_2 z_2(T) + a_{22}$$

at given time moment T. Here coefficients C_A, φ_1, φ_2 are outlined by the classic theory of bimatrix games [14]

$$C_A = a_{11} - a_{12} - a_{21} + a_{22},$$
$$\varphi_1 = a_{22} - a_{12}, \quad \varphi_2 = a_{22} - a_{21}.$$

The terminal functional g_B of the second player as well as its corresponding parameters C_B, ψ_1, and ψ_2 are defined in a similar manner based on the elements of matrix B.

4. “Global” payoff functionals

In this section we consider “global” payoff functionals of players on the infinite time horizon which are defined on the trajectories $(z_1(\cdot), z_2(\cdot))$ of the system (2.1) and have the form

$$\begin{aligned} JI_A^{\infty} &= [JI_A^-,\, JI_A^+], \\ JI_A^- &= JI_A^-(z_1(\cdot), z_2(\cdot)) = \liminf_{T\to\infty} \frac{1}{(T-t_0)} \int_{t_0}^{T} g_A(z_1(t), z_2(t))dt, \\ JI_A^+ &= JI_A^+(z_1(\cdot), z_2(\cdot)) = \limsup_{T\to\infty} \frac{1}{(T-t_0)} \int_{t_0}^{T} g_A(z_1(t), z_2(t))dt, \end{aligned} \tag{4.1}$$

$$\begin{aligned} JI_B^{\infty} &= [JI_B^-,\, JI_B^+], \\ JI_B^- &= JI_B^-(z_1(\cdot), z_2(\cdot)) = \liminf_{T\to\infty} \frac{1}{(T-t_0)} \int_{t_0}^{T} g_B(z_1(t), z_2(t))dt, \\ JI_B^+ &= JI_B^+(z_1(\cdot), z_2(\cdot)) = \limsup_{T\to\infty} \frac{1}{(T-t_0)} \int_{t_0}^{T} g_B(z_1(t), z_2(t))dt. \end{aligned} \tag{4.2}$$

Average integral functionals represented by formulas (4.1) and (4.2) are widely used in classical models of evolutionary economics and biology. Functionals of this type have already been studied in the context of optimal control problems in papers [1, 13], where they were referred to as “time average values”. These approaches assume abandoning the goal of profit maximization at any given moment of time, allowing for temporary failure in profit or even losses at certain stages in order to achieve maximum cumulative results over the entire period under consideration.

5. Dynamic Nash equilibrium

Let us consider the concept of the dynamic Nash equilibrium applied to evolutionary games with dynamic process of the type (2.1) and payoffs specified by average integral functionals of the type (4.1), (4.2). We will consider these constructions within the framework of the approach for non-antagonistic positional differential games [4, 8, 9]. Let us define the dynamic Nash equilibrium in the class of positional strategies (feedbacks) $U = u(t, z_1, z_2, \varepsilon)$, $V = v(t, z_1, z_2, \varepsilon)$.

Definition 1. *The dynamic Nash equilibria* (U^0, V^0), $U^0 = u^0(t, z_1, z_2, \varepsilon)$, $V^0 = v^0(t, z_1, z_2, \varepsilon)$ *from the class of feedbacks* $U = u(t, z_1, z_2, \varepsilon)$, $V = v(t, z_1, z_2, \varepsilon)$ *for the given problem are defined by inequalities*

$$\begin{gathered} JI_A^-(z_1^*(\cdot), z_2^*(\cdot)) \geq JI_A^+(z_1^1(\cdot), z_2^1(\cdot)) - \varepsilon, \\ JI_B^-(z_1^*(\cdot), z_2^*(\cdot)) \geq JI_B^+(z_1^2(\cdot), z_2^2(\cdot)) - \varepsilon, \\ (z_1^*(\cdot), z_2^*(\cdot)) \in X(z_1^0, z_2^0, U^0, V^0), \quad (z_1^1(\cdot), z_2^1(\cdot)) \in X(z_1^0, z_2^0, U, V^0), \quad (z_1^2(\cdot), z_2^2(\cdot)) \in X(z_1^0, z_2^0, U^0, V). \end{gathered}$$

Here symbol X *stands for the set of trajectories, that start form the initial point and are generated by the corresponding positional strategies in the sense of the paper* [9].

6. Zero-sum games

For constructing desired feedbacks U^0, V^0 we use an approach [4]. In accordance with this approach we construct the equilibrium using optimal control feedbacks for differential games

$\Gamma_A = \Gamma_A^- \cup \Gamma_A^+$ and $\Gamma_B = \Gamma_B^- \cup \Gamma_B^+$ with payoffs JI_A^∞ (4.1) and JI_B^∞ (4.2). In the game Γ_A the first player maximizes the functional $JI_A^-(z_1(\cdot), z_2(\cdot))$ with the guarantee, using the feedback $U = u(t, z_1, z_2, \varepsilon)$, and the second player tries, on the contrary, to minimize the functional $JI_A^+(z_1(\cdot), z_2(\cdot))$, using the feedback $V = v(t, z_1, z_2, \varepsilon)$. Vice versa, in the game Γ_B the second player maximizes the functional $JI_B^-(z_1(\cdot), z_2(\cdot))$ with the guarantee, and the first player minimizes the functional $JI_B^+(z_1(\cdot), z_2(\cdot))$.

Let us introduce the following notations. Feedbacks that solve the problem of guaranteed maximization of payoff functionals JI_A^- and JI_B^- we define as $u_A^0 = u_A^0(t, z_1, z_2, \varepsilon)$ and $v_B^0 = v_B^0(t, z_1, z_2, \varepsilon)$. Let us note that such feedbacks represent a guaranteed maximization of players' payoffs in the long term, and can be called "positive". By $u_B^0 = u_B^0(t, z_1, z_2, \varepsilon)$ and $v_A^0 = v_A^0(t, z_1, z_2, \varepsilon)$ we denote the feedbacks that are most unfavorable for the opposing players, namely, those that minimize the payoff functionals JI_B^+ and JI_A^+ of the opposing players. We call such feedbacks "penalizing".

Let us note that inflexible solutions to the above problems can be obtained within the framework of the classical theory of bimatrix games. Indeed, let us assume for definiteness (this assumption is made for illustrative purposes and does not limit the generality of the solution) that the following relations are satisfied for the parameters of the matrices A and B, corresponding to the almost antagonistic structure of the bimatrix game

$$\begin{gathered} C_A > 0, \quad C_B < 0, \\ 0 < z_1^A = \frac{\varphi_2}{C_A} < 1, \quad 0 < z_1^B = \frac{\psi_2}{C_B} < 1, \\ 0 < z_2^A = \frac{\varphi_1}{C_A} < 1, \quad 0 < z_2^B = \frac{\psi_1}{C_B} < 1. \end{gathered} \tag{6.1}$$

The following statement can be proved.

Proposition 1. *Differential games Γ_A^-, Γ_A^+ have equal values*

$$w_A^- = w_A^+ = w_A = \frac{D_A}{C_A},$$

and differential games Γ_B^-, Γ_B^+ have equal values

$$w_B^- = w_B^+ = w_B = \frac{D_B}{C_B}$$

for an arbitrary initial position $(z_1^0, z_2^0) \in [0,1] \times [1,0]$. These values, for example, can be guaranteed by "positive" feedbacks u_A^{cl}, v_B^{cl} corresponding to classical solutions z_1^A, z_2^B

$$u_A^0 = u_A^{cl} = u_A^{cl}(z_1, z_2) = \begin{cases} 0, & z_1^A < z_1 \le 1, \\ 1, & 0 \le z_1 < z_1^A, \\ [0,1], & z_1 = z_1^A, \end{cases}$$

$$v_B^0 = v_B^{cl} = v_B^{cl}(z_1, z_2) = \begin{cases} 0, & z_2^B < z_2 \le 1, \\ 1, & 0 \le z_2 < z_2^B, \\ [0,1], & z_2 = z_2^B. \end{cases}$$

"Penalizing" feedbacks are determined by formulas

$$u_B^0 = u_B^{cl} = u_B^{cl}(z_1, z_2) = \begin{cases} 0, & z_1^B < z_1 \le 1, \\ 1, & 0 \le z_1 < z_1^B, \\ [0,1], & z_1 = z_1^B, \end{cases}$$

$$v_A^0 = v_A^{cl} = v_A^{cl}(z_1, z_2) = \begin{cases} 0, & z_2^A < z_2 \leq 1, \\ 1, & 0 \leq z_2 < z_2^A, \\ [0,1]\,, & z_2 = z_2^A, \end{cases}$$

and correspond to classical static solutions z_1^B, z_2^A (6.1)*, which generate static Nash equilibrium* $NE = (z_1^B, z_2^A)$*.*

The proof of this statement can be obtained by direct substitution of the specified strategies into the corresponding payoff functionals (4.1), (4.2).

Remark 1. Values of payoffs $g_A(z_1, z_2)$, $g_B(z_1, z_2)$ coincide at points (z_1^A, z_2^B), (x_1^B, z_2^A)

$$g_A(z_1^A, z_2^B) = g_A(z_1^B, z_2^A) = w_A, \quad g_B(z_1^A, z_2^B) = g_B(z_1^B, z_2^A) = w_B.$$

Point $NE = (z_1^B, z_2^A)$ is a "mutually penalizing" Nash equilibrium while the point (z_1^A, z_2^B) has no equilibrium properties in the corresponding static game.

7. Basic elements of dynamic Nash equilibrium

Let us construct a pair of feedbacks which constitute the Nash equilibrium. For that we paste "positive" feedbacks u_A^0, v_B^0 and "penalizing" feedbacks u_B^0, v_A^0.

We choose an initial position $(z_1^0, z_2^0) \in [0,1] \times [0,1]$ and an accuracy parameter $\varepsilon > 0$. Next, we choose a trajectory $(z_1^*(\cdot), z_2^*(\cdot)) \in X(z_1^0, z_2^0, u_A^0(\cdot), v_B^0(\cdot))$, generated by "positive" feedbacks $u_A^0 = u_A^0(t, z_1, z_2, \varepsilon)$ and $v_B^0 = v_B^0(t, z_1, z_2, \varepsilon)$. Let's take $T_\varepsilon > 0$ such as

$$\begin{gathered} g_A(z_1^*(t), z_2^*(t)) > JI_A^-(z_1^*(\cdot), z_2^*(\cdot)) - \varepsilon, \\ g_B(z_1^*(t), z_2^*(t)) > JI_B^-(z_1^*(\cdot), z_2^*(\cdot)) - \varepsilon, \\ t \in [T_\varepsilon, +\infty). \end{gathered}$$

We denote as $u_A^\varepsilon(t)\colon\ [0, T_\varepsilon) \to [0,1]$, $v_B^\varepsilon(t)\colon\ [0, T_\varepsilon) \to [0,1]$ step-by-step implementation of strategies u_A^0, v_B^0 such as corresponding step-by-step trajectory $(z_1^\varepsilon(\cdot), z_2^\varepsilon(\cdot))$ satisfy the condition

$$\max_{t \in [0, T_\varepsilon]} \|(z_1^*(t), z_2^*(t)) - (z_1^\varepsilon(t), z_2^\varepsilon(t))\| < \varepsilon.$$

From the results of paper [4] the next proposition follows.

Proposition 2. *A pair of feedbacks* $U^0 = u^0(t, z_1, z_2, \varepsilon)$*,* $V^0 = v^0(t, z_1, z_2, \varepsilon)$*, pasting together "positive" feedbacks* u_A^0*,* v_B^0 *and "penalizing" feedbacks* u_B^0*,* v_A^0 *according to the relations*

$$U^0 = u^0(t, z_1, z_2, \varepsilon) \begin{cases} u_A^\varepsilon(t), & \|(z_1, z_2) - (z_1^\varepsilon(t), z_2^\varepsilon(t))\| < \varepsilon, \\ u_B^0(z_1, z_2), & \textit{otherwise}, \end{cases}$$

$$V^0 = v^0(t, z_1, z_2, \varepsilon) \begin{cases} v_B^\varepsilon(t), & \|(z_1, z_2) - (z_1^\varepsilon(t), z_2^\varepsilon(t))\| < \varepsilon, \\ v_A^0(z_1, z_2), & \textit{otherwise} \end{cases}$$

is the dynamic ε*–Nash equilibrium.*

Below we construct flexible "positive" feedbacks, which generate trajectories $(z_1^{fl}(\cdot), z_2^{fl}(\cdot))$, leading to "better" positions, than unflexible static equilibrium (z_1^B, z_2^A), (z_1^A, z_2^B) by both criteria

$$\begin{aligned} JI_A^\infty(z_1^{fl}(\cdot), z_2^{fl}(\cdot)) \geq w_A, \\ JI_B^\infty(z_1^{fl}(\cdot), z_2^{fl}(\cdot)) \geq w_B. \end{aligned} \tag{7.1}$$

8. Optimal control problems for heterogeneous dynamic game

For constructing "positive" feedbacks $u_A^0 = u_A^{fl}(z_1, z_2)$, $v_B^0 = v_B^{fl}(z_1, z_2)$ we consider in this section an auxiliary two-step optimal control problem with average integral payoff functional for the first player in situation when actions of the second player is most unfavorable. Namely, we analyze an optimal control problem for the dynamic system (2.1)

$$\begin{cases} \dot{z}_1 = -\alpha z_1 + \alpha u, & z_1(0) = z_1^0, \\ \dot{z}_2 = -\beta z_2 + \beta v, & z_2(0) = z_2^0. \end{cases} \tag{8.1}$$

with payoff functional

$$J_A^f = \int_0^{T_f} g_A(z_1(t), z_2(t))dt.$$

Here without loss of generality we consider that $t_0 = 0$, $T = T_f$ and terminal time moment $T_f = T_f(z_1^0, z_2^0)$ we determine later.

Without loss of generality, we consider that the value of the static game equals to zero, and the next conditions fulfilled

$$\begin{gathered} w_A = \frac{D_A}{C_A} = 0, \quad C_A > 0, \\ 0 < x_A = \frac{\alpha_2}{C_A} < 1, \quad 0 < y_A = \frac{\alpha_1}{C_A} < 1. \end{gathered} \tag{8.2}$$

We consider a case, when initial conditions (z_1^0, z_2^0) of the system (8.1) satisfy relations

$$z_1^0 = z_1^A, \quad z_2^0 > z_2^A. \tag{8.3}$$

We assume that actions of the second player is mostly unfavorable for the first player. For trajectories of the system (8.1), that start from initial positions (z_1^0, z_2^0) (8.3), these actions $v_A^0 = v_A^{cl}(z_1, z_2)$ are determined by the relation

$$v_A^{cl}(z_1, z_2) \equiv 0.$$

In this situation optimal actions $u_A^0 = u_A^{fl}(z_1, z_2)$ of the first player according to payoff functionals J_A^f can be presented as two-step impulse control: it equals to unit from the initial time moment $t_0 = 0$ till the moment of optimal switch s and the equals to null till the final time moment T_f

$$u_A^0(t) = u_A^{fl}(z_1(t), z_2(t)) = \begin{cases} 1, & \text{if} \quad t_0 \leq t < s, \\ 0, & \text{if} \quad s \leq t < T_f. \end{cases}$$

Here the value s is the optimization parameter. Final time moment T_f is determined from the next condition. The trajectory $(z_1(\cdot),\ z_2(\cdot))$ of the system (8.1), that starts from the line where $z_1(t_0) = z_1^A$ returns to this line when $z_1(T_f) = z_1^A$.

Let us consider two aggregates of characteristics. The first one is described by the system of differential equations with the value of control parameter $u = 1$

$$\begin{cases} \dot{z}_1 = -\alpha z_1 + \alpha, \\ \dot{z}_2 = -\beta z_2, \end{cases} \tag{8.4}$$

solutions of which are determined by the Cauchy formula

$$\begin{cases} z_1(t) = 1 - (1 - z_1^0) \cdot e^{-\alpha t}, \\ z_2(t) = z_2^0 \cdot e^{-\beta t}. \end{cases} \tag{8.5}$$

Here initial positions (z_1^0, z_2^0) satisfy conditions (8.3), time parameter t satisfies the inequality $0 \le t < s$, where the moment s is the time of switching the characteristics.

The second aggregate of characteristics is given by the system of differential equations with the value of control parameter $u = 0$

$$\begin{cases} \dot{z_1} = -\alpha z_1, \\ \dot{z_2} = -\beta z_2, \end{cases} \tag{8.6}$$

solutions of which are determined by the Cauchy formula

$$\begin{cases} z_1(t) = z_1^1 e^{-\alpha t}, \\ z_2(t) = z_2^1 e^{-\beta t}. \end{cases} \tag{8.7}$$

Here initial positions $(z_1^1, z_2^1) = (z_1^0(s), z_2^0(s))$ are determined by relations

$$\begin{cases} z_1^1 = z_1^0(s) = 1 - (1 - z_1^A)e^{-\alpha s}, \\ z_2^1 = z_2^0(s) = z_2^0 e^{-\beta s}, \end{cases} \tag{8.8}$$

and time parameter t satisfies an inequality $0 \le t < \tau^*$. The final time moment τ^* is given by formulas

$$z_1^0(s)e^{-\alpha\tau^*} = z_1^A, \quad \tau^* = \frac{1}{\alpha}\ln\frac{z_1^0(s)}{z_1^A} = \frac{1}{\alpha}\ln\frac{(1-(1-z_1^A)e^{-\alpha s})}{z_1^A}, \tag{8.9}$$

and the final position $(z_1^2, z_2^2) = (z_1(\tau^*), z_2(\tau^*))$ is determined as follows

$$z_1^2 = z_1^A, \quad z_2^2 = z_2^0(s)e^{-\beta\tau^*}. \tag{8.10}$$

The optimal control problem is to find such time moment s and corresponding switching point $(z_1^1, z_2^1) = (z_1^0(s), z_2^0(s))$ on the trajectory $(z_1(\cdot), z_2(\cdot))$, where the integral $I = I(s)$ reaches its maximum

$$I(s) = I_1(s) + I_2(s), \tag{8.11}$$

$$\begin{gathered} I_1(s) = \int_0^s (C_A \cdot (1 - (1 - z_1^A) \cdot e^{-\alpha\cdot t}) \cdot z_2^0 \cdot e^{-\beta\cdot t} \\ -\varphi_1 \cdot (1 - (1 - z_1^A) \cdot e^{-\alpha\cdot t}) - \varphi_2 \cdot z_2^0 \cdot e^{-\beta\cdot t} + a_{22})dt, \end{gathered} \tag{8.12}$$

$$I_2(s) = \int_0^{\tau^*} (C_A \cdot z_1^0(s) \cdot z_2^0(s) \cdot e^{-(\alpha+\beta)\cdot\tau} - \varphi_1 \cdot z_1^0(s) \cdot e^{-\alpha\cdot t} - \varphi_2 \cdot z_2^0(s) \cdot e^{-\beta\cdot t} + a_{22})d\tau. \tag{8.13}$$

In Fig. 1 we show the saddle point S_A for the matrix A, the switching curve M_A of the control of the first player for the value of the proportion $\beta/\alpha = 2$, the initial position IP chosen on the line $z_1 = z_1^A$ for $z_2 > z_2^A$, the characteristic oriented to the vertex of the square $(1; 0)$ and the characteristic oriented to the vertex of the square $(0; 0)$, the switching point SP of the motion of characteristics and the finite point of motion FP located on the line $z_1 = z_1^A$.

9. Solutions for optimal control problem

We obtain the solution of optimal control problem (8.4)–(8.13) by calculating the derivative $dI(s)/ds$, presenting it as function of optimal switching points $(z_1, z_2) = (z_1^1, z_2^1)$, equating this derivative to zero $dI(s)/ds = 0$ and finding an equation $F(z_1, z_2) = 0$ for the curve that consits of optimal switching points (z_1, z_2). Let us note, that the derivative $dI(s)/ds$ changes its sign from positive to negative at the point (z_1^1, z_2^1) and, hence, the integral reaches its maximum.

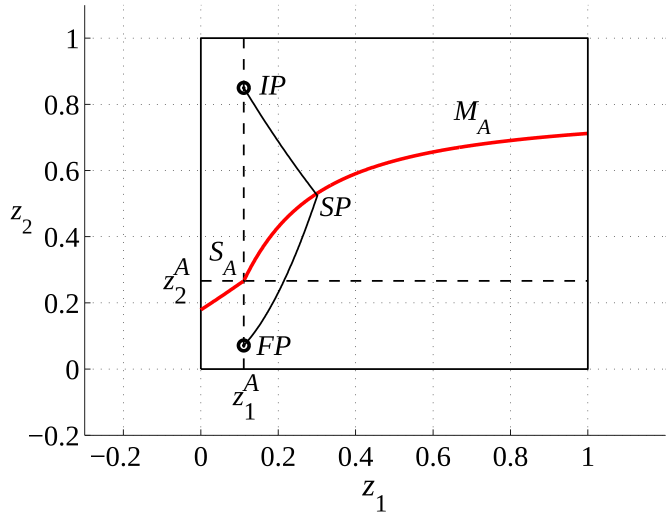

Figure 1. Switching curve for optimal control problem and optimal trajectory.

First, we calculate integrals I_1, I_2

$$I_1 = I_1(s) = C_A \cdot (1 - z_1^A) \cdot z_2^0 \cdot \frac{(e^{-(\alpha+\beta)\cdot s} - 1)}{(\alpha+\beta)} - (C_A - \varphi_2) \cdot z_2^0 \cdot \frac{(e^{-\beta\cdot s} - 1)}{\beta}$$

$$-\varphi_1 \cdot (1 - z_1^A) \cdot \frac{(e^{-\alpha\cdot s} - 1)}{\alpha} - (a_{22} - \varphi_1) \cdot s,$$

$$I_2 = I_2(s) = C_A \cdot z_1^0(s) \cdot z_2^0(s) \cdot \left(\frac{1 - e^{-(\alpha+\beta)\cdot\tau^*}}{\alpha+\beta}\right) - \varphi_1 \cdot z_1^0(s) \cdot \left(\frac{1 - e^{-\alpha\cdot\tau^*}}{\alpha}\right)$$

$$-\varphi_2 \cdot z_2^0(s) \cdot \left(\frac{1 - e^{-\beta\cdot\tau^*}}{\beta}\right) + a_{22} \cdot \tau^*.$$

We calculate derivatives $dI_1(s)/ds$, $dI_2(s)/ds$ and present them as functions of optimal switching points $(z_1^1, z_2^1) = (z_1^0(s), z_2^0(s))$

$$\frac{dI_1(s)}{ds} = -C_A \cdot (1 - z_1^A) \cdot z_2^0 \cdot e^{-(\alpha+\beta)\cdot s} + (C_A - \varphi_2) \cdot z_2^0 \cdot e^{-\beta\cdot s} + \varphi_1 \cdot (1 - z_1^A) \cdot e^{-\alpha s} + a_{22} - \varphi_1$$

$$= C_A \cdot z_1^0(s) \cdot z_2^0(s) - \varphi_1 \cdot z_1^0(s) - \varphi_2 \cdot z_2^0(s) + a_{22},$$

$$\frac{dI_2(s)}{ds} = C_A \cdot \left[\frac{dz_1^0(s)}{ds} \cdot z_2^0(s) \cdot \left(\frac{1 - e^{-(\alpha+\beta)\cdot\tau^*(s)}}{\alpha+\beta}\right) z_1^0(s) \cdot \frac{dz_2^0(s)}{ds} \cdot \left(\frac{1 - e^{-(\alpha+\beta)\cdot\tau^*(s)}}{\alpha+\beta}\right)\right.$$

$$\left. + z_1^0(s) \cdot z_2^0(s) \cdot e^{-(\alpha+\beta)\cdot\tau^*(s)} \cdot \frac{d\tau^*(s)}{ds}\right] - \varphi_1 \cdot \frac{dz_1^0(s)}{ds} \cdot \left(\frac{1 - e^{-\alpha\cdot\tau^*(s)}}{\alpha}\right)$$

$$-\varphi_1 \cdot z_1^0(s) \cdot e^{-\alpha\cdot\tau^*(s)} \cdot \frac{d\tau^*(s)}{ds} - \varphi_2 \cdot \frac{dz_2^0(s)}{ds} \cdot \left(\frac{1 - e^{-\beta\cdot\tau^*(s)}}{\beta}\right)$$

$$-\varphi_2 \cdot z_2^0(s) \cdot e^{-\beta\cdot\tau^*(s)} \cdot \frac{d\tau^*(s)}{ds} - a_{22} \cdot \frac{d\tau^*(s)}{ds}.$$

In the last equation we take into the account next expressions for the derivatives and exponents

$$z_1^0(s) = 1 - (1 - x_A) \cdot e^{-\alpha\cdot s};$$

$$\frac{dz_1^0(s)}{ds} = \alpha \cdot (1 - x_A) \cdot e^{-\alpha s} = \alpha \cdot \big(((1 - x_A) \cdot e^{-\alpha\cdot s} - 1) + 1\big) = \alpha \cdot (1 - z_1^0(s));$$

$$z_2^0(s) = y_0 \cdot e^{-\beta\cdot s};$$

$$\frac{dz_2^0(s)}{ds} = -y_0 \cdot \beta \cdot e^{-\beta\cdot s} = -\beta \cdot z_2^0(s);$$

$$\frac{d\tau^*(s)}{ds} = \Big[\frac{1}{\alpha}\cdot \ln\Big(\frac{(1-(1-x_A)\cdot e^{-\alpha\cdot s})}{x_A}\Big)\Big]_s' = \frac{1}{\alpha}\cdot\Big[\ln\Big(1-(1-x_A)\cdot e^{-\alpha\cdot s}\Big) - \ln x_A\Big]_s'$$

$$= \frac{1}{\alpha}\cdot\frac{1}{(1-(1-x_A)\cdot e^{-\alpha\cdot s})}\cdot\alpha\cdot(1-x_A)\cdot e^{-\alpha\cdot s} = \frac{1-z_1^0(s)}{z_1^0(s)};$$

$$e^{-\alpha\cdot\tau^*(s)} = e^{-\alpha\cdot 1/\alpha\cdot\ln(z_1^0(s)/x_A)} = \frac{x_A}{z_1^0(s)} = \frac{\varphi_2}{C_A\cdot z_1^0(s)};$$

$$e^{-\beta\cdot\tau^*(s)} = e^{-\beta\cdot 1/\alpha\cdot\ln(z_1^0(s)/x_A)} = e^{\ln(x_A/z_1^0(s))^{\beta/\alpha}} = \Big(\frac{x_A}{z_1^0(s)}\Big)^{\beta/\alpha} = \Big(\frac{\varphi_2}{C_A\cdot z_1^0(s)}\Big)^{\beta/\alpha};$$

$$1-e^{-\alpha\cdot\tau^*(s)} = 1-\frac{\varphi_2}{C_A\cdot z_1^0(s)} = \frac{C_A\cdot z_1^0(s)-\varphi_2}{C_A\cdot z_1^0(s)};$$

$$1-e^{-\beta\cdot\tau^*(s)} = 1-\Big(\frac{\varphi_2}{C_A\cdot z_1^0(s)}\Big)^{\beta/\alpha} = \frac{(C_A\cdot z_1^0(s))^{\beta/\alpha}-(\varphi_2)^{\beta/\alpha}}{(C_A\cdot z_1^0(s))^{\beta/\alpha}};$$

$$e^{-(\alpha+\beta)\cdot\tau^*(s)} = e^{-(\alpha+\beta)\cdot 1/\alpha\cdot\ln(z_1^0(s)/x_A)} = e^{-(1+\beta/\alpha)\cdot\ln(z_1^0(s)/x_A)}$$

$$= \Big(\frac{x_A}{z_1^0(s)}\Big)^{(1+\beta/\alpha)} = \Big(\frac{\varphi_2}{C_A\cdot z_1^0(s)}\Big)^{(1+\beta/\alpha)};$$

$$1-e^{-(\alpha+\beta)\cdot\tau^*(s)} = 1-\Big(\frac{\varphi_2}{C_A\cdot z_1^0(s)}\Big)^{(1+\beta/\alpha)} = \frac{(C_A\cdot z_1^0(s))^{(1+\beta/\alpha)}-(\varphi_2)^{(1+\beta/\alpha)}}{(C_A\cdot z_1^0(s))^{(1+\beta/\alpha)}};$$

Finally, we get the next formula for the derivative $dI_2(s)/ds$

$$\frac{dI_2(s)}{ds} = C_A\cdot\Big[\alpha\cdot(1-z_1^0(s))\cdot z_2^0(s)\cdot\frac{\big((C_A\cdot z_1^0(s))^{(1+\beta/\alpha)}-\varphi_2^{(1+\beta/\alpha)}\big)}{(C_A\cdot z_1^0(s))^{(1+\beta/\alpha)}\cdot(\alpha+\beta)}$$

$$+z_1^0(s)\cdot(-\beta\cdot z_2^0(s))\cdot\frac{\big((C_A\cdot z_1^0(s))^{(1+\beta/\alpha)}-\varphi_2^{(1+\beta/\alpha)}\big)}{(C_A\cdot z_1^0(s))^{(1+\beta/\alpha)}\cdot(\alpha+\beta)}$$

$$+z_1^0(s)\cdot z_2^0(s)\cdot\Big(\frac{\varphi_2}{C_A\cdot z_1^0(s)}\Big)^{(1+\beta/\alpha)}\cdot\frac{(1-z_1^0(s))}{z_1^0(s)}\Big]$$

$$-\varphi_1\cdot\alpha\cdot(1-z_1^0(s))\cdot\frac{(C_A\cdot z_1^0(s)-\varphi_2)}{C_A\cdot z_1^0(s)\cdot\alpha} - \varphi_1\cdot z_1^0(s)\cdot\frac{\alpha_2}{C_A\cdot z_1^0(s)}\cdot\frac{(1-z_1^0(s))}{z_1^0(s)}$$

$$-\varphi_2\cdot(-\beta\cdot z_2^0(s))\cdot\frac{\big((C_A\cdot z_1^0(s))^{\beta/\alpha}-\varphi_2^{\beta/\alpha}\big)}{(C_A\cdot z_1^0(s))^{\beta/\alpha}\cdot\beta}$$

$$-\varphi_2\cdot z_2^0(s)\cdot\Big(\frac{\varphi_2}{C_A\cdot z_1^0(s)}\Big)^{\beta/\alpha}\cdot\frac{(1-z_1^0(s))}{z_1^0(s)} + a_{22}\cdot\frac{(1-z_1^0(s))}{z_1^0(s)}.$$

Summarizing derivatives $dI_1(s)/ds$, $dI_2(s)/ds$, we equate the sum to zero and, taking into account the fact that $w_A = 0$ (see (8.2)), we obtain the following equation for the switching curve M_A^1 when $C_A > 0$ in the domain where $z_2 > z_2^A$

$$M_A^1 = \Big\{(z_1^0(s), z_2^0(s)) \in [0,1]\times[0,1]\colon\ z_2^0(s) = [\varphi_1\cdot z_1^0(s) - a_{22}]/$$

$$/\Big[C_A\cdot(z_1^0(s))^2 - \varphi_2\cdot z_1^0(s) + \alpha\cdot z_1^0(s)\cdot(1-z_1^0(s))\cdot\frac{C_A\cdot\big((C_A\cdot z_1^0(s))^{(1+\beta/\alpha)}-\varphi_2^{(1+\beta/\alpha)}\big)}{(C_A\cdot z_1^0(s))^{(1+\beta/\alpha)}\cdot(\alpha+\beta)} -$$

$$-\beta \cdot (z_1^0(s))^2 \cdot \frac{C_A \cdot \Big((C_A \cdot z_1^0(s))^{(1+\beta/\alpha)} - \varphi_2^{(1+\beta/\alpha)}\Big)}{(C_A \cdot z_1^0(s))^{(1+\beta/\alpha)} \cdot (\alpha+\beta)}$$

$$+C_A \cdot z_1^0(s) \cdot (1 - z_1^0(s)) \cdot \Big(\frac{\varphi_2}{C_A \cdot z_1^0(s)}\Big)^{(1+\beta/\alpha)} + \varphi_2 \cdot z_1^0(s) \cdot \frac{\big((C_A \cdot z_1^0(s))^{\beta/\alpha} - \varphi_2^{\beta/\alpha}\big)}{(C_A \cdot z_1^0(s))^{\beta/\alpha}} -$$

$$-\varphi_2 \cdot (1 - z_1^0(s)) \cdot \Big(\frac{\varphi_2}{C_A \cdot z_1^0(s)}\Big)^{\beta/\alpha}\Bigg], \quad z_2^0(s) \geq \frac{\varphi_1}{C_A}\Bigg\}.$$

To construct the final switching curve $M_A = M_A^1 \cup M_A^2$ for optimal strategy of the first player in the case $C_A > 0$, we add to the curve M_A^1 the similar curve M_A^2 in the domain where $z_2 \leq z_2^A$

$$M_A^2 = \Bigg\{(z_1^0(s), z_2^0(s)) \in [0,1] \times [0,1] \colon$$

$$z_2^0(s) = -\big[(C_A - \varphi_1) \cdot (1 - z_1^0(s)) - a_{11}\big] \Big/ \Bigg[C_A \cdot (1 - z_1^0(s))^2 - (C_A - \varphi_2) \cdot (1 - z_1^0(s))$$

$$+\alpha \cdot (1 - z_1^0(s)) \cdot z_1^0(s) \cdot \frac{C_A \cdot \big(C_A^{(1+\beta/\alpha)} \cdot (1 - z_1^0(s))^{(1+\beta/\alpha)} - (C_A - \varphi_2)^{(1+\beta/\alpha)}\big)}{C_A^{(1+\beta/\alpha)} \cdot (1 - z_1^0(s))^{(1+\beta/\alpha)} \cdot (\alpha+\beta)}$$

$$-\beta \cdot (1 - z_1^0(s))^2 \cdot \frac{C_A \cdot \big(C_A^{(1+\beta/\alpha)} \cdot (1 - z_1^0(s))^{(1+\beta/\alpha)} - (C_A - \varphi_2)^{(1+\beta/\alpha)}\big)}{C_A^{(1+\beta/\alpha)} \cdot (1 - z_1^0(s))^{(1+\beta/\alpha)} \cdot (\alpha+\beta)}$$

$$+C_A \cdot (1 - z_1^0(s)) \cdot z_1^0(s) \cdot \Big(\frac{(C_A - \varphi_2)}{C_A \cdot (1 - z_1^0(s))}\Big)^{(1+\beta/\alpha)}$$

$$+(C_A - \varphi_2) \cdot (1 - z_1^0(s)) \cdot \frac{\big(C_A^{\beta/\alpha} \cdot (1 - z_1^0(s))^{c\beta/\alpha} - (C_A - \varphi_2)^{\beta/\alpha}\big)}{C_A^{\beta/\alpha} \cdot (1 - z_1^0(s))^{\beta/\alpha}}$$

$$-(C_A - \varphi_2) \cdot z_1^0(s) \cdot \left(\frac{(C_A - \varphi_2)}{C_A \cdot (1 - z_1^0(s))}\right)^{\beta/\alpha}\Bigg] + 1, \quad z_2^0(s) \leq \frac{\varphi_1}{C_A}\Bigg\}.$$

In case, when $C_A < 0$, curves M_A, M_A^1 and M_A^2 are described by formulas

$$M_A = M_A^1 \cup M_A^2,$$

$$M_A^1 = \Bigg\{(z_1^0(s), z_2^0(s)) \in [0,1] \times [0,1] \colon$$

$$z_2^0(s) = \big[\varphi_1 \cdot (1 - z_1^0(s)) + a_{12}\big] \Big/ \Bigg[C_A \cdot (1 - z_1^0(s))^2 - (C_A - \varphi_2) \cdot (1 - z_1^0(s))$$

$$+\alpha \cdot (1 - z_1^0(s)) \cdot z_1^0(s) \cdot \frac{C_A \cdot \big((C_A \cdot (1 - z_1^0(s)))^{(1+\beta/\alpha)} - (C_A - \varphi_2)^{(1+\beta/\alpha)}\big)}{(C_A \cdot (1 - z_1^0(s)))^{(1+\beta/\alpha)} \cdot (\alpha+\beta)}$$

$$-\beta \cdot (1 - z_1^0(s))^2 \cdot \frac{C_A \cdot \Big((C_A \cdot (1 - z_1^0(s)))^{(1+\beta/\alpha)} - (C_A - \varphi_2)^{(1+\beta/\alpha)}\Big)}{(C_A \cdot (1 - z_1^0(s)))^{(1+\beta/\alpha)} \cdot (\alpha+\beta)}$$

$$+C_A \cdot (1 - z_1^0(s)) \cdot z_1^0(s) \cdot \Big(\frac{C_A - \varphi_2}{C_A \cdot (1 - z_1^0(s))}\Big)^{(1+\beta/\alpha)}$$

$$+(C_A - \varphi_2) \cdot (1 - z_1^0(s)) \cdot \frac{\big((C_A \cdot (1 - z_1^0(s)))^{\beta/\alpha} - (C_A - \varphi_2)^{\beta/\alpha}\big)}{(C_A \cdot (1 - z_1^0(s)))^{\beta/\alpha}}$$

$$-(C_A - \varphi_2) \cdot z_1^0(s) \cdot \Big(\frac{(C_A - \varphi_2)}{C_A \cdot (1 - z_1^0(s))}\Big)^{\beta/\alpha}\Bigg], \quad z_2^0(s) \geq \frac{\varphi_1}{C_A}\Bigg\},$$

$$\begin{aligned}
M_A^2 = \Big\{ & (z_1^0(s), z_2^0(s)) \in [0,1] \times [0,1]\colon \\
z_2^0(s) = & -\big[(C_A - \varphi_1) \cdot z_1^0(s) + a_{21}\big] \Big/ \Big[C_A \cdot (z_1^0(s))^2 - \varphi_2 \cdot z_1^0(s) \\
& + \alpha \cdot z_1^0(s) \cdot (1 - z_1^0(s)) \cdot \frac{C_A \cdot \big((C_A \cdot z_1^0(s))^{(1+\beta/\alpha)} - \varphi_2^{(1+\beta/\alpha)}\big)}{(C_A \cdot z_1^0(s))^{(1+\beta/\alpha)} \cdot (\alpha + \beta)} \\
& - \beta \cdot (z_1^0(s))^2 \cdot \frac{C_A \cdot \Big((C_A \cdot z_1^0(s))^{(1+\beta/\alpha)} - \varphi_2^{(1+\beta/\alpha)}\Big)}{(C_A \cdot z_1^0(s))^{(1+\beta/\alpha)} \cdot (\alpha + \beta)} \\
& + C_A \cdot z_1^0(s) \cdot (1 - z_1^0(s)) \cdot \Big(\frac{\varphi_2}{C_A \cdot z_1^0(s)}\Big)^{(1+\beta/\alpha)} + \varphi_2 \cdot z_1^0(s) \cdot \frac{\Big((C_A \cdot z_1^0(s))^{\beta/\alpha} - \varphi_2^{\beta/\alpha}\Big)}{(C_A \cdot z_1^0(s))^{\beta/\alpha}} \\
& - \varphi_2 \cdot (1 - z_1^0(s)) \cdot \Big(\frac{\varphi_2}{C_A \cdot z_1^0(s)}\Big)^{\beta/\alpha} \Big] + 1, \quad z_2^0(s) \leq \frac{\varphi_1}{C_A} \Big\}.
\end{aligned}$$

Curve M_A divides the unit square $[0,1] \times [0,1]$ into two parts: upper part

$$D_A^u \supset \big\{(z_1, z_2)\colon z_1 = z_1^A,\ z_2 > z_2^A\big\}$$

and lower part

$$D_A^l \supset \big\{(z_1, z_2)\colon z_1 = z_1^A,\ z_2 < z_2^A\big\}.$$

Basing on structure of domains D_A^u and D_A^l we construct the flexible "positive" feedback u_A^{fl} with properties (7.1) according to the following formulas

$$u_A^{fl} \;=\; u_A^{fl}(z_1, z_2) = \begin{cases} \max\{0, -\operatorname{sgn}(C_A)\}, & \text{if } (z_1, z_2) \in D_A^u, \\ \max\{0, \operatorname{sgn}(C_A)\}, & \text{if } (z_1, z_2) \in D_A^l, \\ [0,1], & \text{if } (z_1, z_2) \in M_A. \end{cases}$$

One can obtain similar switching curves M_B for the second player corresponding to the matrix B. More precisely, in case when $C_B > 0$, switching curve M_B is given by relations

$$M_B = M_B^1 \cup M_B^2,$$

$$\begin{aligned}
M_B^1 = \Big\{ & (z_1^0(s), z_2^0(s)) \in [0,1] \times [0,1]\colon \\
z_1^0(s) = & \Big[(\psi_2 + b_{21}) \cdot \frac{(1 - z_2^0(s))}{z_2^0(s)} + b_{21}\Big] \Big/ \Big[C_B \cdot \frac{\alpha}{(\alpha + \beta)} \cdot \Big(1 - \Big(\frac{z_2^B}{z_2^0(s)}\Big)^{(\alpha+\beta)/\beta}\Big) \\
& + C_B \cdot \Big(\Big(\frac{z_2^B}{z_2^0(s)}\Big)^{\alpha/\beta} \cdot \frac{1}{z_2^0(s)} - 1\Big) - (C_B - \psi_1) \cdot \Big(\Big(\frac{z_2^B}{z_2^0(s)}\Big)^{\alpha/\beta} \cdot \frac{1}{z_2^0(s)}\Big)\Big], \quad z_1^0(s) \geq \frac{\psi_2}{C_B} \Big\},
\end{aligned}$$

$$\begin{aligned}
M_B^2 = \Big\{ & (z_1^0(s), z_2^0(s)) \in [0,1] \times [0,1]\colon \\
z_1^0(s) = & -\Big[(C_B - \psi_2 + b_{12}) \cdot \frac{z_2^0(s)}{(1 - z_2^0(s))} + b_{12}\Big] \Big/ \Big[C_B \cdot \frac{\alpha}{(\alpha + \beta)} \cdot \Big(1 - \Big(\frac{(1 - z_2^B)}{(1 - z_2^0(s))}\Big)^{(\alpha+\beta)/\beta}\Big) \\
& + C_B \cdot \Big(\Big(\frac{(1 - z_2^B)}{(1 - z_2^0(s))}\Big)^{\alpha/\beta} \cdot \frac{1}{(1 - z_2^0(s))} - 1\Big) - \psi_1 \cdot \Big(\Big(\frac{(1 - z_2^B)}{(1 - z_2^0(s))}\Big)^{\alpha/\beta} \cdot \frac{1}{(1 - z_2^0(s))}\Big)\Big] + 1, \\
& z_1^0(s) \leq \frac{\psi_2}{C_B} \Big\}.
\end{aligned}$$

In case, when parameter C_B is negative $C_B < 0$, curves M_B, M_B^1 and M_B^2 are determined by formulas

$$M_B = M_B^1 \cup M_B^2,$$

$$\begin{gathered}
M_B^1 = \Big\{(z_1^0(s), z_2^0(s)) \in [0,1] \times [0,1]\colon \\
z_1^0(s) = \Big[(\psi_2 - b_{22}) \cdot \frac{z_2^0(s)}{(1 - z_2^0(s))} - b_{22}\Big] \Big/ \Big[C_B \cdot \frac{\alpha}{(\alpha+\beta)} \cdot \Big(1 - \Big(\frac{(1 - z_2^B)}{(1 - z_2^0(s))}\Big)^{(\alpha+\beta)/\beta}\Big) \\
+ C_B \cdot \Big(\Big(\frac{(1 - z_2^B)}{(1 - z_2^0(s))}\Big)^{\alpha/\beta} \cdot \frac{1}{(1 - z_2^0(s))} - 1\Big) - \psi_1 \cdot \Big(\Big(\frac{(1 - z_2^B)}{(1 - z_2^0(s))}\Big)^{\alpha/\beta} \cdot \frac{1}{(1 - z_2^0(s))}\Big)\Big], \\
z_1^0(s) \geq \frac{\psi_2}{C_B}\Big\},
\end{gathered}$$

$$\begin{gathered}
M_B^2 = \Big\{(z_1^0(s), z_2^0(s)) \in [0,1] \times [0,1]\colon \\
z_1^0(s) = -\Big[(C_B - \psi_2 - b_{11}) \cdot \frac{(1 - z_2^0(s))}{z_2^0(s)} - b_{11}\Big] \Big/ \Big[C_B \cdot \frac{\alpha}{(\alpha+\beta)} \cdot \Big(1 - \Big(\frac{z_2^B}{z_2^0(s)}\Big)^{(\alpha+\beta)/\beta}\Big) \\
+ C_B \cdot \Big(\Big(\frac{z_2^B}{z_2^0(s)}\Big)^{\alpha/\beta} \cdot \frac{1}{z_2^0(s)} - 1\Big) - (C_B - \psi_1) \cdot \Big(\Big(\frac{z_2^B}{z_2^0(s)}\Big)^{\alpha/\beta} \cdot \frac{1}{z_2^0(s)}\Big)\Big] + 1, \quad z_1^0(s) \leq \frac{\psi_2}{C_B}\Big\}.
\end{gathered}$$

The curve M_B divides the unit square $[0,1] \times [0,1]$ into two parts: left part

$$D_B^l \supset \{(z_1, z_2)\colon z_1 < z_1^B,\ z_2 = z_2^B\}$$

and right part

$$D_B^r \supset \{(z_1, z_2)\colon z_1 > z_1^B,\ z_2 = z_2^B\}.$$

In this case, the flexible "positive" feedback v_B^{fl} possessing properties (7.1) has the following structure

$$v_B^{fl} = v_B^{fl}(x,y) = \begin{cases} \max\{0, -\operatorname{sgn}(C_B)\}, & \text{if } (z_1, z_2) \in D_B^l, \\ \max\{0, \operatorname{sgn}(C_B)\}, & \text{if } (z_1, z_2) \in D_B^r, \\ [0,1], & \text{if } (z_1, z_2) \in M_B. \end{cases}$$

10. Test model of investments

For the verification of results of calculations on algorithms of construction of optimal controls and equilibrium trajectories we use the data from the test-example of the paper [6]. As a model example, we consider payoff matrices of two players on a financial markets of stocks and bonds. Matrix A corresponds to the behavior of traders which play on the increase of rates ans are called "bulls". Matrix B corresponds to the behavior of traders which play on the decrease of rates and are called "bears". Parameters of matrices mean the yield on stocks and bonds expressed as interest rates

$$A = \begin{pmatrix} 10 & 0 \\ 1.75 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -5 & 3 \\ 10 & 0.5 \end{pmatrix},$$

$$C_A = a_{11} - a_{12} - a_{21} + a_{22} = 10 - 0 - 1.75 + 3 = 11.25,$$

$$\varphi_1 = a_{22} - a_{12} = 3 - 0 = 3, \quad \varphi_2 = a_{22} - a_{21} = 3 - 1.75 = 1.25,$$

$$z_1^A = \frac{\varphi_2}{C_A} = \frac{1.25}{11.25} = 0.11, \quad z_2^A = \frac{\varphi_1}{C_A} = \frac{3}{11.25} = 0.27,$$

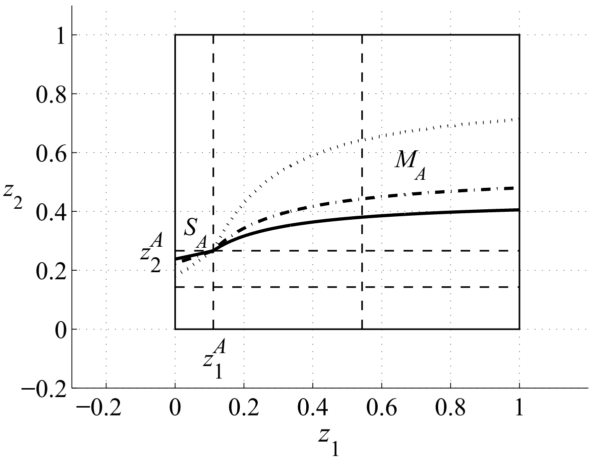

Figure 2. Sensitivity analysis for switching curve M_A of control of the first player.

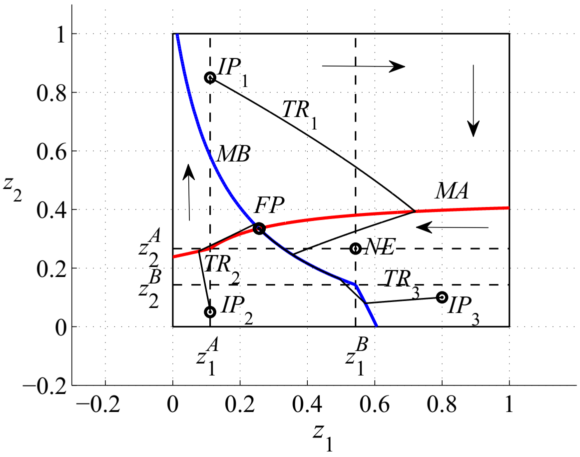

Figure 3. Equilibrium trajectories.

$$C_B = b_{11} - b_{12} - b_{21} + b_{22} = -5 - 3 - 10 + 0.5 = -17.5,$$
$$\psi_1 = b_{22} - b_{12} = 0.5 - 3 = -2.5, \quad \psi_2 = b_{22} - b_{21} = 0.5 - 10 = -9.5,$$
$$z_1^B = \frac{\psi_2}{C_B} = \frac{-9.5}{-17.5} = 0.54, \quad z_2^B = \frac{\psi_1}{C_B} = \frac{-2.5}{-17.5} = 0.14.$$

In Fig. 2 we present the sensitivity analysis for the the switching curve M_A of control of the first player depending on the proportion of the maximal rates β/α. The saddle point S_A for the matrix A is shown. The dashed line denotes the curve M_A at a value of $\beta/\alpha = 2$, the dashed-dotted line — at $\beta/\alpha = 1$, the solid line — at $\beta/\alpha = 2/3$. One can see that when the proportion β/α is declining the shape of the switching curve tends to the horizontal line $z_2 = z_2^A$. This line is the barrier line for the element of the gradient $\partial g_A/\partial z_1$ of the "local" payoff functional $g_A(z_1, z_2)$, which describes the short-term interests of the first player. So, the long-term interests of the first player, presented by the switching curve M_A, merge with the short-term interests given by the line $z_2 = z_2^A$.

In Figure 3 we present the synthesis of guaranteed control strategies based on the feedback principle for the case when $\beta/\alpha = 2/3$. We depict switching curves M_A and M_B of the controls of the first and second players, respectively. The direction of the motion is indicated by arrows. The equilibrium trajectories TR_1, TR_2, TR_3 start from the initial points IP_1, IP_2, IP_3, move along the characteristics of the Hamilton–Jacobi equations, meet the switching curves, where they change the direction of the motion and converge to the point FP.

The values of the players' payoff functionals at the final point FP dominate over the values at the point of the static Nash equilibrium NE:

$$JI_A^\infty = g_A(z_1^{FP}, z_2^{FP}) = 3.0; \quad JI_B^\infty = g_B(z_1^{FP}, z_2^{FP}) = 2.8;$$
$$g_A(z_1^{NE}, z_2^{NE}) = 2.7; \quad g_B(z_1^{NE}, z_2^{NE}) = 1.9.$$

11. Conclusion

We analyse the dynamic bimatrix game with average integral payoff functionals and heterogeneous dynamics of players' interaction. The game dynamics is considered on the infinite time horizon. Equilibrium trajectories are generated within the framework of the dynamic Nash equilibrium based on the concept of guaranteed strategies proposed by N.N. Krasovskii. Switching curves are calculated for guaranteed strategies within the dynamic programming approach and construction of the L.S. Pontryagin maximum principle. The sensitivity analysis is carried out for shapes of switching curves depending on the parameter of proportions for maximal rates of payers. The obtained results are applied to the test model of the dynamic bimatrix game for which the comparative analysis is provided for payoff functional values on equilibrium trajectories.

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