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ENUMERATION OF INTERSECTION ARRAYS OF SHILLA GRAPHS WITH $b = 6$[1]

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Abstract: Let Γ be a distance-regular graph of diameter 3, and let θ_1 be its second eigenvalue. The graph Γ is called a Shilla graph if $\theta_1 = a_3$. In this case, $\theta_1 = (a_1 + \sqrt{a_1^2 + 4k})/2$, and $a = a_3$ divides k. We set $b = b(\Gamma) = k/a$. J.H. Koolen and J. Park found the intersection arrays of Shilla graphs with $b \leq 3$. J. Cai, I.N. Belousov, and A.A. Makhnev enumerated the intersection arrays of Shilla graphs with $b = 4$. H. Li, I.N. Belousov, and A.A. Makhnev found the intersection arrays of Shilla graphs with $b = 5$. In this paper, we enumerate the intersection arrays of Shilla graphs with $b = 6$.

Keywords: Distance-regular graph, Shilla graph, Intersection array.

1. Introduction

We consider undirected graphs without loops or multiple edges. For a vertex a of a graph Γ, denote by $\Gamma_i(a)$ the i-neighborhood of a, that is, the subgraph induced in Γ on the set of vertices at distance i from a. We set $[a] = \Gamma_1(a)$ and $a^{\perp} = \{a\} \cup [a]$.

Let Γ be a graph, and let a and b be two vertices in Γ. The number of vertices in $[a] \cap [b]$ is denoted by $\mu(a, b)$ (or by $\lambda(a, b)$ if $d(a, b) = 1$). Furthermore, the subgraph induced by $[a] \cap [b]$ is called a *μ-subgraph* (or a *λ-subgraph*).

Let Γ be a graph of diameter d, and let $i \in \{1, 2, 3, \ldots, d\}$. The graph Γ_i has the same vertex set, and two vertices u and w are adjacent in Γ_i if $d_\Gamma(u, w) = i$.

If vertices u and w are at distance i in Γ, then $b_i(u, w)$ (respectively, $c_i(u, w)$) is the number of vertices in the intersection of $\Gamma_{i+1}(u)$ (respectively, $\Gamma_{i-1}(u)$) with $[w]$. A graph Γ of diameter d is called *distance-regular* with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if the values $b_i = b_i(u, w)$ and $c_i = c_i(u, w)$ are independent of the choice of vertices u and w at distance i in Γ for every $i = 0, \ldots, d$. Further, let $p^l_{ij}(x, y)$ denote the number of vertices in $\Gamma_i(x) \cap \Gamma_j(y)$ for vertices x and y at distance l in Γ. In a distance-regular graph, the numbers $p^l_{ij}(x, y)$ are independent of the choice of vertices x and y, are denoted by p^l_{ij}, and are called the intersection numbers of Γ (see [3]). We set $a_i = k - b_i - c_i$.

Let Γ be a distance-regular graph of diameter 3, and let θ_1 be the second eigenvalue of Γ. Γ is

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called a Shilla graph if $\theta_1 = a_3$. In this case,

$$\theta_1 = \frac{a_1 + \sqrt{a_1^2 + 4k}}{2}$$

and $a = a_3$ divides k [6, Theorem 7]. We set $b = b(\Gamma) = k/a$.

Koolen and Park found the intersection arrays of Shilla graphs with $b \le 3$ [6]. Cai, Makhnev, and Belousov enumerated the intersection arrays of Shilla graphs with $b = 4$ [4]. Haiyan Li, Makhnev, and Belousov found the intersection arrays of Shilla graphs with $b = 5$ [7].

In this paper, we enumerate the intersection arrays of Shilla graphs with $b = 6$.

A Q-polynomial Shilla graph with $b = 6$ [1] has the intersection array

$$\{42t, 5(7t + 1), 3(t + 3); 1, 3(t + 3), 35t\},$$

where

$$t \in \{7, 12, 17, 27, 57\}, \quad \{312, 265, 48; 1, 24, 260\}, \quad \{372, 315, 75; 1, 15, 310\},$$
$$\{624, 525, 80; 1, 40, 520\}, \quad \{744, 625, 125; 1, 25, 620\}, \quad \{930, 780, 150; 1, 30, 775\},$$
$$\{1794, 1500, 200; 1, 100, 1495\}, \quad \text{or} \quad \{5694, 4750, 600; 1, 300, 4745\}.$$

In [2], it is proved that graphs with intersection arrays $\{312, 265, 48; 1, 24, 260\}$, $\{624, 525, 80; 1, 40, 520\}$, and $\{930, 780, 150; 1, 30, 775\}$ do not exist. In [8], it is proved that graphs with intersection arrays $\{372, 315, 75; 1, 15, 310\}$, $\{744, 625, 125; 1, 25, 620\}$, and $\{1794, 1500, 200; 1, 100, 1495\}$ do not exist. Therefore, to prove the nonexistence of Q-polynomial Shilla graphs with $b = 6$, it suffices to prove that distance-regular graphs with intersection arrays

$$\{42t, 5(7t + 1), 3(t + 3); 1, 3(t + 3), 35t\},$$

where $t \in \{7, 12, 17, 27, 57\}$, and $\{5694, 4750, 600; 1, 300, 4745\}$ do not exist.

Theorem 1. *Distance-regular graphs with intersection arrays*

$$\{42t, 5(7t + 1), 3(t + 3); 1, 3(t + 3), 35t\},$$

where $t \in \{7, 12, 17, 27, 57\}$, *and* $\{5694, 4750, 600; 1, 300, 4745\}$ *do not exist.*

Next, we prove the nonexistence of some Shilla graphs with $b = 6$.

Theorem 2. *A distance-regular graph with intersection array* $\{714, 600, 120; 1, 5, 595\}$ *does not exist.*

Proof. A graph Γ with intersection array $\{714, 600, 120; 1, 5, 595\}$ has the spectrum

$$714^1, 119^{4785}, -1^{51765}, -11^{47124}.$$

The maximal size of a clique in Γ is at most $1 + 714/11$. For $s = 7$, the equality in the Koolen–Park bound [9, Proposition 1] is achieved:

$$84 = 21 \cdot 4 \ge 7 \cdot 114 - 714 = 84.$$

Hence, Γ is an amply regular Terwilliger graph with parameters $(103675, 714, 113, 5)$. By [3, Theorem 1.16.3], the reduction of the graph $\Gamma(u)$ is a strongly regular graph with parameters $(714/s, (114 - s)/s, \lambda, 4/s)$, where

$$\lambda = \frac{(114 - s)(114 - 2s) - 4 \cdot 600}{s(114 - s)}.$$

If $s = 2$, then

$$\lambda = (112 \cdot 110 - 2400)/224$$

is not an integer. If $s = 1$, then

$$\lambda = (113 \cdot 112 - 2400)/113$$

is also not an integer. Theorem 2 is proved. □

Proposition 1. *A graph does not contain* 8*-claws for the following intersection arrays*:

$$\{594, 500, 100; 1, 5, 495\}, \quad \{654, 550, 108; 1, 6, 545\}, \quad \{726, 610, 120; 1, 6, 605\},$$
$$\{1242, 1040, 208; 1, 13, 1035\}, \quad \{2184, 1825, 360; 1, 24, 1820\}.$$

Proof. A graph with intersection array $\{594, 500, 100; 1, 5, 495\}$ has the spectrum

$$\{594^1, 99^{3927}, -1^{35937}, -11^{32130}\}.$$

The size of the Delsarte clique is equal to $1 + 594/11 = 55$. For $s = 8$, the Koolen–Park bound

$$112 = 28 \cdot 4 < 8 \cdot 94 - 594 = 158$$

is violated.

A graph with intersection array $\{654, 550, 108; 1, 6, 545\}$ has the spectrum

$$\{654^1, 109^{3591}, 0^{33250}, -11^{35643}\}.$$

The size of the Delsarte clique is at most $1 + 654/11$. For $s = 8$, the Koolen–Park bound

$$140 = 28 \cdot 5 < 8 \cdot 104 - 654 = 178$$

is violated.

A graph with intersection array $\{726, 610, 120; 1, 6, 605\}$ has the spectrum

$$\{726^1, 121^{4020}, 0^{40870}, -11^{44286}\}.$$

The size of the Delsarte clique is equal to $1 + 726/11 = 67$. For $s = 8$, the Koolen–Park bound

$$140 = 28 \cdot 5 < 8 \cdot 116 - 726 = 202$$

is violated.

A graph with intersection array $\{1242, 1040, 208; 1, 13, 1035\}$ has the spectrum

$$\{1242^1, 207^{3201}, -1^{87009}, -19^{30360}\}.$$

The size of the Delsarte clique is at most $1 + 1242/19$. For $s = 8$, the Koolen–Park bound

$$336 = 28 \cdot 12 < 8 \cdot 202 - 1242 = 374$$

is violated.

A graph with intersection array $\{2184, 1825, 360; 1, 24, 1820\}$ has the spectrum

$$\{2184, 1825, 360; 1, 24, 1820\}.$$

The size of the Delsarte clique is equal to $1 + 2184/26 = 85$. For $s = 8$, the Koolen–Park bound

$$644 = 28 \cdot 23 < 8 \cdot 359 - 2184 = 688$$

is violated.

Proposition 1 is proved. □

Theorem 3. *Distance-regular graphs with intersection arrays*

$$\{594, 500, 100; 1, 5, 495\}, \quad \{654, 550, 108; 1, 6, 545\}, \quad \{726, 610, 120; 1, 6, 605\},$$
$$\{1242, 1040, 208; 1, 13, 1035\}, \quad \{2184, 1825, 360; 1, 24, 1820\}$$

do not exist.

Theorem 4. *A Shilla graph with $b = 6$ has an intersection array from the following list*:

$$\{42, 40, 8; 1, 1, 35\}, \quad \{42, 40, 8; 1, 2, 35\}, \quad \{42, 40, 12; 1, 3, 35\}, \quad \{48, 45, 9; 1, 1, 40\},$$
$$\{66, 60, 18; 1, 2, 55\}, \quad \{90, 80, 16; 1, 4, 75\}, \quad \{114, 100, 20; 1, 5, 95\}, \quad \{120, 105, 21; 1, 7, 100\},$$
$$\{126, 110, 4; 1, 4, 105\}, \quad \{204, 175, 20; 1, 20, 170\}, \quad \{210, 180, 36; 1, 9, 175\},$$
$$\{228, 195, 15; 1, 5, 190\}, \quad \{264, 225, 45; 1, 3, 220\}, \quad \{264, 225, 45; 1, 5, 220\},$$
$$\{294, 250, 20; 1, 20, 245\}, \quad \{294, 250, 32; 1, 4, 245\}, \quad \{306, 260, 36; 1, 9, 255\},$$
$$\{324, 275, 55; 1, 5, 270\}, \quad \{330, 280, 54; 1, 6, 275\}, \quad \{474, 400, 80; 1, 10, 395\},$$
$$\{624, 525, 105; 1, 15, 520\}, \quad \{834, 700, 120; 1, 15, 695\}, \quad \{882, 740, 144; 1, 18, 735\},$$
$$\{924, 775, 155; 1, 25, 770\}, \quad \{930, 780, 150; 1, 30, 775\}, \quad \{1110, 930, 180; 1, 30, 925\},$$
$$\{1128, 945, 168; 1, 24, 940\}, \quad \{1296, 1085, 189; 1, 27, 1080\}, \quad \{1674, 1400, 280; 1, 25, 1395\},$$
$$\{1794, 1500, 288; 1, 24, 1495\}, \quad \textit{or} \quad \{3354, 2800, 480; 1, 75, 2795\}.$$

Let Γ be a Shilla graph with $b = 6$ that is not Q-polynomial. By [6, Lemma 14], we have $k < 6^5 \cdot 7^2$. Using computer calculations, we prove Theorem 4.

Note that for graphs from our list, c_2 does not divide b_2 only for the arrays

$$\{924, 775, 155; 1, 25, 770\}, \quad \{1674, 1400, 280; 1, 25, 1395\}, \quad \text{and} \quad \{3354, 2800, 480; 1, 75, 2795\}.$$

Similarly, for graphs with $b = 4$, the parameter c_2 does not divide b_2 only for the arrays

$$\{140, 108, 24; 1, 9, 105\}, \quad \{152, 117, 39; 1, 9, 114\}, \quad \{236, 180, 48; 1, 9, 177\}, \quad \{260, 198, 66; 1, 9, 195\}.$$

For graphs with $b = 5$, the parameter c_2 does not divide b_2 only for the arrays

$$\{255, 208, 52; 1, 8, 204\}, \quad \{315, 256, 50; 1, 20, 252\}, \quad \{715, 576, 132; 1, 16, 572\},$$
$$\{735, 592, 148; 1, 16, 588\}, \quad \{1045, 840, 180; 1, 24, 836\}.$$

In any case, the parameters c_2 and b_2 are not coprime. Hence, we can propose the following hypothesis.

Hypothesis 1. *Let Γ be a Shilla graph. If c_2 does not divide b_2, then the parameters c_2 and b_2 are not coprime.*

2. Proof of Theorem 3

Assume that Γ is a distance-regular graph with intersection array $\{594, 500, 100; 1, 5, 495\}$. Then the size of the Delsarte clique is $1 + 594/11 = 55$.

Let u be a vertex of Γ, let $\{x_1, \dots, x_5, o_1, o_2\}$ be a 7-coclique in $[u]$, and let $\Delta = [u] - (x_1^\perp \cup \dots \cup x_5^\perp)$. Then Δ does not contain 3-cocliques and $\big|[u] \cap [x_i] \cap [o_j]\big| \leq 4$, so $|\Delta(o_j)| \geq 94 - 5 \cdot 4 = 74$. Since $|\Delta(o_1) \cap [o_2]| \leq 4$, we have $|\Delta(o_1) - [o_2]| \geq 70$. This contradicts the fact that $\{u, o_1\} \cup (\Delta(o_1) - [o_2])$ is a 72-clique.

Assume that Γ is a distance-regular graph with intersection array $\{654, 550, 108; 1, 6, 545\}$. Then the size of the Delsarte clique is at most $1 + 654/11$.

Let u be a vertex of Γ, let $\{x_1, \ldots, x_5, o_1, o_2\}$ be a 7-coclique in $[u]$, and let $\Delta = [u] - (x_1^\perp \cup \cdots \cup x_5^\perp)$. Then Δ does not contain 3-cocliques and $\big|[u] \cap [x_i] \cap [o_j]\big| \leq 5$, so $|\Delta(o_j)| \geq 104 - 5 \cdot 5 = 79$. Since $|\Delta(o_1) \cap [o_2]| \leq 5$, we have $|\Delta(o_1) - [o_2]| \geq 74$. This contradicts the fact that $\{u, o_1\} \cup (\Delta(o_1) - [o_2])$ is a 76-clique.

Assume that Γ is a distance-regular graph with intersection array $\{726, 610, 120; 1, 6, 605\}$. Then the size of the Delsarte clique is $1 + 726/11 = 67$.

Let u be a vertex of Γ, let $\{x_1, \ldots, x_5, o_1, o_2\}$ be a 7-coclique in $[u]$, and let $\Delta = [u] - (x_1^\perp \cup \cdots \cup x_5^\perp)$. Then Δ does not contain 3-cocliques and $|[u] \cap [x_i] \cap [o_j]| \leq 5$, so $|\Delta(o_j)| \geq 104 - 5 \cdot 5 = 91$. Since $|\Delta(o_1) \cap [o_2]| \leq 5$, we have $|\Delta(o_1) - [o_2]| \geq 86$. This contradicts the fact that $\{u, o_1\} \cup (\Delta(o_1) - [o_2])$ is a 88-clique.

Assume that Γ is a distance-regular graph with intersection array $\{1242, 1040, 208; 1, 13, 1035\}$.

Let u be a vertex of Γ, let $\{x_1, \ldots, x_5, o_1, o_2\}$ be a 7-coclique in $[u]$, and let $\Delta = [u] - (x_1^\perp \cup \cdots \cup x_5^\perp)$. Then Δ does not contain 3-cocliques and $|[u] \cap [x_i] \cap [o_j]| \leq 12$, so $|\Delta(o_j)| \geq 104 - 5 \cdot 5 = 142$. Since $|\Delta(o_1) \cap [o_2]| \leq 12$, we have $|\Delta(o_1) - [o_2]| \geq 130$. This contradicts the fact that $\{u, o_1\} \cup (\Delta(o_1) - [o_2])$ is a 132-clique.

Assume that Γ is a distance-regular graph with intersection array $\{2184, 1825, 360; 1, 24, 1820\}$.

Let u be a vertex of Γ, let $\{x_1, \ldots, x_5, o_1, o_2\}$ be a 7-coclique in $[u]$, and let $\Delta = [u] - (x_1^\perp \cup \cdots \cup x_5^\perp)$. Then Δ does not contain 3-cocliques and $|[u] \cap [x_i] \cap [o_j]| \leq 23$, so $|\Delta(o_j)| \geq 359 - 5 \cdot 23 = 254$. Since $|\Delta(o_1) \cap [o_2]| \leq 23$, we have $|\Delta(o_1) - [o_2]| \geq 221$. This contradicts the fact that $\{u, o_1\} \cup (\Delta(o_1) - [o_2])$ is a 223-clique.

Theorem 3 is proved. □

3. Triple intersection numbers

Let Γ be a distance-regular graph of diameter d.

If u_1, u_2, u_3 are vertices of Γ, and r_1, r_2, r_3 are integers from $\{0, 1, \ldots, d\}$, then we define $\left\{\begin{smallmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{smallmatrix}\right\}$ as the set of vertices $w \in \Gamma$ such that $d(w, u_i) = r_i$. Furthermore, $\left[\begin{smallmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{smallmatrix}\right]$ denotes the number of vertices in $\left\{\begin{smallmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{smallmatrix}\right\}$.

The numbers $\left[\begin{smallmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{smallmatrix}\right]$ are called triple intersection numbers.

We abbreviate the latter as $[r_1r_2r_3]$ whenever no confusion about the triple (u_1, u_2, u_3) may arise.

Unlike the case $t = 2$, for $t \geq 3$ there are no formulas for $[r_1r_2r_3]$ that are generally valid for distance-regular graphs. However, certain restrictions for their values can be found in [5].

Let u, v, w be three fixed vertices in Γ, and let $W = d(u, v)$, $U = d(v, w)$, and $V = d(u, w)$.

There exists precisely one vertex $x = u$ such that $d(x, u) = 0$; therefore, $[0jh]$ is either 0 or 1. We can apply the same argument also for v and w. Altogether, we obtain

$$[0jh] = \delta_{jW}\delta_{hV}, \quad [i0h] = \delta_{iW}\delta_{hU}, \quad [ij0] = \delta_{iU}\delta_{jV} \quad (0 \leq i, j, h \leq 3).$$

Another set of equations can be obtained by fixing the distance from two of the vertices u, v, w and counting vertices at all distances from the third vertex:

$$\sum_{l=1}^{d} [ljh] = p_{jh}^{U} - [0jh], \quad \sum_{l=1}^{d} [ilh] = p_{ih}^{V} - [i0h], \quad \sum_{l=1}^{d} [ijl] = p_{ij}^{W} - [ij0]. \tag{3.1}$$

We can use the triangle inequality to conclude that some variables vanish. For example, for $0 \le i, j \le 3$, if $|i-j| > W$ or $i+j < W$, then $p^W_{ij} = 0$, and hence $[ijh] = 0 \ (0 \le h \le 3)$.

If a Krein parameter q^h_{ij} is zero, we can obtain another equation for triple intersection numbers. Define

$$S_{ijh}(u,v,w) = \sum_{r,s,t=0}^{d} Q_{ri} Q_{sj} Q_{th} \begin{bmatrix} uvw \\ rst \end{bmatrix}.$$

If $q^h_{ij} = 0$, then $S_{ijh}(u,v,w) = 0$ (see [5, Theorem 3]).

Fix vertices $u, v, w \in \Gamma$ and set

$$\{ijh\} = \begin{Bmatrix} uvw \\ ijh \end{Bmatrix}, \quad [ijh] = \begin{bmatrix} uvw \\ ijh \end{bmatrix}, \quad [ijh]' = \begin{bmatrix} uvw \\ ihj \end{bmatrix}, \quad [ijh]^* = \begin{bmatrix} uvw \\ jih \end{bmatrix} \quad [ijh]^\sim = \begin{bmatrix} uvw \\ hji \end{bmatrix}.$$

Calculating the parameters $[ijh]'$, $[ijh]^*$, and $[ijh]^\sim$ (the symmetrization of the triple intersection numbers array) can yield new relations that prove the nonexistence of the graph.

4. Proof of Theorem 1

Let Γ be a graph with intersection array $\{5694, 4750, 600; 1, 300, 4745\}$. Then Γ has

$$1 + 5694 + 90155 + 11400 = 107250$$

vertices, the spectrum $5694^1, 949^{450}, 69^{24674}, -26^{82125}$, and the dual eigenmatrix

$$Q = \begin{pmatrix} 1 & 450 & 24674 & 82125 \\ 1 & 75 & 299 & -375 \\ 1 & 0 & -1144/19 & 1125/19 \\ 1 & -75/2 & 12337/38 & -5475/19 \end{pmatrix}.$$

Lemma 1. *The intersection numbers of* Γ *are*

$$p^1_{11} = 943, \quad p^1_{21} = 4750, \quad p^1_{32} = 9500, \quad p^1_{22} = 75905, \quad p^1_{33} = 1900;$$
$$p^2_{11} = 300, \quad p^2_{12} = 4794, \quad p^2_{13} = 600, \quad p^2_{22} = 75760, \quad p^2_{23} = 9600, \quad p^2_{33} = 1200;$$
$$p^3_{12} = 4745, \quad p^3_{13} = 949, \quad p^3_{22} = 75920, \quad p^3_{23} = 9490, \quad p^3_{33} = 960.$$

Proof. This is proved by simple calculations. □

Let u, v, w be vertices of Γ, and let

$$\{rst\} = \begin{Bmatrix} uvw \\ rst \end{Bmatrix}, \quad [rst] = \begin{bmatrix} uvw \\ rst \end{bmatrix}.$$

We set $\Sigma = \Gamma_3(u)$ and $\Lambda = \Sigma_2$. Then Λ is a regular graph of degree $p^2_{32} = 9490$ on $k_3 = 11400$ vertices.

Lemma 2. *Let* $d(u,v) = d(u,w) = 3$ *and* $d(v,w) = 1$. *Then*

$$[122] = 4r_6/3 + 3965, \quad [123] = [132] = -4r_6/3 + 780, \quad [133] = 4r_6/3 + 169;$$
$$[211] = 2r_6/3 + 705, \quad [212] = [221] = -2r_6/3 + 4040, \quad [222] = -5r_6/3 + 64120,$$
$$[223] = [232] = 7r_6/3 + 7760, \quad [233] = -7r_6/3 + 1730;$$
$$[311] = -2r_6/3 + 238, \quad [312] = [321] = 2r_6/3 + 710, \quad [322] = r_6/3 + 7820,$$
$$[323] = [332] = -r_6 + 960, \quad [333] = r_6,$$

where $r_6 \in \{0, 3, \ldots, 357\}$.

Proof. This is proved by applying formulas (3.1). □

By Lemma 2, we have $7820 \leq [322] = r_6/3 + 7820 \leq 7939$.

Lemma 3. *Let $d(u,v) = d(u,w) = d(v,w) = 3$. Then*

$$\begin{gathered}
[122] = -r_{17} + 4745, \quad [123] = [132] = r_{17}, \quad [133] = -r_{17} + 949; \\
[212] = [221] = -r_{17} + 4745, \quad [213] = [231] = r_{17}, \quad [222] = 23r_{17}/4 + 118625/2, \\
[223] = [232] = -19r_{17}/4 + 23725/2, \quad [233] = 15r_{17}/4 - 4745/2; \\
[312] = [321] = r_{17}, \quad [313] = [331] = -r_{17} + 949, \quad [322] = -19r_{17}/4 + 23725/2, \\
[333] = -11r_{17} + 4765/2,
\end{gathered}$$

where $r_{17} \in \{634, 638, \ldots, 866\}$.

Proof. This is proved by applying formulas (3.1). □

By Lemma 3, we have

$$7749 \leq [322] = -19r_{17}/4 + 23725/2 \leq 8851.$$

We find the number d of edges between $\Lambda(v)$ and $\Lambda_2(v)$. Since

$$p^3_{13} = 949, \quad p^3_{23} = 9490, \quad p^3_{33} = 960,$$

we have

$$14860220 = 949 \cdot 7820 + 960 \cdot 7749 \leq d \leq 949 \cdot 7939 + 960 \cdot 8851 = 1603171.$$

On the other hand, $d = 9490(9489 - \lambda)$. Therefore,

$$1565.88 \leq (9489 - \lambda) \leq 1689.26 \quad \text{and} \quad 7799.74 \leq \lambda \leq 7923.12,$$

where λ denotes the average value of the parameter $\lambda(\Lambda)$.

Lemma 4. *Let $d(u,v) = d(u,w) = 3$ and $d(v,w) = 2$. Then*

$$\begin{gathered}
[122] = (4r_{15} - 36r_{16} + 46115/19, \quad [123] = (-4r_{15} + 36r_{16} + 44040)/19, \\
[133] = (4r_{15} - 36r_{16} - 26009)/19; \\
[211] = -r_{16} + 300, \quad [212] = [221] = (4r_{15} + 21r_{16} + 42885)/19, \\
[213] = [231] = (-4r_{15} - 2r_{16} + 41570)/19, \\
[222] = (-23r_{15} + 36r_{16} + 1393325)/19, \quad [223] = [232] = r_{15} - 3r_{16} + 330, \\
[233] = (-15r_{15} + 59r_{16} + 132470)/19; \\
[311] = r_{16}, \quad [312] = [321] = (-4r_{15} - 21r_{16} + 48201)/19, \\
[313] = [331] = (4r_{15} + 2r_{16} - 30170)/19, \quad [322] = r_{15}, \\
[323] = [332] = (-15r_{15} + 21r_{16} + 132090)/19, \quad [333] = (11r_{15} - 23r_{16} - 83680)/19,
\end{gathered}$$

where

$$r_{15} \in \{7608, 7609, \ldots, 9266\}, \quad r_{16} \in \{0, 1, \ldots, 300\},$$

$2r_{15} + r_{16} + 1$ *is divisible by* 19.

Proof. This is proved by applying formulas (3.1). □

By Lemma 4, we have

$$7608 \le [322] = r_{15} \le 9266.$$

Let $d(u,v)=3$.
We find the number e_2 of pairs of vertices y,z at distance 2 such that

$$y \in \left\{ {uv \atop 31} \right\}, \quad z \in \left\{ {uv \atop 33} \right\}.$$

By Lemma 2, we have $[332] = -r_6 + 960$, where $r_6 \in \{0,3,\dots,357\}$, so

$$572247 = 949(960-357) \le e_2 \le 949 \cdot 960 = 911040.$$

On the other hand, by Lemma 3, we have $[312] = r_{17}$, so

$$572247 \le e_2 = \sum_i r^i_{17} \le 911040 \quad \text{and} \quad 596.09 \le \sum_i r^i_{17}/960 \le 949.$$

We find the number f_1 of pairs of vertices y,z at distance 1 such that

$$y \in \left\{ {uv \atop 31} \right\}, \quad z \in \left\{ {uv \atop 32} \right\}.$$

By Lemma 2, we have $[321] = 2r_6/3 + 710$, where $r_6 \in \{0,3,\dots,357\}$, so

$$0 \le f_1 \le 949 \cdot 948 = 899652.$$

On the other hand, by Lemma 4, we have $[311] = r_{16}$, so

$$0 \le f_1 = \sum_i r^i_{16} \le 899652 \quad \text{and} \quad 0 \le \sum_i r^i_{16}/960 \le 94.8.$$

We find the number f_2 of pairs of vertices y,z at distance 2 such that

$$y \in \left\{ {uv \atop 31} \right\}, \quad z \in \left\{ {uv \atop 32} \right\}.$$

By Lemma 2, we have $[322] = r_6/3 + 7820$, where $r_6 \in \{0,3,\dots,357\}$, so

$$7421180 = 949 \cdot 7820 \le f_2 \le 949 \cdot 7939 = 7534111.$$

On the other hand, by Lemma 4, we have $[312] = (-4r_{15} - 21r_{16} + 48201)/19$, so

$$7421180 \le f_1 = -\sum_i (4r^i_{15} + 21r^i_{16}) - 48201)/19 \le 7534111,$$

$$314279381 \le \sum_i (4r^i_{15} + 21r^i_{16}) \le 316425070,$$

$$33116.9 \le \sum_i (4r^i_{15} + 21r^i_{16})/9490 \le 33343.$$

We find the number f_3 of pairs of vertices y,z at distance 3 such that

$$y \in \left\{ {uv \atop 31} \right\}, \quad z \in \left\{ {uv \atop 32} \right\}.$$

By Lemma 2, we have $[323] = -r_6 + 960$, where $r_6 \in \{0, 3, \ldots, 357\}$, so

$$572247 = 949 \cdot 603 \leq f_3 \leq 949 \cdot 960 = 911040.$$

On the other hand, by Lemma 4, we have $[313] = (4r_{15} + 2r_{16} - 30170)/19$, so

$$10872693 \leq f_3 = \sum_i (4r^i_{15} + 2r^i_{16}) - 457427490)/19 \leq 17309760,$$
$$468300183 \leq \sum_i (4r^i_{15} + 2r^i_{16}) \leq 474737250,$$
$$49346.7 \leq \sum_i (4r^i_{15} + 2r^i_{16})/9490 \leq 50025,$$

a contradiction with

$$49346.7 \leq \sum_i (4r^i_{15} + 21r^i_{16})/9490 \leq 33343.$$

Let Γ be a graph with intersection array $\{42t, 5(7t + 1), 3(t + 3); 1, 3(t + 3), 35t\}$, where $t \in \{7, 12, 17, 27, 57\}$. Then Γ has

$$1 + 42t + 70(7t + 1)t/(t + 3) + 42t + 6 = 70(7t + 1)t/(t + 3) + 84t + 7$$

vertices, the spectrum

$$\begin{aligned} 42\,t,&\text{ multiplicity: } 1,\\ 7\,t,&\text{ multiplicity: } \frac{6\,(41\,t+3)}{t+3},\\ t-3,&\text{ multiplicity: } \frac{1470\,(7\,t+1)t}{(t+18)(t+3)},\\ -21,&\text{ multiplicity: } \frac{2\,(41\,t+3)(7\,t+1)t}{(t+18)(t+3)}, \end{aligned}$$

and the dual eigenmatrix

$$Q = \begin{pmatrix} 1 & \frac{6\,(41\,t+3)}{t+3} & \frac{1470\,\left(7\,t^2+t\right)}{t^2+21\,t+54} & \frac{2\,\left(287\,t^3+62\,t^2+3\,t\right)}{t^2+21\,t+54} \\ 1 & \frac{41\,t+3}{t+3} & \frac{35\,\left(7\,t^2-20\,t-3\right)}{t^2+21\,t+54} & -\frac{287\,t^2+62\,t+3}{t^2+21\,t+54} \\ 1 & 0 & -\frac{21\,(2\,t+1)}{t+18} & \frac{41\,t+3}{t+18} \\ 1 & -\frac{41\,t+3}{t+3} & \frac{245\,t}{t+18} & -\frac{5\,\left(41\,t^2+3\,t\right)}{t^2+21\,t+54} \end{pmatrix}.$$

Lemma 5. *The intersection numbers of* Γ *are*

$$p^1_{11} = 7t - 6, \quad p^1_{21} = 35t + 5, \quad p^1_{32} = 35t + 5,$$
$$p^1_{22} = 30(14t^2 - 5t - 1)/(t + 3), \quad p^1_{33} = 7t + 1;$$
$$p^2_{11} = 3t + 9, \quad p^2_{12} = 36t - 18, \quad p^2_{13} = 3t + 9,$$
$$p^2_{22} = (418t^2 - 117t + 87)/(t + 3), \quad p^2_{23} = 36t - 12, \quad p^2_{33} = 3t + 9;$$
$$p^3_{12} = 35t, \quad p^3_{13} = 7t, \quad p^3_{22} = 140(3t^2 - t)/(t + 3),$$
$$p^3_{23} = 35t, \quad p^3_{33} = 5.$$

Proof. This is proved by simple calculations. ☐

Let u, v, w be vertices of Γ,

$$\{rst\} = \left\{ \begin{matrix} uvw \\ rst \end{matrix} \right\}, \quad \text{and} \quad [rst] = \left[\begin{matrix} uvw \\ rst \end{matrix} \right].$$

We set $\Sigma = \Gamma_3(u)$ and $\Lambda = \Sigma_2$. Then Λ is a regular graph of degree $p^3_{32} = 35t$ on $k_3 = 42t + 6$ vertices.

Lemma 6. *Let $d(u, v) = d(u, w) = 3$ and $d(v, w) = 1$. Then*

$$[122] = r_6 + 28t, \quad [123] = [132] = -r_6 + 7t, \quad [133] = r_6;$$
$$[211] = r_6 + 3t, \quad [212] = [221] = -r_6 + 32t, \quad [222] = (r_6t + 1078t^2 + 3r_6 - 966t)/(3(t + 3)),$$
$$[223] = [232] = 2r_6/3 + 86t/3, \quad [233] = -2r_6/3 + 19t/3;$$
$$[311] = -r_6 + 4t - 6, \quad [312] = [321] = r_6 + 3t + 5,$$
$$[322] = -4r_6/3 + 98t/3 - 10, \quad [323] = [332] = r_6/3 - 2t/3 + 5, \quad [333] = -r_6/3 + 2t/3,$$

where $2t - 15 \le r_6 \le 2t$.

Proof. This is proved by applying formulas (3.1). ☐

By Lemma 6, we have

$$30t - 10 \le [322] = -4r_6/3 + 98t/3 - 10 \le 30t + 10.$$

Lemma 7. *Let $d(u, v) = d(u, w) = d(v, w) = 3$. Then*

$$[122] = r_{17} + 28t, \quad [123] = [132] = -r_{17} + 7t, \quad [133] = r_{17};$$
$$[212] = [221] = 3r_{17}/8 + 245/2, \quad [213] = [231] = r_{17} + 28t,$$
$$[222] = -(11r_{17}t - 1092t^2 + 33r_{17} + 924t)/(3(t + 3)),$$
$$[231] = -r_{17} + 7t, \quad [223] = [232] = 8r_{17}/3 + 28t, \quad [233] = -5r_{17}/3;$$
$$[312] = [321] = -r_{17} + 7t, \quad [313] = [331] = r_{17}, \quad [322] = 8r_{17}/3 + 28t,$$
$$[323] = [332] = -5r_{17}/3, \quad [333] = 2r_{17}/3 + 4,$$

where $r_{17} = 0$.

Proof. This is proved by applying formulas (3.1). ☐

By Lemma 7, we have $[322] = 28t$.

We find the number d of edges between $\Lambda(v)$ and $\Lambda_2(v)$. Since $p^3_{13} = 7t$, $p^3_{23} = 35t$, and $p^3_{33} = 5$, we have

$$210t^2 + 70t \le d \le 210t^2 + 210t.$$

On the other hand,

$$d = 35t(35t - 1 - \lambda),$$

so

$$6t + 2 \le (35t - 1 - \lambda) \le 6t + 6 \quad \text{and} \quad 29t - 7 \le \lambda \le 29t - 3,$$

where λ is the average value of the parameter $\lambda(\Lambda)$.

Lemma 8. *Let* $d(u,v) = d(u,w) = 3$ *and* $d(v,w) = 2$*. Then*

$$[122] = 3r_{15} - 3r_{16} + 29t - 3, \quad [123] = [132] = -3r_{15} + 3r_{16} + 6t + 3,$$
$$[133] = 3r_{15} - 3r_{16} + t - 3;$$
$$[211] = 2r_{15} - 3r_{16} + 2t + 6, \quad [212] = [221] = -r_{15} + 3r_{16} + 30t - 15,$$
$$[213] = [231] = -r_{15} + 3t + 9,$$
$$[222] = -(3r_{15}t + r_{16}t - 360t^2 + 9r_{15} + 3r_{16} + 285t - 105)/(t+3),$$
$$[223] = [232] = 4r_{15} - 2r_{16} + 30t - 20, \quad [233] = -3r_{15} + 2r_{16} + 2t + 11;$$
$$[311] = -2r_{15} + 3r_{16} + t + 3, \quad [312] = [321] = r_{15} - 3r_{16} + 6t - 3, \quad [313] = [331] = r_{15},$$
$$[322] = 4r_{16} + 29t - 3, \quad [323] = [332] = -r_{15} - r_{16} + 5, \quad [333] = r_{16},$$

where

$$r_{15} + r_{16} \leq 5, \quad 2r_{15} - 3r_{16} \leq t + 3, \quad 3r_{16} - 3r_{15} \leq t - 3.$$

Proof. This is proved by applying formulas (3.1). □

By Lemma 8, we have

$$29t - 3 \leq [322] = 4r_{16} + 29t - 3 \leq 29t + 17.$$

Since $29t - 7 \leq \lambda \leq 29t - 3$, we have $\lambda = 29t - 3$. Thus, Λ is an edge-regular graph with parameters $(42t + 6, 35t, 29t - 3)$.

Let $d(u,v) = 3$.

We find the number f_1 of pairs of vertices y, z at distance 1 such that

$$y \in \left\{ \begin{matrix} uv \\ 31 \end{matrix} \right\}, \quad z \in \left\{ \begin{matrix} uv \\ 32 \end{matrix} \right\}.$$

By Lemma 6, we have $[321] = r_6 + 3t + 5$, where

$$2t - 15 \leq r_6 \leq 2t,$$

so

$$7t(5t - 10) \leq f_1 \leq 7t(5t + 5).$$

On the other hand, by Lemma 8, we have $[311] = -2r_{15} + 3r_{16} + t + 3$, so

$$7t(5t - 10) \leq f_1 = \sum_i (3r^i_{16} - 2r^i_{15}) + (t+3)p^3_{23} \leq 7t(5t + 5)$$

and

$$\sum_i (3r^i_{16} - 2r^i_{15})/35t \leq (t+1) - (t+3),$$

a contradiction.

Theorem 1 is proved. □

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