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FIXED POINTS IN THE CONSTRUCTION OF A MINIMAX SOLUTION FOR A CLASS OF BOUNDARY VALUE PROBLEMS FOR HAMILTON–JACOBI EQUATIONS[1]

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Abstract: This paper deals with analytical and numerical methods for constructing a minimax (generalized) solution to the Dirichlet problem for the Hamilton–Jacobi equation. The case of a closed planar nonconvex boundary set is considered, where the boundary points have a smoothness defect in the coordinate functions with respect to third-order derivatives. These points belong to the pseudo-vertices of the boundary set. Pseudo-vertices generate branches of a singular set, which are one-dimensional manifolds where the smoothness of the minimax solution breaks down. To construct a branch of a singular set, it is necessary to find markers, i.e., numerical characteristics of the corresponding pseudo-vertex. The markers (left and right ones) establish a link between the characteristics of the Hamilton–Jacobi equation and the geometry of the boundary set. For the markers, a relation with the structure of the equation at a fixed point is obtained. An iterative procedure for calculating a solution based on Newton's method is proposed. The convergence of the procedure to the pseudo-vertex marker is proved. An example of constructing a minimax solution is given, demonstrating the effectiveness of the developed approaches for solving nonsmooth boundary value problems.

Keywords: Hamilton–Jacobi equation, Minimax solution, Speed-in-action, Singular set, Wavefront, Diffeomorphism, Eikonal, Pseudo-vertex.

1. Introduction

Hamilton–Jacobi equations describe and simulate processes in mechanics, optimal control theory, differential games, geometric optics, economics, and other branches of science and their applications. In the most general case, the solutions to these equations are understood as generalized (see [5, 18]). The present research follows the minimax solution concept [18], which originates from the theory of positional differential games [10]. The constructions of the theory of singularities of smooth mappings are also involved [2]; in particular, the jet technique is used for localizing singularities [4].

The paper examines the problem of identifying and constructing singularities of the minimax solution to the Hamilton–Jacobi equation. The solution has the meaning of the optimal result function in the corresponding time-optimal control problem with a nonconvex boundary set. This property of the boundary set geometry causes a violation of the smoothness of the minimax solution. It should be noted that the problem under study has much in common with the problem of violating the smoothness of the evolution of wavefronts in problems of geometric optics (for more details, see, for example, [21]). The connection of the Hamilton–Jacobi equations with the eikonal equation, the basic equation in geometric optics, as well as the eikonal structure for some special cases are clarified in the renowned paper [11].

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It is possible to construct an analytical minimax solution to the problem under consideration only in simplified or model situations. In general, it is required to develop both theory and numerical procedures aimed at identifying and constructing a singular solution set. In the planar problem under study, the pseudo-vertices of the boundary set responsible for generating the branches of the singular set are often defined by analytical methods. At the same time, these branches are usually constructed using numerical algorithms, whose implementation requires finding certain numerical markers of pseudo-vertices and local diffeomorphisms generating these pseudo-vertices (for more details, see [21, 22]). It is important to note that the branches of a singular set are determined by pseudo-vertices with various differential properties. In this paper, one of the most interesting cases for the boundary of such a set is considered. The boundary has a classical curvature. However, there are points at which the smoothness of the boundary's curvature is violated. The branches of the singular set of the minimax solution correspond to these points under certain conditions. The main theoretical result of the study is the derivation of the equation for the pseudo-vertex marker. The equation structure is that of a fixed-point equation, while its right-hand side is a rational function, which is the ratio of cubic polynomials. The differentiability of the right-hand side of the equation enables the use of the classical Newton's method for its numerical solution. An iterative scheme of Newton's method is proposed. Moreover, the initial point is localized, which ensures the convergence of the procedure to the pseudo-vertex marker.

The obtained results are illustrated by an example of constructing a minimax solution to the Dirichlet boundary value problem, which represents the optimal result function in the corresponding time-optimal control problem.

2. Main focus of study and basic concepts

The Dirichlet problem is considered for the Hamilton–Jacobi equation:

$$\min_{\nu\colon \|\nu\|\leqslant 1}\left(\nu_1\frac{\partial u}{\partial x_1}+\nu_2\frac{\partial u}{\partial x_2}\right)+1=0,\quad u|_\Gamma=0. \tag{2.1}$$

Here, $\|\nu\|=\sqrt{\nu_1^2+\nu_2^2}$ is the Euclidean norm of the vector $\nu=(\nu_1,\nu_2)$. The boundary condition in (2.1) is imposed on the boundary $\Gamma=\partial A$ of a closed set $A\subset\mathbb{R}^2$. The curve Γ has no self-intersection points. The minimax solution to problem (2.1) is known to be (see [13])

$$u(x)=\rho\left(x,A\right),$$

where

$$\rho\left(x,A\right)=\inf_{a\in A}\|x-a\|$$

is the Euclidean distance from the point $x=(x_1,x_2)$ to the set A.

Problem (2.1) is associated with a speed-in-action problem with a spherical velocity vectogram of unit radius centered at the origin and a boundary set A. According to R. Isaacs's classification, the differential equation in (2.1) is the Bellman–Isaacs equation [9] for the speed-in-action problem under consideration. The restriction of the optimal result function to the complement of the set A in the plane coincides with the generalized (minimax) solution to problem (2.1) (see [13]).

It should be emphasized that the differential properties of Γ significantly affect the singular set of the generalized solution to problem (2.1). We define a class of curves to which the boundary set belongs. Let $\gamma\colon T\to\mathbb{R}^2$ be a continuous mapping from a numerical interval $T=(\hat{t},\check{t})$, where $-\infty\leqslant\hat{t}<\check{t}\leqslant+\infty$, onto the plane. The vector function $\gamma(t)=\big(\gamma_1(t),\gamma_2(t)\big)$ is twice continuously differentiable everywhere on T. The image $\Gamma=\{\gamma(t)\colon t\in T\}$ of the mapping is a plane curve. Curves defined on finite intervals $-\infty<\hat{t}<\check{t}<+\infty$ and allowing extension to the end points $t=\hat{t}$ and $t=\check{t}$ so that $\gamma(\hat{t})=\gamma(\check{t})$ are known as contours and will also be considered.

Denote by $\det(a,b)$ the second-order determinant based on the vectors $a=(a_1,a_2)$ and $b=(b_1,b_2)$ written line by line. Denote by $\langle a,b\rangle$ the dot product of these vectors. Henceforth, we assume that the following conditions are met:

(Γ1) $\gamma(t)=\big(\gamma_1(t),\gamma_2(t)\big)$ is a twice continuously differentiable function everywhere on T;
(Γ2) $\forall t\in T \quad \gamma'(t)\neq(0,0)$;
(Γ3) there exists a finite set $T^0\subset T$ of points $t_0\in T^0$, where the one-sided third-order derivatives are finite and at least one of the equalities

$$\gamma_1'''(t_0-0)=\gamma_1'''(t_0+0)\,,\quad \gamma_2'''(t_0-0)=\gamma_2'''(t_0+0)$$

is violated;
(Γ4) $\forall t\in T \quad \det\big(\gamma'(t),\gamma''(t)\big)\neq 0$.

Let $\{\Gamma\}_T$ denote the set of curves Γ which have no self-intersection points and satisfy the differential properties (Γ1)–(Γ4).

We now define the key constructive elements of the developed approach.

Let us introduce an equation for two parameters $t_1\in T$ and $t_2\in T$ that correspond to two arbitrary points of the curve Γ:

$$G(t_1,t_2)\triangleq\frac{\gamma_2(t_2)-\gamma_2(t_1)}{\gamma_1(t_2)-\gamma_1(t_1)}-\frac{-\gamma_1'(t_1)\gamma_1'(t_2)+\gamma_2'(t_1)\gamma_2'(t_2)+s(t_1)s(t_2)}{\gamma_2'(t_1)\gamma_1'(t_2)+\gamma_2'(t_2)\gamma_1'(t_1)}=0. \tag{2.2}$$

The equation (here $s(t)=\|\gamma'(t)\|$) results from reducing the basic equation of the golden ratio, which plays a key role in identifying pseudo-vertices of the curve (for details, see [20]). Equation (2.2) is introduced to reparametrize the curve. A distinctive feature of the equation is that its left-hand side is defined by a symmetric function. Therefore, the equation always has the identity function as a solution, which is trivial. When studying the equation, it is necessary to exclude (remove, isolate) this particular solution and find other solutions that are related to the reparametrization of the curve and determine its singular points. Solutions to the equation in the class of diffeomorphisms with strictly negative derivatives generate pseudo-vertices. To clarify the concepts, let us fix a point $t_0\in T$ and smallness parameters $\delta_1>0$ and $\delta_2>0$. The solutions to the equation are understood as local diffeomorphisms [4] defined on one side of the point under consideration. We say that a local diffeomorphism $t_2=t_2(t_1)$ defined by equation (2.2) is left-semicontinuous at the point $t_1=t_0$ and maps the left semi-neighborhood of $t_1=t_0$ to its right semi-neighborhood if the following conditions are met:

($A1$) $t_2\big((t_0-\delta_1,t_0)\big)=(t_0,t_0+\delta_2),\quad \delta_1>0,\quad \delta_2>0$;
($A2$) $\lim\limits_{t_1\to t_0-0}t_2(t_1)=t_0$.

Definition 1 [22]. *A pseudo-vertex of a curve* Γ *is a point*

$$x^{(0)}=\big(\gamma_1(t_0),\gamma_2(t_0)\big)\triangleq\lim_{t_1\to t_0-0}(x_1^*,x_2^*)\,,$$

where $(x_1^*,x_2^*)=\big(x_1^*(t_1,t_2(t_1)),x_2^*(t_1,t_2(t_1))\big)$ *is a one-parametric subset of solutions* $(x_1^*,x_2^*)=\big(x_1^*(t_1,t_2),x_2^*(t_1,t_2)\big)$ *to the system of equations*

$$\begin{cases}\big(x_1^*-\gamma_1(t_1)\big)\gamma_2'(t_1)=\big(x_2^*-\gamma_2(t_1)\big)\gamma_1'(t_1),\\ \big(x_1^*-\gamma_1(t_2)\big)\gamma_2'(t_2)=\big(x_2^*-\gamma_2(t_2)\big)\gamma_1'(t_2),\end{cases}$$

defined by a local diffeomorphism $t_2=t_2(t_1)$ *given by equation* (2.2) *and meeting conditions* ($A1$) *and* ($A2$).

Let us use the operator $P_A(x)$ to project points $x \in \mathbb{R}^2 \setminus A$ onto A. The solution to equation (2.1) is smooth over the entire analyzed domain in the case of single-element values $P_A(x)$, $x \in \mathbb{R}^2 \setminus A$, i.e., when $\operatorname{card} P_A(x) = 1$ [1]. This case is realized under convexity of the boundary set A. Then, A is a "sun" [7]. In this paper, we study the general case where the set A is not a "sun". Here, A is a nonconvex set, i.e., there may exist points $x \in \mathbb{R}^2 \setminus A$ for which $\operatorname{card} P_A(x) > 1$.

Definition 2 [22]. *A bisector of a set $A \subset \mathbb{R}^2$ is*

$$L = \left\{ x \in \mathbb{R}^2 \setminus A \colon \operatorname{card} P_A(x) > 1 \right\}.$$

In this definition, $\operatorname{card} P_A(x)$ denotes the cardinality of the set $P_A(x)$.

The bisector L is a singular set of the minimax solution to problem (2.1) and is related to symmetry sets whose topological properties are under study, among other things, in the theory of singularities of smooth mappings (see, e.g., [17]).

Definition 3 [22]. *A branch $L\left(x^{(0)}\right)$ of the bisector L of a curve Γ is the set of points satisfying the system of equations*

$$\begin{cases} \big(x_1 - \gamma_1(t_1)\big)\gamma_2'(t_1) + \big(x_2 - \gamma_2(t_1)\big)\gamma_1'(t_1) = 0, \\ \big(x_1 - \gamma_1(t_2)\big)\gamma_2'(t_2) + \big(x_2 - \gamma_2(t_2)\big)\gamma_1'(t_2) = 0, \end{cases}$$

where the point $(x_1, x_2) \in \mathbb{R}^2$ is a pseudo-vertex of Γ and $t_2 = t_2(t_1)$ is a local diffeomorphism that is left-continuous at the point $t_1 = t_0$, mapping the left half-neighbourhood of $t_1 = t_0$ to its right half-neighbourhood defined by equation (2.2), and satisfying conditions (A1) and (A2).

The bisector of a closed set in Euclidean space divides the complement of the set under consideration into subsets, within each of which the Euclidean distance to the set is a differentiable function (see [6, Ch. 2, Sect. 8]). The concept introduced here, in the presence of several nearest points, is close to the concepts generalizing the definition of a bisector in [8, 16]. It should be noted that Definition 3 is aimed at isolating a single set. In those papers, generalized notions of a bisector are applied to two or more sets.

Pseudo-vertices and bisector branches are the key structural elements for constructing — analytically, numerically, or in a combined manner — the singular set of problem (2.1). Here, it is appropriate to emphasize the connection of these elements with the concept of an alpha-set, which is one of the well-known generalized concepts of a convex set. Pseudo-vertices are special points on the boundary of a boundary set associated with the characterization of a set in terms of its nonconvexity (alpha value). The bisector is a set where this value is attained [19].

Let us introduce scalar characteristics of a pseudo-vertex.

Definition 4 [22]. *The left one-sided derivative*

$$\lambda \triangleq t_2'(t_0 - 0) = \lim_{t_1 \to t_0 - 0} \frac{t_2(t_1) - t_0}{t_1 - t_0}$$

is the left marker of a pseudo-vertex $x^{(0)} \in \Gamma$, where $t_2 = t_2(t_1)$ is the local diffeomorphism generating the pseudo-vertex $x^{(0)}$.

The range of possible values (spectrum) of one-sided markers is the closure of the negative half-line $\Lambda = (-\infty, 0]$. The markers take their corresponding values from Λ, depending on the order of the curve's smoothness at the pseudo-vertex. Determining the markers makes it possible to construct branches of a singular set either in accordance with Definition 3 or by using integral curves (see [15]).

3. Main theoretical result

The case of a nonstationary pseudo-vertex is described below. Moreover, it is shown that its marker is a fixed point of a rational function.

Theorem 1. *Let* $x^{(0)} = \big(\gamma_1(t_0), \gamma_2(t_0)\big)$ *be a pseudo-vertex of the curve* $\Gamma \subset \Gamma_T$ *in the Dirichlet problem* (2.1). *Suppose that* $x^{(0)}$ *is defined by a local diffeomorphism* $t_2 = t_2(t_1)$ *in* (2.2), *conditions* (Γ1)–(Γ4) *hold, and the coordinate functions of the pseudo-vertex satisfy the nonstationarity conditions*:

$$\gamma_1'(t_0) \neq 0, \quad \gamma_2'(t_0) \neq 0. \tag{3.1}$$

Then, if there exists a finite left marker $\lambda = t_2'(t_0 - 0) \leqslant 0$, *it satisfies the equality*

$$\lambda = \frac{-\lambda^3 k_+' + \big(3\lambda^2 - 3\lambda + 1\big) k_-'}{-\left(\lambda^3 - 3\lambda^2 + 3\lambda\right) k_+' + k_-'}, \tag{3.2}$$

where

$$k_\pm' = \det\big(\gamma', \gamma_\pm'''\big) - \frac{3\,\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}$$

are the one-sided (*left* k_-' *and right* k_+') *derivatives of the curvature calculated at* $t = t_0 \in T^0$.

Proof. The proof of (3.2) is substantially based on the constructions and results in [22].

According to the conditions of Theorem 1, there exists a finite left marker. It can be determined as follows [22]:

$$\lambda = -\lim_{t_1 \to t_0 - 0} \left(\frac{\partial G(t_1, t_2(t_1))}{\partial t_1} \cdot \left(\frac{\partial G(t_1, t_2(t_1))}{\partial t_2} \right)^{-1} \right). \tag{3.3}$$

Equality (3.3) expresses the condition for a transverse intersection at the closure of the graph of a local diffeomorphism $t_2 = t_2(t_1)$ with the graph of the identity diffeomorphism $t_2 = t_1$ at a common limit point $(t_1, t_2) = (t_0, t_0)$.

The technique of local expansions of functions (the jet technique) made it possible to obtain linear partial differential asymptotics in (3.3) calculated along the graph of the local diffeomorphism to the left of $t_0 \in \mathbb{R}$:

$$\frac{\partial G\big(t_1, t_2(t_1)\big)}{\partial t_1} \sim C_1(t_0 - t_1), \quad C_1 = \text{const}, \quad t_1 \to t_0 - 0, \tag{3.4}$$

$$\frac{\partial G\big(t_1, t_2(t_1)\big)}{\partial t_2} \sim C_2(t_0 - t_1), \quad C_2 = \text{const}, \quad t_1 \to t_0 - 0. \tag{3.5}$$

Here, the numerical coefficients are given by [22, Eqs. (29), (30)]:

$$C_1 = \bigg(-\gamma_2'' \det(\gamma', \gamma'') + \frac{-\lambda^3}{3(1-\lambda)^2}\gamma_2' \det(\gamma', \gamma_+''') + \frac{3\lambda^2 - 3\lambda + 1}{3(1-\lambda)^2}\gamma_2' \cdot \det(\gamma', \gamma_-''')$$
$$+\gamma_1' \frac{(\det(\gamma', \gamma''))^2}{s^2} + \lambda \frac{\gamma_2' \langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2} \bigg) / (2(\gamma_1')^2 \gamma_2'),$$

$$C_2 = \bigg(-\lambda\gamma_2'' \det(\gamma', \gamma'') - \frac{1}{3(1-\lambda)^2}\gamma_2' \det(\gamma', \gamma_-''') + \frac{\lambda^3 - 3\lambda^2 + 3\lambda}{3(1-\lambda)^2}\gamma_2' \cdot \det(\gamma', \gamma_+''')$$
$$+\lambda\gamma_1' \frac{(\det(\gamma', \gamma''))^2}{s^2} + \frac{\gamma_2' \langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2} \bigg) / (2(\gamma_1')^2 \gamma_2').$$

Transforming (3.3) and taking into account (3.1), (3.4), and (3.5) yields:

$$\begin{aligned}\lambda = -\Big(& -\gamma_2'' \det(\gamma', \gamma'') + \frac{-\lambda^3}{3(1-\lambda)^2}\gamma_2' \det(\gamma', \gamma_+''') + \frac{3\lambda^2 - 3\lambda + 1}{3(1-\lambda)^2}\gamma_2' \cdot \det(\gamma', \gamma_-''') \\ & +\lambda\gamma_1' \frac{\big(\det(\gamma', \gamma'')\big)^2}{s^2} + \frac{\gamma_2'\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}\Big) \\ /\Big(& -\lambda\gamma_2'' \det(\gamma', \gamma'') - \frac{1}{3(1-\lambda)^2}\gamma_2' \det(\gamma', \gamma_-''') + \frac{\lambda^3 - 3\lambda^2 + 3\lambda}{3(1-\lambda)^2}\gamma_2' \cdot \det(\gamma', \gamma_+''') \\ & +\lambda\gamma_1' \frac{\big(\det(\gamma', \gamma'')\big)^2}{s^2} + \frac{\gamma_2'\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}\Big).\end{aligned}$$

It should be noted that

$$\gamma_2'' \det(\gamma', \gamma'') - \gamma_1' \frac{(\det(\gamma', \gamma''))^2}{s^2} = \det(\gamma', \gamma'')\Big(\gamma_2'' - \gamma_1' \frac{\det(\gamma', \gamma'')}{s^2}\Big),$$

$$\det(\gamma', \gamma'')\frac{\gamma_2''(\gamma_2')^2 + \gamma_1'\gamma_2'\gamma_1''}{s^2} = \frac{\gamma_2' \det(\gamma', \gamma'')\langle\gamma', \gamma''\rangle}{s^2}.$$

After simplification, we obtain:

$$\begin{aligned}\lambda = -\Big(& \frac{\lambda^3}{3(1-\lambda)^2} \det(\gamma', \gamma_+''') + \frac{-3\lambda^2 + 3\lambda - 1}{3(1-\lambda)^2} \det(\gamma', \gamma_-''') + (1-\lambda)\frac{\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}\Big) \\ /\Big(& \frac{1}{3(1-\lambda)^2} \det(\gamma', \gamma_-'') - \frac{\lambda^3 - 3\lambda^2 + 3\lambda}{3(1-\lambda)^2} \det(\gamma', \gamma_+''') + (\lambda - 1)\frac{\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}\Big).\end{aligned}$$

Performing algebraic transformations yields:

$$\begin{aligned}\lambda = \Big(& -\frac{\lambda^3}{3(1-\lambda)^2} \det(\gamma', \gamma_+''') - \frac{-3\lambda^2 + 3\lambda - 1}{3(1-\lambda)^2} \det(\gamma', \gamma_-''') + (\lambda - 1)\frac{\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}\Big) \\ /\Big(& \frac{1}{3(1-\lambda)^2} \det(\gamma', \gamma_-'') - \frac{\lambda^3 - 3\lambda^2 + 3\lambda}{3(1-\lambda)^2} \det(\gamma', \gamma_+''') + (\lambda - 1)\frac{\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}\Big),\end{aligned}$$

$$\begin{aligned}\lambda = \Big(& -\frac{\lambda^3}{(1-\lambda)^3} \det(\gamma', \gamma_+''') - \frac{-3\lambda^2 + 3\lambda - 1}{(1-\lambda)^3} \det(\gamma', \gamma_-''') - \frac{3\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}\Big) \\ /\Big(& -\frac{\lambda^3 - 3\lambda^2 + 3\lambda}{(1-\lambda)^3} \det(\gamma', \gamma_+''') + \frac{1}{(1-\lambda)^3} \det(\gamma', \gamma_-'') - \frac{3\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}\Big),\end{aligned}$$

$$\begin{aligned}\lambda = \Big(& -\frac{\lambda^3}{t(1-\lambda)^3}\Big(\det(\gamma', \gamma_+''') - \frac{3\langle\gamma', \gamma'\rangle \det(\gamma', \gamma'')}{s^2}\Big) \\ & +\frac{3\lambda^2 - 3\lambda + 1}{(1-\lambda)^3}\Big(\det(\gamma', \gamma_-''') - \frac{3\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}\Big)\Big) \\ /\Big(& -\frac{\lambda^3 - 3\lambda^2 + 3\lambda}{(1-\lambda)^3}\Big(\det(\gamma', \gamma_+''') - \frac{3\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}\Big) \\ & +\frac{1}{(1-\lambda)^3}\Big(\det(\gamma', \gamma_-''') - \frac{3\langle\gamma', \gamma''\rangle \det(\gamma', \gamma'')}{s^2}\Big)\Big),\end{aligned}$$

$$\lambda = \Big(-\frac{\lambda^3}{(1-\lambda)^3}k_+' + \frac{3\lambda^2 - 3\lambda + 1}{(1-\lambda)^3}k_-'\Big)/\Big(-\frac{\lambda^3 - 3\lambda^2 + 3\lambda}{t(1-\lambda)^3}k_+' + \frac{1}{(1-\lambda)^3}k_-'\Big).$$

Approximating both the numerator and denominator by $(1-\lambda)^{-3} \neq 0$, we finally obtain (3.2). □

4. The left marker as a fixed point of a smooth map

Let us investigate the problem of approximate marker calculation based on (3.2).

Define

$$\varphi\left(\lambda\right) = \frac{-\lambda^3 k'_+ + \left(3\lambda^2 - 3\lambda + 1\right) k'_-}{-\left(\lambda^3 - 3\lambda^2 + 3\lambda\right) k'_+ + k'_-}$$

and consider (3.2) as an equation for λ:

$$\lambda = \varphi\left(\lambda\right). \tag{4.1}$$

We define the left marker as a fixed point in (4.1) on the spectrum Λ [12]. We restrict ourselves to the case where the one-sided derivatives of curvature are strictly separated from zero and have different signs:

$$\sigma = \frac{k'_+}{k'_-} < 0. \tag{4.2}$$

Essentially, this corresponds to the case where a pseudo-vertex is a point of a nonsmooth extremum of the curvature of the boundary set.

Proposition 1. *If condition* (4.2) *holds, then equation* (4.1) *has a unique solution on the set* Λ.

Proof. When approximating the numerator

$$\varphi_1(\lambda) = (1-\lambda)^3 + \lambda^3(1-\sigma)$$

and the denominator

$$\varphi_2(\lambda) = -\sigma(\lambda-1)^3 - \sigma + 1$$

of $\varphi(\lambda)$, we obtain

$$\varphi(\lambda) = \frac{(1-\lambda)^3 + \lambda^3(1-\sigma)}{-\sigma(\lambda-1)^3 + 1 - \sigma}.$$

By direct calculation, it is easy to verify that under condition (4.2), each of the cubic polynomials in the numerator and the denominator has only one real root.

Here,

$$\overline{\lambda} = -\frac{1}{\sqrt[3]{1-\sigma} - 1}$$

is the real root of $\varphi_1(\lambda)$ and

$$\widetilde{\lambda} = 1 + \sqrt[3]{1/\sigma - 1}$$

is the real root of $\varphi_2(\lambda)$. Since $\overline{\lambda} < 0$ and $\widetilde{\lambda} < 0$, both roots belong to Λ, and $\widetilde{\lambda}$ is the break point of the function $\varphi(\lambda)$, where

$$\overline{\lambda} = -\frac{1}{\sqrt[3]{1-\sigma} - 1} < \widetilde{\lambda} = 1 + \sqrt[3]{\frac{1}{\sigma} - 1}.$$

This induces a partition of the spectrum:

$$\Lambda = \Lambda_1 \cup \{\widetilde{\lambda}\} \cup \Lambda_2 \cup \{-\infty\}, \quad \text{where} \quad \Lambda_1 = (-\infty, \widetilde{\lambda}), \quad \Lambda_2 = (\widetilde{\lambda}, 0].$$

Differentiating, we obtain

$$\varphi'(\lambda) = 3(1-\sigma)\frac{-(\lambda-1)^2 - \sigma\lambda^2(\lambda-1)^3 + \lambda^2(1-\sigma) + \sigma\lambda^3(\lambda-1)^2}{(-\sigma(\lambda-1)^3 + 1 - \sigma)^2}.$$

A simplification yields

$$\varphi'(\lambda) = 3(1-\sigma)\frac{2\lambda - 1 + \sigma\lambda^2((\lambda-1)^2-1)}{(-\sigma(\lambda-1)^3+1-\sigma)^2} < 0, \quad \lambda \in \Lambda_1 \cup \Lambda_2. \tag{4.3}$$

We now investigate the existence of a solution to (4.1) in each of the two selected spectral intervals. On the set

$$\Lambda_2 = (\widetilde{\lambda}, 0] = \left(1 + \sqrt[3]{1/\sigma - 1}, 0\right] \subset \Lambda,$$

the function $\varphi(\lambda)$ is strictly decreasing, taking values from the half-interval $[1,+\infty)$. Since

$$[1,+\infty) \cap (\widetilde{\lambda}, 0] = \varnothing,$$

equation (4.1) has no solution on $\Lambda_2 = (\widetilde{\lambda}, 0]$. Therefore, a solution to (4.1) on Λ can exist only on its subset

$$\Lambda_1 = (-\infty, \widetilde{\lambda}) = \left(-\infty, 1 + \sqrt[3]{1/\sigma - 1}\right).$$

Indeed, the function $\varphi(\lambda)$ takes values from the interval $(-\infty, 1)$, i.e., it attains all possible values from the spectrum Λ except for the improper value $-\infty$. Moreover,

$$\varphi(\overline{\lambda}) = 0 > \lim_{\lambda \to \widetilde{\lambda} - 0} \varphi(\lambda) = -\infty, \quad \overline{\lambda} < \widetilde{\lambda} < 0. \tag{4.4}$$

Define $\Phi(\lambda) = \varphi(\lambda) - \lambda$. The problem of the existence of a fixed point for (4.1) is equivalent to the problem of the existence of a root of the function on Λ_1. Then,

$$\Phi(\lambda) = \frac{(1-\lambda)^3 + \lambda^3(1-\sigma)}{-\sigma(\lambda-1)^3+1-\sigma} - \lambda = (\lambda - 1)\frac{\sigma\lambda^3 - 3\sigma\lambda^2 + 3\lambda - 1}{-\sigma(\lambda-1)^3+1-\sigma}.$$

Note that, according to (4.3),

$$\Phi'(\lambda) = \varphi'(\lambda) - 1 < 0, \quad \lambda \in \Lambda_1 \subset \Lambda. \tag{4.5}$$

Therefore, the function $\Phi(\lambda) = \varphi(\lambda) - \lambda$ is strictly decreasing on Λ_1. From (4.4) and the form of the function, it follows that

$$\Phi\left(\overline{\lambda}\right) = -\overline{\lambda} > 0, \qquad \lim_{\lambda \to \widetilde{\lambda} - 0} \Phi(\lambda) = -\infty.$$

Then, by the classical intermediate value theorem for continuous functions, there exists $\lambda_* \in \Lambda_1$ such that $\Phi(\lambda_*) = 0$, or equivalently, $\lambda_* = \varphi(\lambda_*)$. Therefore, on the interval $\Lambda_1 \subset \Lambda$ equation (4.1) has a solution, which is unique due to the strict monotonicity of

$$\Phi(\lambda) = \varphi(\lambda) - \lambda.$$

□

To find a numerical solution to equation (4.1), we use the iterative formula of the classical Newton's method [3]:

$$\lambda_{n+1} = \lambda_n - \frac{\Phi(\lambda_n)}{\Phi'(\lambda_n)}, \quad n = 0, 1, 2, \ldots \tag{4.6}$$

We now calculate the derivative

$$\Phi'(\lambda) = 3(1-\sigma)\frac{2\lambda - 1 + \sigma\lambda^2((\lambda-1)^2-1)}{(-\sigma(\lambda-1)^3+1-\sigma)^2} - 1$$

$$= \frac{3(1-\sigma)(2\lambda - 1 + \sigma\lambda^2((\lambda-1)^2-1)) - \sigma^2(\lambda-1)^6 - (1-\sigma)^2 + 2\sigma(1-\sigma)(\lambda-1)^3}{\left(-\sigma(\lambda-1)^3+1-\sigma\right)^2}.$$

We define the initial approximation $\lambda_0 \in \Lambda_1$. To this end, we find the point of the local minimum of the polynomial

$$P_3(\lambda) = \sigma\lambda^3 - 3\sigma\lambda^2 + 3\lambda - 1$$

in the numerator of $\Phi(\lambda)$. This point is

$$\breve{\lambda} = 1 - \sqrt{1 - \frac{1}{\sigma}},$$

which is the negative zero of the derivative

$$P_3'(\lambda) = 3\sigma\lambda^2 - 6\sigma\lambda + 3.$$

Then

$$\Phi(\breve{\lambda}) = -2\sqrt{\frac{\sigma - 1}{\sigma}} < 0.$$

Since

$$\breve{\lambda} = 1 - \sqrt{1 - \frac{1}{\sigma}} < \widetilde{\lambda} = 1 + \sqrt[3]{\frac{1}{\sigma} - 1} = 1 - \sqrt[3]{1 - \frac{1}{\sigma}},$$

it follows that $\breve{\lambda} \in \Lambda_1$. As a result, the function takes values of opposite signs:

$$\Phi(\overline{\lambda}) = -\overline{\lambda} > 0, \quad \Phi(\breve{\lambda}) = -2\sqrt{\frac{\sigma - 1}{\sigma}} < 0.$$

Thus, we have found an interval that localizes the solution to (4.1). Specifically,

$$\lambda_* \in \left[\overline{\lambda}, \breve{\lambda}\right] = \left[-\frac{1}{\sqrt[3]{1-\sigma} - 1}, 1 - \sqrt{1 - 1/\sigma}\right].$$

Let the initial approximation be

$$\lambda_0 = \breve{\lambda} = 1 - \sqrt{1 - \frac{1}{\sigma}}.$$

Then,

$$\frac{\Phi(\lambda_0)}{\Phi'(\lambda_0)} > 0,$$

and according to (4.6), we obtain $\lambda_1 < \lambda_0$, $\lambda_1 \in \Lambda_1$ in the next iteration. Due to (4.5) and (4.6), each subsequent term of the sequence generated by rule (4.6) is strictly less than the previous one. This results in a bounded, negative, and monotone numerical sequence λ_n, $n = 0, 1, 2, \ldots$, which converges to the root of the function $\Phi(\lambda)$. Therefore,

$$\lim_{n\to\infty} \lambda_n = \widehat{\lambda} \in \Lambda_1.$$

Since $\Phi(\lambda)$ has a unique root on the set Λ, the left marker is $\lambda_* = \widehat{\lambda}$.

Transforming formula (4.5), we calculate

$$\begin{aligned}\frac{\Phi(\lambda)}{\Phi'(\lambda)} &= (\lambda - 1)\frac{(\sigma\lambda - 1)(\lambda - 1)^2 + (1 - \sigma)(\lambda^2 + \lambda)}{-\sigma(\lambda - 1)^3 + 1 - \sigma} \\ &\cdot \left(\frac{3(1-\sigma)\big(2\lambda - 1 + \sigma\lambda^2((\lambda-1)^2 - 1)\big) - \big(-\sigma(\lambda-1)^3 + 1 - \sigma\big)^2}{\big(-\sigma(\lambda-1)^3 + 1 - \sigma\big)^2}\right)^{-1} \\ &= (\lambda - 1)\frac{(\sigma\lambda^3 - 3\sigma\lambda^2 + 3\lambda - 1)\big(1 - \sigma\lambda(\lambda^2 - 3\lambda + 3)\big)}{3(1-\sigma)\big(2\lambda - 1 + \sigma\lambda^3(\lambda - 2)\big) - \big(1 - \sigma\lambda(\lambda^2 - 3\lambda + 3)\big)^2}.\end{aligned}$$

As a result, the iterative process for approximating the left marker is as follows:

$$\begin{gathered}\lambda_{n+1}=\lambda_n-(\lambda_n-1)\cdot\frac{(\sigma\lambda_n^3-3\sigma\lambda_n^2+3\lambda_n-1)\big(1-\sigma\lambda_n(\lambda_n^2-3\lambda_n+3)\big)}{3(1-\sigma)\big(2\lambda_n-1+\sigma\lambda_n^3(\lambda_n-2)\big)-\big(1-\sigma\lambda_n(\lambda_n^2-3\lambda_n+3)\big)^2},\\ n=0,1,2,\dots,\\ \lambda_0=1-\sqrt{1-1/\sigma}.\end{gathered}\tag{4.7}$$

5. Numerical and analytical simulation of a generalized solution

Example 1. As an example, we consider the Dirichlet problem (2.1) for a closed unbounded convex set $A\subset\mathbb{R}^2$ with a twice smooth boundary

$$\Gamma=\big\{\gamma(t)\in\mathbb{R}^2\colon\gamma(t)=(\gamma_1(t),\gamma_2(t)),\ t\in T\big\},$$

where $T=\mathbb{R}$,

$$\gamma_1(t)=t,\tag{5.1}$$

$$\gamma_2(t)=\begin{cases}\dfrac{t^3+t^2+2t+2}{2}, & t\in(-\infty,0],\\ e^t, & t\in(0,+\infty).\end{cases}\tag{5.2}$$

The curve $\Gamma=\partial A$ has the unique pseudo-vertex $x^{(0)}=(\gamma_1(t_0),\gamma_2(t_0))$ corresponding to the parameter $t_0=0$. Here, $T^0=\{0\}$. It is easy to show that the smoothness of the curvature of the curve is violated at this point, since

$$\gamma_{2,-}'''(t_0)=3\neq\gamma_{2,+}'''(t_0)=1.$$

In this case, the pseudo-vertex is not a stationary point in either coordinates:

$$\gamma_1'(t_0)=1\neq0,\quad\gamma_2'(t_0)=1\neq0.$$

We calculate the ratio of one-sided curvatures at the pseudo-vertex and compare it to zero:

$$\sigma=\frac{k_+'}{k_-'}=\frac{\gamma_2'''(t_0+0)\big(1+(\gamma_2'(t_0))^2\big)-3(\gamma_2''(t_0))^2\gamma_2'(t_0)}{\gamma_2'''(t_0-0)\big(1+(\gamma_2'(t_0))^2\big)-3(\gamma_2''(t_0))^2\gamma_2'(t_0)}=-\frac{1}{3}<0.$$

Thus, all assumptions of Theorem 1 are satisfied. Moreover, condition (4.2) holds. To find an approximate value of the left marker, we apply the iterative procedure (4.7) with the initial approximation

$$\lambda_0=1-\sqrt{1-\sigma^{-1}}=1-\sqrt{1-(-1/3)^{-1}}=-1.$$

Sequentially, we obtain the following values in the table. The Table 1 contains the iteration index in the first column, the current value of the root approximation in the second column, and the change percentage from the previous step in the third column.

Let the fifth term of the sequence, $\lambda\approx\lambda_5=-2.0642$, serve as an approximate value of the marker λ. This marker λ at the pseudo-vertex $x^{(0)}$ is unique according to Proposition 1.

The unique pseudo-vertex of the boundary condition corresponds to a single branch $L(x^{(0)})$ of the singular set. To construct this branch, it is crucial to determine the diffeomorphism $t_2=t_2(t_1)$. In general, finding such functions analytically is possible only in rare cases. Therefore, further constructions require the use of computational procedures (see, e.g., [21]).

Table 1. Iterative calculation of the marker λ.

i	λ_i	$\dfrac{\lVert\lambda_i - \lambda_{i-1}\rVert}{\lVert\lambda_{i-1}\rVert} \times 100\%$
0	-1	–
1	-1.4	40%
2	-1.8764	34.03%
3	-2.0543	9.48%
4	-2.0642	0.48%
5	-2.0642	0%

Then, knowing the left marker for the pseudo-vertex and the local diffeomorphism $t_2 = t_2(t_1)$ that generates it, we construct the branches of the singular set by solving the system of equations from Definition 3 (see Fig. 1). It should be noted that another approach can be used to construct the branch of the singular set, which involves integrating an ordinary differential equation. The initial conditions for this equation are determined by the pseudo-vertex markers, while its dynamics are governed by the local diffeomorphisms generating the pseudo-vertex $x^{(0)}$. Fig. 2 shows the curve defined by (5.1) and (5.2), the singular set, and the level lines of the minimax solution to problem (2.1) found by approximate methods. In other words, the nonsmoothness sets and the level line map of the optimal result function are found in the corresponding speed-in-action problem. Fig. 3 provides an approximate graph of the optimal result function generated by the software package [14].

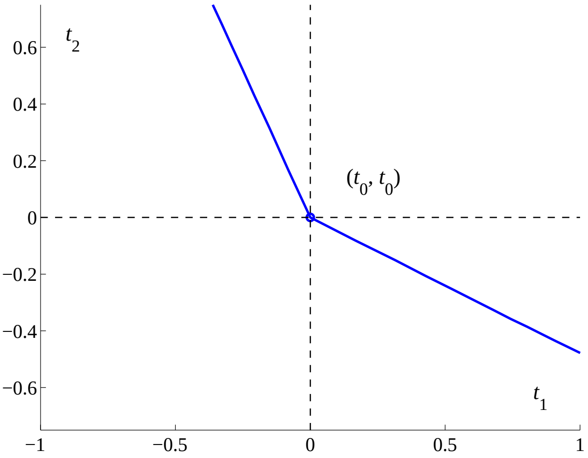

Figure 1. Gluing of the graphs of local diffeomorphisms $t_2 = t_2(t_1)$ and $t_1 = t_1(t_2)$, which determines the pseudo-vertex of the curve Γ at $t_0 = 0$.

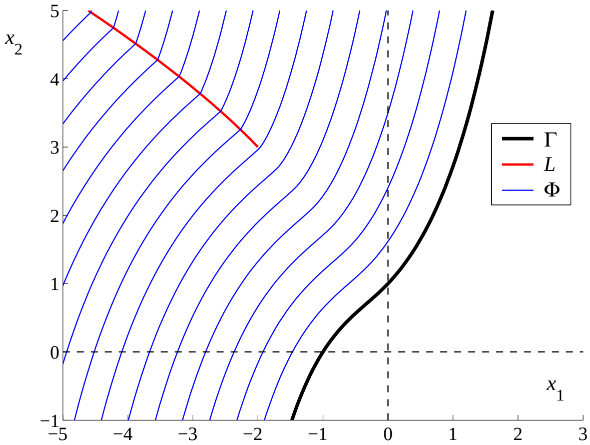

Figure 2. The boundary Γ of the set, the singular set L and the level lines Φ of the minimax solution $u(x_1, x_2)$.

6. Conclusion

Theorem 1 and Proposition 1 allow us to reduce the construction of the bisector of a set to solving a Cauchy problem for a differential equation relating the projection parameters t of points

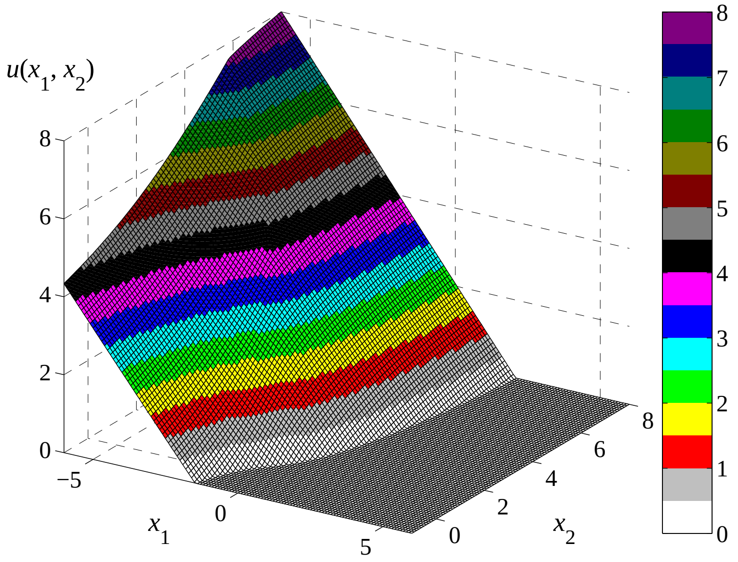

Figure 3. Graph of the minimax solution $u(x_1, x_2)$.

of the set L. In this case, the initial condition in the Cauchy form is specified at a pseudo-vertex as $t_2(t_0) = t_0$, and the limit value of the derivative $t_2'(t_0)$ equals the pseudo-vertex marker λ. The proposed iterative scheme for calculating λ is illustrated by Example 1, in which the boundary condition in problem (2.1) is defined on a parametrically specified curve Γ. A pseudo-vertex is found on this curve at which a discontinuity in the derivative of the curvature $k(t)$ occurs.

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