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APPROXIMATION OF ONE CLASS OF SMOOTH FUNCTIONS BY ANOTHER CLASS OF SMOOTHER FUNCTIONS ON THE AXIS[1]

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Abstract: This paper investigates the problem of best and best linear approximation in the space of functions on the real axis with bounded Fourier transform. The study focuses on approximating the class $B_1^{n-k}(1)$ of functions whose derivatives of order $n-k-1$ have variation bounded by 1 by the class $B_2^n(N)$ of functions whose nth-order derivative $(0\le k<n)$ belongs to the space $L_2(-\infty,\infty)$ with norm bounded by $N>0$. This problem is related to Stechkin's problem and the corresponding sharp Kolmogorov inequality, both previously studied by the author. Stechkin's problem concerns the best approximation in the uniform norm on the real axis of kth-order differentiation operators by bounded linear operators from L_2 to C, considered on the class of functions whose Fourier transform of the nth-order derivative $(0\le k<n)$ is summable.

Keywords: Differentiation operator, Stechkin's problem; Kolmogorov inequality, Approximation of one class of functions by another.

1. Introduction

1.1. Notation and preliminaries

It is well-known that the following four extremal problems for smooth functions in Lebesgue spaces on the real axis (and on the half-axis) are interrelated:

1. Stechkin's problem on the best approximation of the differentiation operator of order k on the class of n-times differentiable functions (where $0\le k<n$);
2. sharp Kolmogorov inequalities for the norm of intermediate derivatives in terms of the norm of the function and the norm of its higher-order derivative;
3. the problem of the best approximation of one class of differentiable functions by another, smoother class;
4. the problem of linear approximation of one class by another; see [1, 3, 6, 9, 22], and the references therein.

In particular, it is known [1] that, under appropriate conditions on the problem parameters, the problem of approximating one class of differentiable functions by another, smoother class is dual to the problem of finding the sharp constant in the Kolmogorov inequality, while the problem of linear approximation of one class by another is dual to Stechkin's problem. These results were obtained

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by the author [1] for classical derivatives in the (classical) Lebesgue norms. The interrelations among these four problems have proven to be an effective method for studying each of them.

The results obtained in these problems are applied in studying the optimal recovery of differentiation operators for functions specified with an error (i.e., the numerical differentiation of approximately given functions). Extensive research has been devoted to these problems; see, for example, [2, 3, 16, 20], and the references therein.

In this paper, we use the standard notation for spaces of complex-valued functions on the real axis: $L_\gamma = L_\gamma(-\infty,\infty)$, where γ is a real number with $1 \le \gamma < \infty$, denotes the Lebesgue space of measurable functions f such that the function $|f|^\gamma$ is summable on the axis. The space L_γ is equipped with the norm

$$\|f\|_\gamma = \|f\|_{L_\gamma} = \Big(\int |f(t)|^\gamma dt\Big)^{1/\gamma}.$$

Hereinafter, the domain of integration will be omitted in integrals over the axis. The space $L_\infty = L_\infty(-\infty,\infty)$ consists of essentially bounded measurable functions on the axis and is equipped with the norm

$$\|f\|_\infty = \|f\|_{L_\infty} = \operatorname{ess\,sup}\big\{|f(t)|\colon t \in (-\infty,\infty)\big\}.$$

Let $C = C(-\infty,\infty)$ be the space of bounded continuous functions on the axis with the uniform norm

$$\|f\|_C = \sup\{|f(t)|\colon t \in (-\infty,\infty)\},$$

and let $C_0 = C_0(-\infty,\infty)$ be the subspace of C consisting of functions that vanish at $\pm\infty$. The symbol V denotes the space of (complex) bounded Borel measures (charges) on $(-\infty,\infty)$. This space will be identified with the set of (complex) functions μ of bounded variation on $(-\infty,\infty)$, whose real and imaginary parts take values at points of discontinuity that lie between their right and left limits. The norm in the space V is given by the total variation $\bigvee \mu = \bigvee_{-\infty}^{\infty} \mu$ of the measure (function) $\mu \in V$.

Hereinafter, for a pair of functions f and g such that fg is summable, the notation

$$\langle f, g\rangle = \int f(t)g(t)\,dt$$

will be used.

The direct and inverse Fourier transforms of functions on the real axis are defined by the formulas

$$\widehat{f}\,(t) = \int e^{-2\pi t\eta i} f(\eta)\,d\eta, \quad \check{g}(t) = \int e^{2\pi t\eta i} g(\eta)\,d\eta = \widehat{g}\,(-t).$$

For these integrals to exist in the classical sense, f and g must be summable. The Fourier transform is a natural and particularly useful operator in the space L_2; its definition and properties can be found, for example, in [23, Ch. 1, Sects. 1 and 2]. In L_2, the Parseval identity holds for the Fourier transform:

$$\|\widehat{f}\,\|_2 = \|f\|_2, \quad f \in L_2. \tag{1.1}$$

Moreover, in L_2, the relation

$$\langle f, \widehat{g}\,\rangle = \langle \widehat{f}, g\rangle, \quad f, g \in L_2,$$

also holds, which is essentially equivalent to the Parseval identity (1.1).

Let $\mathscr{S}$ be the Schwartz space of infinitely differentiable, rapidly decreasing (together with all derivatives) functions on the real axis equipped with the standard topology, and let $\mathscr{S}'$ be the corresponding dual space of generalized functions (see, for example, [23, 21]). The value of a functional $\theta \in \mathscr{S}'$ on a function $\phi \in \mathscr{S}$ will also be denoted by $\langle \theta, \phi \rangle$. The space $\mathscr{S}'$ contains the set $\mathscr{L} = \mathscr{L}(\mathbb{R})$ of measurable, locally summable functions f on $\mathbb{R}$ satisfying the condition

$$\int (1+|t|)^d |f(t)| dt < \infty$$

for some exponent $d = d(f) \in \mathbb{R}$; such functions $f \in \mathscr{L}$ are called slowly growing (or classical) functions. To each function $f \in \mathscr{L}$ there corresponds a functional $f \in \mathscr{S}'$ defined by

$$\langle f, \phi \rangle = \int f(t)\phi(t)dt, \quad \phi \in \mathscr{S}.$$

The convolution $\theta * \phi$ of an element $\theta \in \mathscr{S}'$ and a function $\phi \in \mathscr{S}$ is defined by the function

$$y(\eta) = \langle \theta, \sigma_\eta \phi \rangle,$$

where

$$\sigma_\eta \phi(t) = \phi(\eta - t), \quad t \in (-\infty, \infty).$$

If $\theta \in \mathscr{L}$ is a classical function, then

$$(\theta * \phi)(\eta) = \int \theta(t)\phi(\eta - t)\, dt. \tag{1.2}$$

The Fourier transform $\widehat{\theta}$ of a functional $\theta \in \mathscr{S}'$ is a functional $\widehat{\theta} \in \mathscr{S}'$ defined by

$$\langle \widehat{\theta}, \phi \rangle = \langle \theta, \widehat{\phi} \rangle, \quad \phi \in \mathscr{S}. \tag{1.3}$$

For $m \geq 1$ and $1 < p \leq \infty$, let W_p^m be the set of functions x such that the derivative $x^{(m-1)}$ is locally absolutely continuous on the real axis $\mathbb{R} = (-\infty, \infty)$ and $x^{(m)} \in L_p$. For $p = 1$, the space W_1^m is defined as follows: if $m = 1$, it is the set of functions of bounded variation; if $m > 1$, it is the set of functions x such that $x^{(m-2)}$ is locally absolutely continuous and $x^{(m-1)}$ coincides almost everywhere with a function η of bounded variation. The function η is denoted by $x^{(m-1)}$. For $x \in W_1^m$, it is sometimes customary to write $\|x^{(m)}\|_1$ for the total variation $\bigvee x^{(m-1)}$ of the $(m-1)$th derivative. Finally, let $W_{p,r}^n = W_p^n \cap L_r$ be the intersection of the spaces W_p^n and L_r, and let $Q_{p,r}^n$ be the set of functions $x \in W_{p,r}^n$ satisfying $\|x^{(n)}\|_p \leq 1$.

1.2. Brief historical information

Approximation of one class of functions by another, simpler class is one of the problems in function theory. In connection with the study of Stechkin's problem, a related question arose: the approximation of a class of smooth functions by another class of smoother functions. We now describe this problem precisely.

Let $1 \leq p', r', q' \leq \infty$ and let $0 < m \leq n$ be integers. For $N \in \mathbb{R}$, $N > 0$, let $B_{r'}^n(N)$ be the set of functions $\varphi \in W_{r'}^n$ such that $\|\varphi^{(n)}\|_{r'} \leq N$, and let $B_{q'}^m = B_{q'}^m(1)$. For a function $\psi \in W_{q'}^m$, we set

$$F\big(\psi, B_{r'}^n(N)\big) = \inf\big\{\|\psi - \varphi\|_{p'} \colon \varphi \in B_{r'}^n(N)\big\};$$

this is the approximation of a function $\psi \in W_{q'}^m$ by the set $B_{r'}^n(N)$. Then the quantity

$$F(N) = F(B_{q'}^m, B_{r'}^n(N)) = \sup\big\{F(\psi, B_{r'}^n(N)) \colon \psi \in B_{q'}^m\big\}$$

is the approximation (in the norm of the space $L_{p'}$) of the class $B_{q'}^m$ by the class $B_{r'}^n(N)$.

The problem $F(N)$ and the corresponding linear problem $G(N)$ were studied in [1, 8, 24–26, 28]. In particular, it is known that $F(N)$ is finite if (Yu. N. Subbotin [25]) and only if (see the result of V. N. Gabushin in [8])

$$\frac{m}{r'} + \frac{n-m}{p'} \le \frac{n}{q'}. \tag{1.4}$$

Let $1 \le p, q, r \le \infty$ and $0 \le k < n$. There exists extensive research on sharp inequalities of the form

$$\|x^{(k)}\|_q \le K \|x\|_r^{\alpha}\, \|x^{(n)}\|_p^{\beta}, \quad x \in W_{p,r}^n, \tag{1.5}$$

between the norms of successive derivatives of functions on the real axis and half-axis with a constant K independent of x, where

$$\alpha = \frac{n-k-p^{-1}+q^{-1}}{n-p^{-1}+r^{-1}}, \quad \beta = 1-\alpha.$$

Such inequalities (on the axis and half-axis) were first studied by G. H. Hardy and J. E. Littlewood [15], E. Landau [19], and J. Hadamard [14]. Important results in the study of inequality (1.5) were obtained by A. N. Kolmogorov [17], B. Szőkefalvi-Nagy [27], and N. P. Kuptsov [18]. For a range of values of the parameters k, n, p, q, and r (on the axis and half-axis), the sharp constants K and the set of extremal functions for which (1.5) becomes an equality are now known (see the bibliographies in [3, 9, 10, 30, 31]). Inequality (1.5) does not hold for all parameter values; Gabushin [12] proved that the constant K in (1.5) is finite if and only if

$$\frac{n-k}{r} + \frac{k}{p} \ge \frac{n}{q}. \tag{1.6}$$

In the subject under discussion, an important role is played by Stechkin's problem [22] on the best approximation of the differentiation operator of order k by bounded linear operators T from L_r to L_q on the class of n-times differentiable functions, $0 \le k < n$:

$$E(N) = \inf_{\|T\|_{L_r \to L_q} \le N} \ \sup_{x \in Q_{p,r}^n} \|x^{(k)} - Tx\|_q. \tag{1.7}$$

The history and current state of research on Stechkin's problem can be found in [22, 9, 3, 6, 7].

As first noted by L. V. Taikov [28], in some cases $G(N)$, and hence $F(N)$, can be estimated from above by the quantity $E(N)$. Taikov's idea is as follows. Suppose that

$$m = n-k, \quad p = r = q', \quad p' = r' = q,$$

and assume that problem (1.7) has an extremal operator T, which is defined, linear, commutes with the differentiation operator of order n on the set W_p^n, and satisfies

$$\sup\left\{\|x^{(k)} - Tx\|_q :\ x \in B_p^n\right\} = E(N). \tag{1.8}$$

Then

$$G(N) \le E(N), \tag{1.9}$$

and hence,

$$F(N) \le E(N).$$

Indeed, to each function $\psi \in W_p^{n-k}$ we assign a function $x \in W_p^n$ such that $x^{(k)} = \psi$ and define the operator S by the formula

$$S\psi = Tx, \quad \psi \in W_p^{n-k}.$$

Then

$$(S\psi)^{(n)} = (Tx)^{(n)} = Tx^{(n)} = T\psi^{(n-k)},$$

therefore,

$$\|(S\psi)^{(n)}\|_{r'} \le N\|\psi^{(m)}\|_{q'}, \quad \psi \in W^m_{q'}.$$

Combining this with (1.8) yields

$$\|\psi - S\psi\|_q = \|x^{(k)} - Tx\|_q \le E(N)\|x^{(n)}\|_p = E(N)\|\psi^{(n-k)}\|_p,$$

which implies (1.9). This idea was used to solve problems $G(N)$ and $F(N)$ in several special cases by Taikov (1967) [28], Subbotin (1967) [24], Subbotin and Taikov (1968) [26], Subbotin (1971) [25], and Arestov and Gabushin (1971) [8].

In the author's paper [1] (1975), it was proved that $F(N)$ is actually the dual of the problem for inequality (1.5), and the linear problem $G(N)$ is the dual of Stechkin's problem (1.7). Specifically, under the relations

$$m = n - k, \quad 1/p + 1/p' = 1, \quad 1/q + 1/q' = 1, \quad 1/r + 1/r' = 1,$$

and when (1.6) (or equivalently (1.4)) is satisfied, we have

$$F(N) = \Delta(N), \quad \Delta(N) = \beta\alpha^{\alpha/\beta}K^{1/\beta}N^{-\alpha/\beta},$$
$$G(N) = E(N)$$

with K being the smallest (best) constant in inequality (1.5).

In the author's recent paper [5] (2025), a solution is obtained to Stechkin's problem on best approximation in the uniform norm on the real axis. The problem concerns the approximation of fractional (more precisely, real)-order k differentiation operators by bounded linear operators from L_2 to C, acting on the class $\mathcal{Q}^n$ of functions whose fractional derivative of order n $(0 \le k < n)$ has a summable Fourier transform. The corresponding sharp Kolmogorov inequality is also established. In the present paper, we examine the case of classical (nonnegative integer-order) derivatives. Two corresponding function classes are introduced, and we study problems of best and best linear approximation of one class by the other, smoother class. In the formulation of Stechkin's problem and the Kolmogorov inequality considered in [5], the norm of the highest-order derivative is not the classical Lebesgue norm. Consequently, our results are not covered by [1] and earlier literature. The function space endowed with the L_1-norm of the Fourier transform serves as the predual to the multiplier space of L_2. Problems in such predual spaces for multipliers of Lebesgue spaces arise naturally in the study of Stechkin's problem [4].

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2. Problem statement and previous results

2.1. Stechkin's problem and the corresponding Kolmogorov inequality

Let $\mathcal{W}^n$, $n \ge 1$, denote the space of functions $f \in L_2$ that are n times continuously differentiable on the real axis and whose nth derivative is the inverse Fourier transform of a summable function:

$$f^{(n)} = \check{y}, \quad y \in L.$$

This condition means that $f^{(n)} \in C_0$ and that its Fourier transform $\widehat{f^{(n)}}$, understood in the sense of generalized functions, is summable; in other words, $\widehat{f^{(n)}} \in L$.

Regarding the space $\mathcal{W}^n$, the following statement was proved in [5, Lemma 1].

Lemma 1. *For any integer* $n \ge 0$, *the space* $\mathcal{W}^n$ *consists of functions* $f \in L_2$ *having the property that the function* $y_n(\eta) = (2\pi\eta i)^n \widehat{f}(\eta)$ *is summable*: $y_n \in L(-\infty, \infty)$, *and moreover,*

$$f^{(n)}(t) = \widecheck{y_n}(t) = \int e^{2\pi t\eta i}(2\pi\eta i)^n \widehat{f}(\eta)\, d\eta.$$

Denote by $\mathfrak{B}(L_2, C)$ the set of all bounded linear operators from L_2 to C, and by $\mathfrak{B}(N; L_2, C)$, $N > 0$, the subset of operators $T \in \mathfrak{B}(L_2, C)$ with norm $\|T\|_{L_2 \to C} \le N$. For an operator $T \in \mathfrak{B}(L_2, C)$, the quantity

$$U(T) = \sup\left\{\|f^{(k)} - Tf\|_{C(\mathbb{R})} \colon f \in \mathcal{Q}^n\right\}$$

is the deviation in the space C of the operator T from the differentiation operator of order k on the class

$$\mathcal{Q}^n = \{f \in \mathcal{W}^n \colon \|\widehat{f^{(n)}}\|_1 \le 1\}.$$

Then

$$E(N) = E(N; n, k) = \inf\{U(T) \colon T \in \mathfrak{B}(N; L_2, C)\} \tag{2.1}$$

is the best approximation (in the space C) of the differentiation operator of order k on the class $\mathcal{Q}^n$ by the set of (bounded linear) operators $\mathfrak{B}(N; L_2, C)$. The problem is to study the quantity (2.1) and to find an extremal operator attaining the infimum in (2.1); this is a specific variant of Stechkin's problem on the approximation of an unbounded linear operator by bounded linear ones [22].

In the author's paper [5] (2025), the following two statements are proved. Their formulation uses the function

$$\theta_h(t) = (2\pi t i)^k \max\left\{0, \left(1 - (2\pi h)^{n-k}|t|^{n-k}\right)\right\}, \quad t \in (-\infty, \infty), \tag{2.2}$$

depending on the parameter $h > 0$. Prepare in advance a convolution-type operator (1.2), whose kernel is the Fourier transform $\widehat{\theta_h}$ of the function θ_h:

$$(\mathbf{T}_h f)(x) = (f \star \widehat{\theta_h})(x) = \int f(x+t)\widehat{\theta_h}(t)\, dt, \quad f \in L_2. \tag{2.3}$$

Theorem 1. *For problem* (2.1) *with* $0 \le k < n$ *and*

$$N = N(h) = \left(\frac{2}{\pi h^{2k+1}}\right)^{1/2} \frac{(n-k)}{\left\{(2k+1)(2n+1)(n+k+1)\right\}^{1/2}}, \quad h > 0, \tag{2.4}$$

the following statements hold:

(1) *the exact value is*

$$E(N(h)) = h^{n-k};$$

(2) *the operator* $\mathbf{T}_h$ *defined by formula* (2.3) *is extremal for problem* (2.1).

Theorem 2. *On the set* $\mathcal{W}^n$ *for* $0 \le k < n$, *the following inequality holds*:

$$\|f^{(k)}\|_C \le \mathcal{K}\|f\|_2^{\alpha}\,\|\widehat{f^{(n)}}\|_1^{\beta}, \quad f \in \mathcal{W}^n, \tag{2.5}$$

$$\alpha = \frac{2(n-k)}{2n+1}, \quad \beta = 1 - \alpha = \frac{2k+1}{2n+1}.$$

In this inequality, the best constant $\mathcal{K} = \mathcal{K}(n, k)$ *has the value*

$$\mathcal{K} = \left(\frac{1}{2\pi(n+k+1)}\right)^{(n-k)/(2n+1)} \left(\frac{2n+1}{2k+1}\right)^{(n+k+1)/(2n+1)}, \tag{2.6}$$

and the function $f_h = \widehat{\theta_h}$, $h > 0$, *where* θ_h *is defined in* (2.2), *is extremal.*

In [5], these two results were obtained for fractional derivatives of orders $0 \le k < n$. Here, we state them for classical (nonnegative integer) derivatives.

In this case, the well-known Stechkin estimate relating problem (2.1) to inequality (2.5) takes the form of an equality:

$$E(N) = \Delta(N), \quad \Delta(N) = \beta\alpha^{\alpha/\beta}\mathcal{K}^{1/\beta}N^{-\alpha/\beta}, \quad N > 0. \tag{2.7}$$

We will now describe and study the problem of best and best linear approximation of one class of functions by another, smoother class. This problem is related to Stechkin's problem (2.1) and the Kolmogorov inequality (2.5).

2.2. Two classes of differentiable functions. Approximation of one class by another

Two classes of functions will be considered further: the approximating class

$$B_2^n(N) = \big\{\varphi \in W_2^n \colon \|\varphi^{(n)}\|_2 \le N\big\}, \quad N \in \mathbb{R}, \quad N > 0,$$

and the approximated class

$$B_1^m = B_1^m(1) = \big\{\psi \in W_1^m \colon \bigvee \psi^{(m-1)} \le 1\big\}.$$

Denote by $\widehat{\mathscr{L}_\infty}$ the set of functions $\theta \in \mathscr{L}$ whose Fourier transform $\widehat{\theta}$ (understood in $\mathscr{S}'$) is a classical function from L_∞; equip this space with the norm $\|\theta\| = \|\widehat{\theta}\|_\infty$, $\theta \in \widehat{\mathscr{L}_\infty}$. For a function $\psi \in W_1^m$, $0 < m \le n$, define

$$F(\psi, N) = F(\psi, B_2^n(N)) = \inf\big\{\|\psi - \varphi\|_{\widehat{\mathscr{L}_\infty}} \colon \varphi \in B_2^n(N)\big\}; \tag{2.8}$$

this is the quantity of the best approximation of a particular function $\psi \in W_1^m$ by the class $B_2^n(N)$. It is standard to assume that if $\psi - \varphi \notin \widehat{\mathscr{L}_\infty}$, then $\|\psi - \varphi\|_{\widehat{\mathscr{L}_\infty}} = +\infty$. Finally, we set

$$F(N) = F(B_1^m, B_2^n(N)) = \sup\big\{F(\psi, B_2^n(N)) \colon \psi \in B_1^m\big\}; \tag{2.9}$$

this is the value of the best approximation of the class B_1^m by the class $B_2^n(N)$ relative to the norm of the space $\widehat{\mathscr{L}_\infty}$. In what follows, (2.9) will be called the approximation of the class B_1^m by the class $B_2^n(N)$, or simply the class–approximation problem.

Let us now formulate the corresponding linear class–approximation problem. Let $\mathfrak{M}(N)$ be the set of (homogeneous and additive) linear operators S from W_1^m to W_2^n such that

$$\|(S\psi)^{(n)}\|_2 \le N \cdot \bigvee \psi^{(m-1)} \quad \text{for} \quad \psi \in W_1^m.$$

Define

$$J(S) = \sup\big\{\|\psi - S\psi\|_{\widehat{\mathscr{L}_\infty}} \colon \psi \in B_1^m\big\}, \quad S \in \mathfrak{M}(N). \tag{2.10}$$

Then

$$G(N) = \inf\big\{J(S) \colon S \in \mathfrak{M}(N)\big\} \tag{2.11}$$

is the linear approximation of the class B_1^m by the class $B_2^n(N)$ that corresponds to (2.9). It is clear that

$$F(N) \le G(N).$$

2.3. Brief formulation of the main result

Theorem 3. *For all* $0 \le k < n$, $m = n-k$, *and* $N > 0$, *the values of problems* (2.9), (2.11), *and* (2.1) *coincide*:

$$F(N) = G(N) = E(N).$$

The extremal function for problem (2.9) (which attains the supremum in (2.9)) and the best method for the linear class–approximation problem (2.11) will be given in Section 5.

3. Approximation of individual functions from $B_1^{n-k}(1)$ by the class $B_2^n(N)$

3.1. Stechkin's problem for intermediate functionals

Given $0 \le k < n$ and a function $\psi \in W_1^{n-k}$, we define a functional on the space $\mathcal{W}^n$ by

$$\Upsilon_\psi^k x = (-1)^{n-k} \int x^{(k)}(t)\, d\psi^{(n-k-1)}(t), \quad x \in \mathcal{W}^n. \tag{3.1}$$

By Lemma 1, the Fourier transform of any $x \in \mathcal{W}^{(n)}$ and of all its derivatives up to order n are summable. Hence, $x \in \mathcal{W}^n$ and its derivatives up to order n lie in C_0; that is, $x^{(l)} \in C_0$ for $0 \le l \le n$. In particular, the functional Υ_ψ^k given by (3.1) is well-defined on $\mathcal{W}^n$ for all $0 \le k < n$.

As an auxiliary (intermediate) step, we consider a variant of Stechkin's problem concerning the approximation of functional (3.1) by the set $L_2^*(N)$ of bounded linear functionals $T \in L_2^*$ on the space L_2 with norm at most a given number $N > 0$. For a function $\psi \in W_1^{n-k}$ and a bounded linear functional $T \in L_2^*$, the quantity

$$u(\psi, T) = \sup\left\{|\Upsilon_\psi^k x - Tx| \colon x \in \mathcal{Q}^n\right\} \tag{3.2}$$

is the deviation of the functional T from the functional Υ_ψ^k on the class $\mathcal{Q}^n$. Stechkin's problem is to study the best approximation

$$\varepsilon(\psi, N) = \inf\left\{u(\psi, T) \colon \|T\|_{L_2^*} \le N\right\} \tag{3.3}$$

of functional (3.1) by the set $L_2^*(N)$. A functional $\widetilde{T}$ achieving the infimum in (3.3) is called extremal for problem (3.3).

Recall that $L_2^* = L_2$; more precisely, these two spaces are isometric, and the formula

$$Tx = \int x(t)\lambda(t)\, dt \tag{3.4}$$

establishes a bijection between $T \in L_2^*$ and $\lambda \in L_2$ with $\|T\|_{L_2^*} = \|\lambda\|_2$. The function λ in representation (3.4) will be called the weight of the functional T.

3.2. Auxiliary constructions and statements

The next two propositions describe the connection between problems (3.2) and (2.8) for any function $\psi \in W_1^{n-k}$.

Lemma 2. *Let* $\psi \in W_1^{n-k}$ *and* $\varphi \in W_2^n$ *be such that the difference* $z = \psi - \varphi$ *belongs to the space* $\widehat{\mathscr{L}_\infty}$. *Then the following identity holds on* $\mathcal{W}^n$:

$$\Upsilon_\psi^k x - T_\varphi x = \int \check{z}(t)\widehat{x^{(n)}}(t)\, dt, \quad x \in \mathcal{W}^n, \tag{3.5}$$

where T_φ *is the functional*

$$T_\varphi x = \int \lambda(t) x(t) dt, \quad x \in L_2, \tag{3.6}$$

with weight $\lambda = (-1)^n \varphi^{(n)}$.

Proof. 1. For functions $x \in \mathscr{S}$, integrating by parts the appropriate number of times, we obtain

$$\Upsilon^k_\psi x = (-1)^{n-k} \int x^{(k)}(t)\, d\psi^{(n-k-1)}(t)$$

$$= (-1)^{n-k-1} \int x^{(k+1)}(t)\, \psi^{(n-k-1)}(t)\, dt = \cdots = \int x^{(n)}(t)\, \psi(t)\, dt,$$

$$T_\varphi x = (-1)^n \int \varphi^{(n)}(t) x(t) dt = \int x^{(n)}(t) \varphi(t)\, dt.$$

Consequently,

$$\Upsilon^k_\psi x - T_\varphi x = \int x^{(n)}(t)\, z(t)\, dt, \quad \text{where} \quad z = \psi - \varphi. \tag{3.7}$$

According to definition (1.3), for $\phi \in \mathscr{S}$, we have $\langle \widehat{z}, \phi \rangle = \langle z, \widehat{\phi} \rangle$, or equivalently,

$$\langle z, \phi \rangle = \langle \check{z}, \widehat{\phi} \rangle, \quad \phi \in \mathscr{S}.$$

For $\phi = x^{(n)}$, $x \in \mathscr{S}$, this relation takes the form

$$\int z(t)\, x^{(n)}(t)\, dt = \int \check{z}(t)\, \widehat{x^{(n)}}(t)\, dt.$$

Together with (3.7), this implies representation (3.5) on the set $\mathscr{S}$.

2. We now extend representation (3.5) to the set of infinitely differentiable functions $x \in \mathcal{W}^n$ on the real axis. Let $\varkappa$ be an infinitely differentiable function on the real axis with the properties: $0 \le \varkappa(t) \le 1$ for all $t \in (-\infty, \infty)$, $\varkappa(t) = 0$ for $|t| \ge 2$, and $\varkappa(t) = 1$ for $|t| \le 1$. Take an infinitely differentiable on the axis function $x \in \mathcal{W}^n$ and define

$$x_\delta(t) = \varkappa(t\delta^{-1})\, x(t), \quad \delta > 0.$$

The function x_δ is compactly supported and belongs to $\mathscr{S}$. For x_δ, identity (3.5) holds:

$$\Upsilon^k_\psi x_\delta - T_\varphi x_\delta = \int \check{z}(t)\, \widehat{x_\delta{}^{(n)}}(t) dt. \tag{3.8}$$

We study the limit of this relation as $\delta \to +\infty$.

We start with the latter integral in (3.8). Applying Leibniz's formula, we have

$$x_\delta^{(n)}(t) = x^{(n)}(t) \varkappa(t\delta^{-1}) + \varrho(t;\delta), \tag{3.9}$$

where

$$\varrho(t;\delta) = \sum_{j=1}^{n} C_n^j x^{(n-j)}(t) \delta^{-j} \varkappa^{(j)}(t\delta^{-1}).$$

Let us verify that

$$\int \check{z}(t)\, \widehat{\varrho(t;\delta)}(t) dt \to 0, \quad \delta \to +\infty. \tag{3.10}$$

It suffices to prove that for $1 \le j \le n$,

$$I_j(\delta) = \delta^{-j} \int \check{z}(t)\, \widehat{\varrho_j(t;\delta)}(t)dt \to 0, \quad \delta \to +\infty, \tag{3.11}$$

where

$$\varrho_j(t;\delta)(t) = x^{(n-j)}(t)\delta^{-j}\varkappa^{(j)}(t\delta^{-1}).$$

Using the property that the Fourier transform of a product equals the convolution of the Fourier transforms of the factors, we have

$$\widehat{\varrho_j(t;\delta)}(t) = \delta \int \widehat{x^{(n-j)}}(t-\eta)\widehat{\varkappa^{(j)}}(\eta\delta)d\eta.$$

Lemma 1 implies that the Fourier transforms $\widehat{x^{(l)}}$ of the derivatives $x^{(l)}$, $0 \le l \le n$, of functions $x \in \mathcal{W}^{(n)}$ are summable. Therefore,

$$|I_j(\delta)| \le \delta^{-j}\|\check{z}\|_\infty \int \left|\widehat{\varrho_j(t;\delta)}(t)\right|dt \le \delta^{-j}\|\check{z}\|_\infty \int \left|\delta \int \widehat{x^{(n-j)}}(t-\eta)\widehat{\varkappa^{(j)}}(\eta\delta)d\eta\right|dt$$
$$\le \delta^{-j}\|\check{z}\|_\infty\, \|\widehat{x^{(n-j)}}\|_L\, \|\widehat{\varkappa^{(j)}}\|_L.$$

It follows that each limit relation (3.11) holds, whence (3.10) also holds.

Consider the behavior of the integral

$$I_0(\delta) = \int \check{z}(t)\, \widehat{\varrho_0(t;\delta)}(t)dt,$$

where

$$\varrho_0(t;\delta)(t) = x^{(n)}(t)\varkappa(t\delta^{-1}).$$

We have

$$\widehat{\varrho_0(t;\delta)} = \delta \int \widehat{x^{(n)}}(t-x)\widehat{\varkappa}(x\delta)dx. \tag{3.12}$$

As noted above, $\widehat{x^{(n)}} \in L$. The function $\widehat{\varkappa}$, being the Fourier transform of a function from $\mathscr{S}$, belongs to $\mathscr{S}$ and is therefore summable. Moreover,

$$\int \widehat{\varkappa}(t)dt = \varkappa(0) = 1.$$

Hence (see, e.g., [23, Ch. 1, Theorem 1.18]), the family of functions (3.12) converges in L to $\widehat{x^{(n)}}$ as $\delta \to +\infty$. Combining the obtained results, we conclude that

$$\lim_{\delta\to+\infty} \int \check{z}(t)\, \widehat{{x_\delta}^{(n)}}(t)dt = \int \check{z}(t)\, \widehat{x^{(n)}}(t)dt. \tag{3.13}$$

Now let us discuss the behavior of the integral

$$\Upsilon_\psi^k x_\delta = (-1)^{n-k} \int {x_\delta}^{(k)}(t)\, d\psi^{(n-k-1)}(t).$$

Specifically, we will show that

$$\Upsilon_\psi^k x_\delta \to \Upsilon_\psi^k x, \quad \delta \to +\infty. \tag{3.14}$$

In the case $k = 0$, this is obvious. For $1 \le k < n$, similarly to (3.9), we have

$$x_\delta^{(k)}(t) = x^{(k)}(t)\varkappa(t\delta^{-1}) + \rho(t;\delta),$$

where

$$\rho(t;\delta) = \sum_{j=1}^{k} C_k^j x^{(k-j)}(t)\delta^{-j}\varkappa^{(j)}(t\delta^{-1}).$$

Since the derivatives $x^{(j)}$ for $j = 0, 1, \ldots, n-1$ are (continuous and) bounded, we have

$$\|\rho\|_C \leq c\delta^{-1}$$

for some constant $c > 0$. Hence, the convergence (3.14) also holds for $1 \leq k < n$.

Finally, it is obvious that

$$T_\varphi x_\delta \to T_\varphi x, \quad \delta \to +\infty. \tag{3.15}$$

The limit relations (3.13), (3.14), and (3.15) imply that as $\delta \to +\infty$, equality (3.8) turns into (3.5) for infinitely differentiable functions $x \in \mathcal{W}^n$.

3. We now justify relation (3.5) on the whole space $\mathcal{W}^n$. As before, set $\kappa = \widehat{\varkappa}$ and, for $\delta > 0$, define

$$\kappa_\delta(t) = \delta\,\widehat{\varkappa}(\delta t), \quad t \in \mathbb{R}.$$

Then $\kappa_\delta \in L$ and

$$\int \kappa_\delta(t)dt = \varkappa(0) = 1, \quad \delta > 0.$$

For an arbitrary function $x \in \mathcal{W}^n$, consider its convolution with the function κ_δ:

$$y_\delta(t) = (x * \kappa_\delta)(t) = \int x(t-\theta)\kappa_\delta(\theta)\,d\theta = \int x(\theta)\kappa_\delta(t-\theta)\,d\theta.$$

The function y_δ is infinitely differentiable on the real axis and belongs to $\mathcal{W}^n$. By the previous step of the proof, the relation

$$\Upsilon_\psi^k y_\delta - T_\varphi y_\delta = \int \check{z}(t)\,\widehat{y_\delta^{(n)}}(t)dt \tag{3.16}$$

holds. We will verify that as $\delta \to +\infty$, this relation yields (3.5).

We have $y_\delta{}^{(n)} = x^{(n)} * \kappa_\delta$, and thus

$$\widehat{y_\delta^{(n)}} = \widehat{x^{(n)}}\,\widehat{\kappa}_\delta.$$

This equality means that

$$\widehat{y_\delta^{(n)}}(t) = \widehat{x^{(n)}}(t)\,\varkappa(\delta t), \quad t \in \mathbb{R}.$$

Now it is seen that

$$\int \check{z}(t)\,\widehat{y_\delta^{(n)}}(t)dt \to \int \check{z}(t)\,\widehat{x^{(n)}}(t)dt \quad \text{as} \quad \delta \to +\infty.$$

Furthermore, since $x^{(k)} \in C_0$ and $y_\delta^{(k)} = x^{(k)} * \kappa_\delta$, Theorem 1.18 of Chapter 1 in [23] implies that $y_\delta^{(k)}$ converges to $x^{(k)}$ in C_0 as $\delta \to +\infty$. Consequently,

$$\Upsilon_\psi^k y_\delta \to \Upsilon_\psi^k x \quad \text{as} \quad \delta \to +\infty.$$

Similar arguments yield $T_\varphi y_\delta \to T_\varphi x$ as $\delta \to +\infty$. Therefore, for every $x \in \mathcal{W}^{(n)}$, equality (3.16) indeed passes in the limit $\delta \to +\infty$ to (3.5).

The proof of Lemma 2 is complete. $\square$

The next proposition can be considered, in a sense, the converse of Lemma 2.

Lemma 3. *If a function* $\psi \in W_1^{n-k}$ *and a functional* $T \in L_2^*$ *are such that* $u(\psi, T) < \infty$, *then there exists a unique function*

$$\varphi = \varphi(T) \in W_2^n \tag{3.17}$$

with the following properties:

(1) *the formula*

$$(-1)^n \varphi^{(n)} = \lambda \tag{3.18}$$

holds;

(2) *the Fourier transform* $\widehat{z}$ *(in* $\mathscr{S}'$*) of the difference* $z = \psi - \varphi$ *is an essentially bounded function*: $\widehat{z} \in L_\infty$;

(3) *on the set* $\mathcal{W}^n$, *the representation*

$$\Upsilon_\psi^k x - Tx = \int \check{z}(t)\, \widehat{x^{(n)}}(t) dt, \quad x \in \mathcal{W}^n; \tag{3.19}$$

holds;

(4) *the following equality*

$$u(\psi, T) = \|\psi - \varphi\|_{\widehat{\mathscr{L}_\infty}} \tag{3.20}$$

is satisfied.

Proof. Denote by $\mathscr{Q}^n$ the set of functions $x \in \mathscr{S}$ with

$$\|\widehat{x^{(n)}}\|_1 \le 1,$$

and let

$$v(\psi, T) = \sup \big\{ |\Upsilon_\psi^k x - Tx| \colon x \in \mathscr{Q}^n \big\}.$$

Since $\mathscr{Q}^n \subset Q^n$, we have

$$v(\psi, T) \le u(\psi, T). \tag{3.21}$$

Functions $x \in \mathscr{S}$ are uniquely determined by the Fourier transforms of their derivatives of order n, therefore

$$\Upsilon_\psi^k x - Tx = R\widehat{x^{(n)}}, \tag{3.22}$$

where R is some functional on the space Y of Fourier transforms of derivatives $\widehat{x^{(n)}}$ of order n of functions $x \in \mathscr{S}$. The space Y is a (normed) subspace of the space L. It is easy to see that the functional R on Y is linear and $\|R\|_{Y^*} = v(\psi, T)$. Denote by $\mathcal{R}$ the extension of R from Y to L with preservation of the norm. This extension has the form

$$\mathcal{R}y = \int y(t) \xi(t)\, dt, \quad y \in L,$$

where $\xi \in L_\infty$ and

$$\|\xi\|_\infty = \|\mathcal{R}\|_{L^*} = \|R\|_{Y^*} = v(\psi, T). \tag{3.23}$$

Denote by $\underline{\varphi} \in W_2^n$ an (arbitrary) function associated with the weight λ of representation (3.4) of the functional T through relation (3.18). Integrating by parts the integral in (3.4), we obtain the representation

$$Tx = \int x^{(n)}(t) \underline{\varphi}(t)\, dt, \quad x \in \mathscr{S}. \tag{3.24}$$

Using integration by parts again, we find a similar representation for functional (3.1):

$$\Upsilon_\psi^k x = \int x^{(n)}(t)\, d\psi(t), \quad x \in \mathscr{S}.$$

As a result, formula (3.22) takes the form

$$\int \{\psi(t) - \underline{\varphi}(t)\} x^{(n)}(t)\, dt = \int \xi(t) \widehat{x^{(n)}}(t)\, dt, \quad x \in \mathscr{S}. \tag{3.25}$$

Let $\Xi = \widehat{\xi}$ be the Fourier transform (in $\mathscr{S}$, see (1.3)) of the function ξ. Using (1.3), we can write the right-hand side of (3.25) as

$$\langle \xi, \widehat{x^{(n)}} \rangle = \langle \Xi, x^{(n)} \rangle.$$

This relation together with (3.25) implies the equality

$$\langle \mathscr{N}, x^{(n)} \rangle = 0, \quad x \in \mathscr{S}, \tag{3.26}$$

where

$$\mathscr{N} = \Xi - (\psi - \underline{\varphi}) \in \mathscr{S}'.$$

Equality (3.26) implies $\mathscr{N}^{(n)} = 0$ in $\mathscr{S}'$. Consequently (see, e.g., [21, Ch. 1, Sect. 5]), $\mathscr{N}$ belongs to $\mathscr{L}$ and is in fact a polynomial p_{n-1} of degree $n-1$. The function $\varphi = \underline{\varphi} + p_{n-1}$ then also yields representation (3.24) for the functional T and possesses the properties (3.17) and (3.18).

Thus, we have $\Xi = z \in \mathscr{S}'$, $z = \psi - \varphi$, and consequently, $\xi = \check{z}$. Formula (3.25) implies representation (3.19) for functions $x \in \mathscr{S}$.

The pair of functions consisting of the original function $\psi \in W_1^{n-k}$ and the constructed function $\varphi \in W_2^n$ satisfies the conditions of Lemma 2. By Lemma 2, relation (3.5) holds, which in this situation coincides with (3.19). Representation (3.19) gives the estimate

$$u(\psi, T) \le \|\check{z}\|_\infty$$

for the quantity $u(\psi, T)$ defined by formula (3.2). This estimate, together with inequality (3.21) and equalities (3.23), implies property (3.20). All assertions of Lemma 3, except for the uniqueness property of the function φ, are proved.

Suppose that, in addition to the function φ, there exists another function $\varphi_0 \in W_2^n$ satisfying the properties listed in Lemma 3. Property (3.18) implies that the difference $p = \varphi_0 - \varphi$ is an algebraic polynomial of degree $n-1$. The functions $z_0 = \psi - \varphi_0$ and $z = \psi - \varphi$ are related by $z_0 = z - p$. The Fourier transforms of both functions z_0 and z belong to L_∞. Since the Fourier transform of a nonzero polynomial is a linear combination of the Dirac delta function and its derivatives, $\widehat{p}$ is a classical function only if $p \equiv 0$. Consequently, $z_0 = z$ and $\varphi_0 = \varphi$. The uniqueness of the function φ is thus proved. The proof of Lemma 3 is complete. ☐

3.3. Equivalence of problems (2.8) and (3.3)

The following proposition essentially shows the equivalence of problems (2.8) and (3.3).

Lemma 4. *For any function $\psi \in W_1^{n-k}$, the following statements hold.*

(1) *The value of approximation (3.3) and the deviation quantity (2.8) coincide; that is, the following equality holds:*

$$\varepsilon(\psi, N) = F(\psi, N). \tag{3.27}$$

(2) *Extremal elements exist for both problems (3.3) and (2.8): a functional $\widetilde{T} \in L_2^*$ attaining the infimum in (3.3) and a function $\widetilde{\varphi} \in B_2^n(N)$ attaining the infimum in (2.8).*

(3) *Formula* (3.6) *establishes a correspondence between the extremal solutions of problems* (3.3) *and* (2.8).

Proof. (1) Indeed, suppose that $F(\psi,N)<\infty$ and a function $\varphi\in B_2^n(N)$ is such that

$$\psi-\varphi\in\widehat{\mathscr{L}_\infty}.$$

By Lemma 2, there exists a functional T_φ satisfying properties (3.5) and (3.6). Formula (3.6) provides the norm estimate

$$\|T_\varphi\|_{L_2^*}=\|\varphi^{(n)}\|_{L_2}\le N.$$

Relation (3.5) implies the inequality

$$u(\psi,T_\varphi)\le\|\psi-\varphi\|_{\widehat{\mathscr{L}_\infty}}.$$

Hence, for the quantity defined by (3.3), we also have the inequality

$$\varepsilon(\psi,N)\le\|\psi-\varphi\|_{\widehat{\mathscr{L}_\infty}}.$$

Taking the infimum over all $\varphi=\varphi(T)\in B_2^n(N)$ yields

$$\varepsilon(\psi,N)\le F(\psi,N).$$

We now prove the reverse inequality. Assume that $\varepsilon(\psi,N)<\infty$ and let $T\in L_2^*$ be a functional with $\|T\|_{L_2^*}\le N$ such that $u(\psi,T)<\infty$. By Lemma 3, there exists a function $\varphi\in B_2^n(N)$ with property (3.20). From (3.20), we obtain $F(\psi,N)\le u(\psi,T)$. Hence, $F(\psi,N)\le\varepsilon(\psi,N)$.

Therefore, we have shown that finiteness of either $F(\psi,N)$ or $\varepsilon(\psi,N)$ implies the finiteness of the other, and that the two quantities coincide. This proves the first statement of Lemma 4.

(2) Problem (3.3) is a special case of Stechkin's problem on the best approximation of a linear operator (in particular, a functional) by bounded linear operators (in this case, functionals). According to the result of Gabushin, in the functional problem represented by (3.3), an extremal functional $\widetilde{T}$ exists; the real case is treated in [13], and the complex case in [9, Sect. 3, Theorem 3.1, Corollary 3.4].

Lemma 3 guarantees the existence of a function $\widetilde{\varphi}=\varphi(\widetilde{T})$ in W_2^n; this function is clearly extremal for problem (2.8).

(3) The correspondence between the extremal elements of problems (3.3) and (2.8) is a consequence of equality (3.27) and the assertion of Lemma 3.

The proof of Lemma 4 is complete. □

4. Best linear method for class–approximation

4.1. Functional problem equivalent to operator problem (2.1)

Along with (2.1), consider the functional problem

$$e(N)=\inf_{\|T\|_{L_2^*\le N}}\ \sup_{x\in\mathcal{Q}^n}|x^{(k)}(0)-Tx|. \tag{4.1}$$

By well-known arguments, problems (4.1) and (2.1) are equivalent, namely, their values coincide:

$$e(N)=E(N). \tag{4.2}$$

Moreover, if

$$T_0 x = \int \lambda_0(t) x(t) dt, \quad \lambda_0 \in L_2, \quad x \in L_2, \tag{4.3}$$

is an extremal functional for problem (4.1), then the convolution operator

$$\mathbf{T}_0 x(t) = \int x(t+\theta)\lambda_0(\theta)\, d\theta, \quad x \in L_2,$$

is extremal for problem (2.1). Conversely, if $\mathbf{T}_0$ is an extremal operator for problem (2.1), then

$$T_0 x = \mathbf{T}_0 x(0), \quad x \in L_2,$$

is an extremal functional for problem (4.1).

By these considerations and the assertions of Theorem 1, the functional (4.3) with weight

$$\lambda_0 = \widehat{\theta_h}, \tag{4.4}$$

where θ_h is the function defined by formula (2.2), will indeed be extremal in problem (4.1) for $N = N(h),\ h > 0$.

It can be shown that the extremal functional for problem (4.1) is unique for any $N > 0$; however, we omit this fact.

4.2. Construction of a linear method for class–approximation

An important role further will be played by the function χ_{n-k}, $0 < n-k \le n$, defined by the relation

$$\chi_{n-k}(t) = \frac{(-1)^{n-k}}{(n-k-1)!}\, t_+^{n-k-1}, \quad t_+ = \begin{cases} t, & t > 0, \\ 0, & t \le 0. \end{cases} \tag{4.5}$$

Since

$$\chi_{n-k}^{(n-k-1)}(t) = (-1)^{n-k} \cdot \begin{cases} 1, & t > 0, \\ 0, & t < 0, \end{cases} \tag{4.6}$$

the function χ_{n-k} belongs to the class B_1^{n-k}.

Based on definition (3.1) and formula (4.6), we conclude that

$$\Upsilon^k_{\chi_{n-k}} x = x^{(k)}(0), \quad x \in \mathcal{W}^n.$$

Therefore (see (4.1), (3.1), (3.2), (3.3), and (3.27)),

$$e(N) = \varepsilon(\chi_{n-k}, N) = F(\chi_{n-k}, N). \tag{4.7}$$

By (4.2) and Theorem 1, each of these three quantities is finite for any $N > 0$.

From now on, T_0 will be functional (4.3) whose weight is given by (4.4). It is extremal for problem (4.1) and hence for the problem $\varepsilon(\chi_{n-k}, N)$ when $N = N(h)$; in this case, $u(\chi_{n-k}, T_0) = e(N) < \infty$. By Lemma 3, applied to the function $\psi = \chi_{n-k}$ and the functional T_0, there exists a function $\varphi_0 \in W_2^n$ related to T_0 by (3.18) such that formula (3.19) holds on $\mathcal{W}^n$. In the present setting this formula becomes

$$x^{(k)}(0) - T_0 x = \int \widecheck{z_0}(t)\, \widehat{x^{(n)}}(t)\, dt, \quad x \in \mathcal{W}^n, \tag{4.8}$$

where

$$z_0 = \chi_{n-k} - \varphi_0. \tag{4.9}$$

Moreover, (3.20) gives

$$\|\widecheck{z_0}\|_\infty = \|\chi_{n-k} - \varphi_0\|_{\widehat{\mathscr{L}_\infty}} = e(N). \tag{4.10}$$

For $x \in \mathcal{W}^n$ and any $\theta \in \mathbb{R}$, the function $x_\theta(t) = x(t+\theta)$ belongs to the space $\mathcal{W}^n$ and satisfies the relation

$$\widehat{x_\theta^{(n)}}(t) = e^{2\pi t\theta i}\widehat{x^{(n)}}(t), \quad t \in \mathbb{R}.$$

Hence, for the function x_θ, equality (4.8) takes the form

$$x^{(k)}(\theta) - T_0 x_\theta = \int e^{2\pi t\theta i}\widecheck{z_0}(t)\,\widehat{x^{(n)}}(t)\,dt. \tag{4.11}$$

For a function $\psi \in W_1^{n-k}$, let

$$\mu = \mu_\psi = (-1)^{n-k}\psi^{(n-k-1)}. \tag{4.12}$$

Integrate equality (4.11) with respect to the measure $d\mu$. According to (3.1),

$$\int x^{(k)}(\theta)\,d\mu(\theta) = \Upsilon_\psi^k x.$$

Further, the formula

$$T[\psi]x = \int T_0 x_\theta\,d\mu(\theta), \quad x_\theta(t) = x(t+\theta),$$

defines a bounded linear functional $T[\psi]$ on L_2; its norm satisfies the estimate

$$\|T[\psi]\|_{L_2^*} \le \|T_0\|_{L_2^*}\bigvee\mu \le N\cdot\bigvee\psi^{(n-k-1)}.$$

The functional $T[\psi]$ admits the representation

$$T[\psi]x = \int \lambda[\psi](t)x(t)\,dt, \quad x \in L_2,$$

where

$$\lambda[\psi](t) = (\lambda_0 * d\mu)(t) = \int \lambda_0(t-\theta)\,d\mu(\theta)$$

and λ_0 is the weight in representation (4.3) of the extremal functional T_0 for problem (4.1). Thus, at this stage we have

$$\int \{x^{(k)}(\theta) - T_0 x_\theta\}\,d\mu(\theta) = \Upsilon_\psi^k x - T[\psi]x. \tag{4.13}$$

Consider the integral

$$\mathcal{I}(x,\psi) = \iint e^{2\pi t\theta i}\widecheck{z_0}(t)\,\widehat{x^{(n)}}(t)dt\,d\mu(\theta)$$

from the right-hand side of (4.11). By the Fubini–Tonelli theorem (see, for example, [11, Ch. III, Sect. 11]), the order of integration in this integral can be changed, resulting in

$$\mathcal{I}(x,\psi) = \int \breve{\mu}(t)\widecheck{z_0}(t)\,\widehat{x^{(n)}}(t)dt, \quad \text{where} \quad \breve{\mu}(t) = \int e^{2\pi t\theta i}\,d\mu(\theta). \tag{4.14}$$

We have $\breve{\mu} \in C$ and $\|\breve{\mu}\|_C \le \bigvee\mu$.

Combining (4.11), (4.13), and (4.14), we arrive at the relation

$$\Upsilon_\psi^k x - T[\psi]x = \int \breve{\mu}(t)\widecheck{z_0}(t)\,\widehat{x^{(n)}}(t)dt, \quad x \in \mathcal{W}^n. \tag{4.15}$$

This relation implies for $\psi \in B_1^{n-k}$ the estimates

$$u(\psi, T[\psi]) \le \|\check{\mu}\, \widecheck{z}_0\|_\infty \le \|\widecheck{z}_0\|_\infty = e(N) < \infty.$$

By Lemma 3, for a function $\psi \in W_1^{n-k}$ and a functional $T[\psi] \in L_2^*$, there exists a (unique) function $\varphi = \varphi[\psi] \in W_2^n$ such that

$$(-1)^n \varphi[\psi]^{(n)} = \lambda[\psi] = \lambda_0 * d\mu, \tag{4.16}$$

$$z = z[\psi] = \psi - \varphi[\psi] \in \widehat{\mathscr{L}}_\infty, \tag{4.17}$$

and the following equality holds:

$$\Upsilon_\psi^k x - T[\psi]x = \int \widecheck{z[\psi]}(t)\, \widehat{x^{(n)}}(t)dt, \quad x \in \mathcal{W}^n. \tag{4.18}$$

Comparing (4.15) and (4.18), we conclude that

$$\int \Big(\widecheck{z[\psi]}(t) - \check{\mu}(t)\widecheck{z}_0(t)\Big)\widehat{x^{(n)}}(t)dt = 0, \quad x \in \mathcal{W}^n. \tag{4.19}$$

The function

$$Z(t) = \widecheck{z[\psi]}(t) - \check{\mu}(t)\widecheck{z}_0(t)$$

belongs to L_∞. From (4.19) it follows, in particular, that

$$\int Z(t) t^n \widehat{x}(t)dt = 0, \quad x \in \mathscr{S}. \tag{4.20}$$

The Fourier transform is a bijection of the space $\mathscr{S}$ onto itself. Therefore, (4.20) implies that $Z(t) = 0$ almost everywhere on the axis. Hence,

$$\widehat{Z} = z[\psi] - \widehat{\check{\mu}\widecheck{z}_0} = 0 \quad \text{a.e. on the axis.}$$

Thus,

$$z[\psi] = \widehat{\check{\mu}\, \widecheck{z}_0} \quad \text{a.e. on the axis.} \tag{4.21}$$

Comparing (4.17) and (4.21), we conclude that

$$z[\psi] = \widehat{\check{\mu}\, \widecheck{z}_0} \in \widehat{\mathscr{L}}_\infty.$$

Define the mapping $\mathbf{S}$ by the relation

$$\mathbf{S}\psi = \psi - z[\psi], \quad z[\psi] = \widehat{\check{\mu}\, \widecheck{z}_0}, \quad \psi \in W_1^{n-k}. \tag{4.22}$$

Recall that the measure μ is given by formula (4.12).

For convenience of further references, we collect the properties of the constructed mapping (4.22) in the following proposition.

Lemma 5. *The operator* $\mathbf{S}$ *given by formula* (4.22) *defines a linear mapping from the space* W_1^{n-k} *into the space* W_2^n. *Its deviation* (2.10) *satisfies the estimate*

$$J(\mathbf{S}) = \sup\{\|\psi - \mathbf{S}\psi\|_{\widehat{\mathscr{L}}_\infty} : \psi \in B_1^{n-k}\} \le e(N). \tag{4.23}$$

Proof. In the notation used above, according to (4.22) and (4.17), we have $\mathbf{S}\psi = \varphi[\psi] \in W_2^n$, so $\mathbf{S}$ is a mapping from W_1^{n-k} to W_2^n. Relation (4.16) implies that $\mathbf{S}$ is, in fact, a mapping from B_1^{n-k} into $B_2^n(N)$. It follows directly from definition (4.22) that $\mathbf{S}\psi$ is a linear mapping in the argument ψ. Furthermore, from (4.22), (4.17), and (4.21) it follows that $\psi - \mathbf{S}\psi = z[\psi] \in \widehat{\mathscr{L}_\infty}$ for any function $\psi \in W_1^{n-k}$ and

$$\|\psi - \mathbf{S}\psi\|_{\widehat{\mathscr{L}_\infty}} = \|z[\psi]\|_{\widehat{\mathscr{L}_\infty}} = \|\check{\mu}\, \widecheck{z_0}\|_\infty \le \|\check{\mu}\|_C\, \|\widecheck{z_0}\|_\infty \le \bigvee \mu \cdot \|\widecheck{z_0}\|_\infty.$$

This and (4.10) imply (4.23). The proof of Lemma 5 is complete. □

4.3. Discussion of the method S

1. In the sequel, we will use the following simple proposition.

Lemma 6. *Let $f \in L_2$ and $\mu \in V$. The function*

$$w = \widehat{\check{f} \cdot \widecheck{d\mu}} \tag{4.24}$$

satisfies

$$w(t) = (f * d\mu)(t) = \int f(t-\eta)\, d\mu(\eta). \tag{4.25}$$

Proof. Define

$$h(t) = (f * d\mu)(t) = \int f(t-\eta)\, d\mu(\eta). \tag{4.26}$$

Together with the function f, the function defined by (4.26) belongs to the space L_2. Let us find the inverse Fourier transform of this function:

$$\check{h}(\tau) = \int e^{2\pi\tau t i} h(t)\, dt = \iint e^{2\pi\tau t i} f(t-\eta)\, d\mu(\eta)\, dt.$$

Changing the order of integration, we get

$$\check{h}(\tau) = \iint e^{2\pi\tau t i} f(t-\eta)\, dt\, d\mu(\eta) = \Big[t - \eta = t\Big] = \iint e^{2\pi i\tau(t+\eta)} f(t)\, dt\, d\mu(\eta)$$

$$= \int e^{2\pi\tau t i} f(t)\, dt \int e^{2\pi\tau\eta i}\, d\mu(\eta) = \check{f}(\tau) \cdot \widecheck{d\mu}(\tau).$$

Thus, we have $\check{h} = \check{f} \cdot \widecheck{d\mu}$.

The function (4.24) has the same property: $\check{w} = \check{f} \cdot \widecheck{d\mu}$. Consequently, the functions w and h coincide (almost everywhere); this is the assertion of (4.25).

Lemma 6 is proved.

2. The operator $\mathbf{S}$ defined on $\mathcal{W}_1^{n-k}$ by (4.22) will be shown below to be extremal for both the best approximation problem (2.9) and the best linear class–approximation problem (2.11). We now present an alternative representation of this operator that, in the author's view, reveals its nature more clearly.

With the function θ_h defined by (2.2) we associate the function

$$\zeta_h(\eta) = \frac{(2\pi\eta i)^k - \theta_h(\eta)}{(2\pi\eta i)^n}, \quad \eta \in (-\infty, \infty). \tag{4.27}$$

As follows from the results of [5] (see Lemmas 3 and 4, formulas (2.18) and (3.2)), the function ζ_h equals the inverse Fourier transform $\zeta_h = \check{\check{z}}_0$ of the function z_0 that produces representation (4.8) with properties (4.9) and (4.10) for the parameter $N = N(h)$, $h > 0$, given by (2.4).

Lemma 4 of [5] yields $\zeta_h \in L_\infty$ together with the norm equality $\|\zeta_h\|_\infty = h^{n-k}$. Because θ_h has compact support, (4.27) implies that

$$\zeta_h(\eta) = O(\eta^{-(n-k)}), \quad \eta \to \pm\infty.$$

Hence, $\check{\check{z}}_0 = \zeta_h$ belongs to L_2 for $0 \le k < n$, and for $0 \le k \le n-2$ it also lies in L_1. Therefore, $z_0 \in L_2$, and when moreover $0 \le k \le n-2$ we have $z_0 \in C_0$ as well. Under the latter condition, Lemma 6 allows us to express the function $z[\psi]$ appearing in (4.22) as the (classical) convolution (1.2):

$$z[\psi] = \widehat{\check{\mu}\, \check{\check{z}}_0} = z_0 * d\mu = (-1)^{n-k}\bigl\{z_0 * d\psi^{(n-k-1)}\bigr\}.$$

This leads to the following representation for the operator $\mathbf{S}$:

$$(\mathbf{S}\psi)(t) = \psi(t) - (-1)^{n-k} \int z_0(t-\tau)\, d\psi^{(n-k-1)}(\tau), \quad \psi \in W_1^{n-k}. \tag{4.28}$$

As can be seen from (4.9) and (4.6), the function z_0 on each of the two half-axes $I_- = (-\infty, 0]$ and $I_+ = [0, \infty)$ is n-times differentiable, more precisely, $z_0 \in W^n_{22}(I_\pm)$. From this, by the corresponding Kolmogorov inequality for the half-axis (see [12] and the bibliography therein), it follows that $z_0^{(k)} \in L_2 \cap C_0$ on the half-axes $I_\pm$ for $0 \le k \le n-1$. By (4.5) and (4.6), the derivative $z_0^{(n-k-1)}$ of the function z_0 at 0 has a first kind discontinuity with a unit jump. If $n-k \le j \le n$, then $z_0^{(j)} = -\varphi_0^{(j)}$ on the axis.

For a function $\psi \in W_1^{n-k}$, let Ψ denote a function with the property $\Psi^{(k)} = \psi$. We have $\Psi \in W_1^n$ and $\psi^{(n-k-1)} = \Psi^{(n-1)}$. Consequently,

$$\int z_0(t-\tau)\, d\psi^{(n-k-1)}(\tau) = \int z_0(t-\tau)\, d\Psi^{(n-1)}(\tau). \tag{4.29}$$

Suppose that for every real t and every integer j such that $0 \le j \le n-1$,

$$\lim_{\tau\to\pm\infty} z_0^{(j)}(t-\tau)\, \Psi^{(n-1-j)}(\tau) = 0.$$

This condition is automatically satisfied when $\Psi \in \mathscr{S}$. Under these hypotheses, after integrating the latter integral in (4.29) by parts $n-k-1$ times we get

$$\begin{aligned}
\int z_0(t-\tau)\, d\Psi^{(n-1)}(\tau) &= z_0(t-\tau)\, \Psi^{(n-1)}(\tau)\Big|_{-\infty}^{\infty} + \int z_0'(t-\tau)\, \Psi^{(n-1)}(\tau)\, d\tau \\
&= \int z_0'(t-\tau)\, d\Psi^{(n-1-1)}(\tau) = \cdots = \int z_0^{(n-k-1)}(t-\tau)\, d\Psi^{(k)}(\tau) \\
&= z_0^{(n-k-1)}(t-\tau)\, \psi(\tau)\Big|_{-\infty}^{\infty} - \int \psi(\tau)\, d\left(z_0^{(n-k-1)}(t-\tau)\right) \\
&= (-1)^{n-k}\psi(t) - \int \Psi^{(k)}(\tau)\, \varphi_0^{(n-k)}(t-\tau)\, d\tau.
\end{aligned}$$

Integrating the resulting integral by parts k more times, we obtain

$$\begin{aligned}
\int z_0(t-\tau)\, d\Psi^{(n-1)}(\tau) &= (-1)^{n-k}\psi(t) - \int \varphi_0^{(n-k)}(t-\tau)\, d\Psi^{(k-1)}(\tau) \\
= (-1)^{n-k}\psi(t) - &\left(\varphi_0^{(n-k)}(t-\tau)\Psi^{(k-1)}(\tau)\right)\Big|_{-\infty}^{\infty} - \int \varphi_0^{(n-k+1)}(t-\tau)\, \Psi^{(k-1)}(\tau)\, d\tau \\
&= \cdots = (-1)^{n-k}\psi(t) - \int \varphi_0^{(n)}(t-\tau)\, \Psi(\tau)\, d\tau.
\end{aligned}$$

Combining this result with (4.29) and (4.28), we obtain

$$\mathbf{S}\psi(t) = \psi(t) - (-1)^{n-k}\left((-1)^{n-k}\psi(t) - \int \varphi_0^{(n)}(t-\tau)\,\Psi(\tau)\,d\tau\right).$$

Thus, operator (4.22) admits the representation

$$\mathbf{S}\psi(t) = (-1)^{n-k}\int \varphi_0^{(n)}(t-\tau)\,\Psi(\tau)\,d\tau.$$

According to equality (3.18) of Lemma 3, the weight λ_0 of the extremal functional (4.3) of problem (4.1) and the function φ_0 are related by the equality $\lambda_0 = (-1)^n\varphi_0^{(n)}$. Furthermore, the function θ_h has the following parity on the axis:

$$\theta_h(-\eta) = (-1)^k\theta_h(\eta), \quad \eta \in (-\infty, \infty);$$

weight (4.4) has the same property. Based on this, we find

$$\mathbf{S}\psi(t)=(-1)^k\int \lambda_0(t-\tau)\,\Psi(\tau)\,d\tau=\int \lambda_0(\tau-t)\,\Psi(\tau)\,d\tau=\int \lambda_0(\tau)\,\Psi(\tau+t)\,d\tau=\int \widehat{\theta_h}(\tau)\,\Psi(\tau+t)\,d\tau.$$

As a result, we obtain the formula

$$\mathbf{S}\psi(t) = \int \widehat{\theta_h}(\tau)\,\Psi(\tau+t)\,d\tau.$$

Comparing this formula with (2.3), we conclude that under the made assumptions, the operator $\mathbf{S}$ for $N = N(h)$, $h > 0$, can be represented as

$$\mathbf{S}\psi = \mathbf{T}_h\Psi, \quad \Psi^{(k)} = \psi, \quad \psi \in W_1^{n-k}. \tag{4.30}$$

Representation (4.30) was the goal of this section.

5. Main result

The following proposition extends Theorem 3.

Theorem 4. *For all $0 \le k < n$, $m = n-k$, and $N > 0$, the following assertions are true for problems (2.9), (2.11), and (2.1).*

1. *The values of problems (2.9), (2.11), and (2.1) coincide:*
$$F(N) = G(N) = E(N). \tag{5.1}$$
2. *In the class–approximation problem (2.9), the function $\chi_{n-k} \in B_1^{n-k}$ is extremal; that is, it attains the supremum in (2.9):*
$$F(N) = F(\chi_{n-k}, N). \tag{5.2}$$
3. *For the best linear class–approximation problem (2.11), the operator $\mathbf{S}$ given by (4.22) is extremal; i.e., it attains the infimum in (2.11):*
$$G(N) = J(\mathbf{S}).$$

Proof. Indeed, from definitions (2.9) and (2.11) together with relations (4.7) and (4.23), we obtain

$$e(N) = F(\chi_{n-k}, N) \leq F(N) \leq G(N) \leq J(\mathbf{S}) \leq e(N). \tag{5.3}$$

Therefore, all inequalities in (5.3) are in fact equalities. Combining this with equality (4.2) yields all statements of Theorem 4. ☐

Theorem 3 is a special case of Theorem 4.
Theorem 4 implies the following proposition.

Corollary 1. *For $0 \leq k < n$, the following sharp inequality holds on the set W_2^n:*

$$\|\chi_{n-k} - \varphi\|^{\beta}_{\widehat{\mathscr{L}_\infty}} \|\varphi^{(n)}\|_2^{\alpha} \geq \beta^{\beta}\alpha^{\alpha}\mathcal{K}, \quad \varphi \in W_2^n; \tag{5.4}$$

here the function χ_{n-k} is defined by (4.5), and $\mathcal{K}$ denotes the best constant in inequality (2.5), given explicitly in (2.6).

Proof. Let $\varphi \in W_2^n$; denote $N = N(\varphi) = \|\varphi^{(n)}\|_2$. We have

$$\|\chi_{n-k} - \varphi\|_{\widehat{\mathscr{L}_\infty}} \geq \inf\left\{\|\chi_{n-k} - \varphi\|_{\widehat{\mathscr{L}_\infty}} :\ \varphi \in B_2^n(N)\right\} = F(\chi_{n-k}, N).$$

Combining this relation with equalities (5.1), (5.2) and (2.7) yields

$$\|\chi_{n-k} - \varphi\|_{\widehat{\mathscr{L}_\infty}} \geq \beta\alpha^{\alpha/\beta}\mathcal{K}^{1/\beta}N^{-\alpha/\beta},$$

which is equivalent to inequality (5.4). For the function $\varphi_0 \in W_2^n$ with property (4.10), inequality (5.4) turns into equality. Hence the corollary is proved.

We note that an inequality similar to (5.4) appears in [1, Corollary 3]. A result of this form was implicitly contained in the earlier paper of Taikov [29].

Conclusion

In Stechkin's problem (2.1) and the corresponding Kolmogorov inequality (2.5), the norm of the highest-order derivative is not the classical Lebesgue norm. Consequently, the main result of this paper, presented in Theorem 4, is not covered by the results in [1]. Nevertheless, our approach employs certain constructions and ideas from [1]. The proof technique for Theorem 4, however, differs fundamentally from that used to establish the main results in [1].

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