URAL MATHEMATICAL JOURNAL, Vol. 11, No. 2, 2025, pp. 4–20

DOI: 10.15826/umj.2025.2.001

ON OBSERVABILITY CONTROL IN DIFFERENTIAL EQUATIONS[1]

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Abstract: We consider a controlled linear differential equation with constraints as in the author's previous paper. The controller's goal is to displace an initial state of x_0 to a specified final state x_T. An observer, unaware of the system's state vector, attempts to determine x_T by analyzing the vector $y(t)$, which is linked to $x(t)$. Using $y(t)$, the observer constructs a set of possible values for x_T. When specific constraints are used for the controls (or disturbances, from the observer's opinion), this set becomes an ellipsoid, characterized by a set of differential equations. The controller, in turn, aims to achieve its own objectives while simultaneously generating the most challenging signals for the observer. Unlike the previous article of the author not scalar, but two-criterion control observation problem is considered here. It is solved in functional spaces in two ways, without passing to sampling of a system. The solution boils down to determination of finite-dimensional parameters of optimal control from the system of linear algebraic equations. As the third option the problem can be solved also by sampling, but then the solution turns out piecewise-constant. We explore an example to illustrate these concepts.

Keywords: Guaranteed estimation, Information set, Reachable set, Observation control.

1. Introduction and preliminaries

In this paper, we adopt an approach to guaranteed estimation derived from [11] and continue investigations [1, 2, 4]. Many estimation problems in mechanics, economics, biology, and financial mathematics with uncertain disturbances may be considered in the framework of set-membership description of uncertainty, as discussed in [5, 16]. In this work, a controller exploits uncertain disturbances in the system to generate worst-case signals for an observer. Alternatively, it may pursue its own objectives, which remain unknown to the observer. Conversely, the observer employs a minimax state estimation algorithm without awareness of the controller's goals. Such challenges emerge, for instance, in aviation, where planes must perform tasks discreetly. Similarly, they appear in other science problems. Researchers have explored optimization problems related to observation processes in various contexts, as detailed in [8–10, 12, 13].

Here, as in [2], we consider a more general form of the system and constraints than in [1, 4]. Namely, suppose that the dynamics of our partly observed system is described by the equations

$$\dot{x}(t) = A(t)x(t) + b(t)v(t), \quad y(t) = G(t)x(t) + cv(t), \quad t \in [0, T], \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is a state vector, $y(t) \in \mathbb{R}^m$ is an output, $v(t) \in \mathbb{R}^l$ is an uncertain disturbance; $A(\cdot)$, $G(\cdot)$, $b(\cdot)$ are continuous matrices. The observer does not know the initial state x_0 and believes

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[1] The work was performed as part of research conducted in the Ural Mathematical Center with the financial support of the Ministry of Science and Higher Education of the Russian Federation (Agreement number 075-02-2025-1549).

that uncertain functions $v(\cdot) \in L_2^l[0,T]$ in (1.1) are restricted by the integral constraints

$$\left\langle |v(\cdot)|^2 + 2s'(\cdot)x(\cdot) - 2r'(\cdot)v(\cdot) \right\rangle \leq 1. \tag{1.2}$$

Hereinafter,

$$\langle F(\cdot) \rangle = \int_0^T F(u)du, \quad \langle F(\cdot) \rangle_\tau^t = \int_\tau^t F(u)du$$

for vector- or matrix-functions $F(\cdot)$; the symbol $'$ denotes the transposition; elements of vector-functions $s(\cdot)$, $r(\cdot)$ belong to the space $L_2[0,T]$. By $|x|_P^2$, we denote a quadratic form $x'Px$, where the matrix P is such that $P' = P \geq 0$. The matrix c in (1.1), being constant, has a full rank, i.e. $\operatorname{rank}(c) = m \wedge l$, where $m \wedge l = \min\{m, l\}$. For brevity, we will write $\langle F(\cdot) \rangle_\tau^T = \langle F(\cdot) \rangle_\tau$ and $\langle F(\cdot) \rangle_0^t = \langle F(\cdot) \rangle^t$.

First, for convenience, we will simplify constraints (1.2). For this purpose, we make the replacement of variables. Let $x_T^v(t)$ be a solution of (1.1) under the boundary condition $x_T^v(T) = 0$. Then we make a substitution of variables:

$$\begin{gathered} \bar{v}(t) = v(t) - r(t) - \widetilde{v}(t), \quad \bar{x}(t) = x(t) - x_T^r(t) - x_T^{\widetilde{v}}(t), \quad \bar{y}(t) = G(t)\bar{x}(t) + c\bar{v}(t), \\ \widetilde{v}(t) = b'(t)\mathbf{s}(t), \quad \mathbf{s}(t) = \left\langle X'(\cdot,t)s(\cdot) \right\rangle^t. \end{gathered} \tag{1.3}$$

Here $X(u,t)$ is a fundamental matrix, $\partial X(u,t)/\partial u = A(u)X(u,t)$, $X(t,t) = I_n$. In new variables the system becomes

$$\begin{gathered} \dot{\bar{x}} = A(t)\bar{x}(t) + b(t)\bar{v}(t), \quad \bar{x}(T) = x(T), \\ \|\bar{v}(\cdot)\|^2 + 2x(T)'\mathbf{s}(T) + \mathbf{h}(T) \leq 1, \quad \|\bar{v}(\cdot)\|^2 = \left\langle |\bar{v}(\cdot)|^2 \right\rangle, \\ \mathbf{h}(T) = \left\langle 2s'(\cdot)x_T^r(\cdot) - |r(\cdot)|^2 - |\widetilde{v}(\cdot)|^2 \right\rangle. \end{gathered} \tag{1.4}$$

From now on, we omit the bars over variables and deal with system (1.4) because we can always return to initial variables. One can see that the variables x and y from equations (1.3) and (1.4) are bound with each other by means of the function $v(t)$. Let us present the system in an equivalent form. Consider the pseudoinverse matrix c^- of c, [7, 14]. It is known that c^-c is an orthogonal projection onto the subspace

$$\operatorname{im} c' = \left\{ v : v = c'y, \ y \in \mathbb{R}^m \right\}.$$

Introduce the matrix $C_1 = I_l - c^-c$ which is the orthogonal projection onto the null subspace

$$\ker c = \{v : cv = 0\}.$$

Then

$$v(t) = c^-cv(t) + C_1v(t), \quad cv(t) = y(t) - Gx(t).$$

If we introduce a notation

$$\mathbf{b}(\cdot) = b(\cdot)c^-, \quad \mathbf{A}(\cdot) = A(\cdot) - \mathbf{b}(\cdot)G(\cdot), \tag{1.5}$$

and substitute the orthogonal expansion of $v(t)$ into (1.4), then this equation is converted to the following one:

$$\dot{x}(t) = \mathbf{A}(t)x(t) + \mathbf{b}(t)y(t) + b(t)C_1v(t). \tag{1.6}$$

The constraints may be rewritten as

$$\begin{gathered} J_T(v,y) + 2x'(T)\mathbf{s}(T) + \mathbf{h}(T) \leq 1, \quad C = (c^-)'c^-, \\ J_T(v,y) = \left\langle |y(\cdot) - G(\cdot)x(\cdot)|_C^2 + |v(\cdot)|_{C_1}^2 \right\rangle. \end{gathered} \tag{1.7}$$

If $\operatorname{rank}(c) = m < l$, we have $c^- = c'(cc')^{-1}$ and $C = (cc')^{-1}$. If

$$\operatorname{rank}(c) = l \leq m,$$

we obtain $c^- = (c'c)^{-1}c'$ and $C_1 = O_l$, i.e., zero matrix. In the last case we deal with the unique uncertain element x_0. This last case is of no interest to the controller because he knows x_0 and cannot change the signal. Therefore, suppose that $\operatorname{rank}(c) = m < l$. But then we can pass to lower dimension of disturbances according to the following remark.

Remark 1. Since

$$\ker c + \operatorname{im} c' = \mathbb{R}^l, \quad \operatorname{im} C_1 = \ker c, \quad \operatorname{im} c' = \mathbb{R}^m,$$

it follows that $\operatorname{rank} C_1 = l - m$. Using the expansion $C_1 = T\widetilde{C}_1T'$, where T is an orthogonal matrix such that

$$TT' = T'T = I_l,$$

and $\widetilde{C}_1$ is a diagonal matrix with 0 and 1 on the diagonal, we can eliminate m zero columns from $\widetilde{C}_1$ and obtain a matrix $\widetilde{D}_1$. Then $\widetilde{C}_1 = \widetilde{D}_1\widetilde{D}_1'$ and $C_1 = D_1D_1'$, where $D_1 = T\widetilde{D}_1$. Define vector-function $w(t) = D_1'v(t) \in \mathbb{R}^{l-m}$, then the following relations

$$C_1v(t) = D_1w(t), \quad D_1 \in \mathbb{R}^{l\times(l-m)}, \quad \operatorname{rank} D_1 = l - m, \quad D_1'D_1 = I_{l-m},$$

are obtained. Therefore, one can use $D_1w(t)$ in (1.6) and (1.7) instead of $C_1v(t)$.

2. The problem for the observer

We consider the same problem for the observer as in [2]. For this purpose, recall some definitions.

Definition 1. *Let the signal $y(t)$ be generated by (1.1) (or (1.3) if the replacement is fulfilled) with the help of unknown pair $(v^*(\cdot), x_T^*)$ satisfying the constraints. A pair $(v(\cdot), x_T)$ is called* compatible *with the measured signal $y(t)$ on $[0, T]$ if the solution $x(t)$ of equation (1.1) (or (1.4)) and the function $v(t)$ satisfy inequality (1.2) (or (1.7)) and the measurement equation with given $y(t)$.*

Definition 2. *The set $\mathbb{X}_T(y)$ is called the* information set *(shortly IS) if it consists of all vectors $x(T)$ for each of which there exists a generating compatible pair $(v(\cdot), x_T)$ such that the corresponding trajectory $x(t)$ satisfies the boundary condition $x(T) = x_T$.*

Thus, the observer's problem is to find $\mathbb{X}_T(y)$ and to give an analytical description of this set. As it was proved by dynamic programming methods in [3], the IS $\mathbb{X}_T(y)$ under restrictions (1.7) is the set given by the inequality

$$\mathbb{X}_T(y) = \left\{x \in \mathbb{R}^n : x'P(T)x - 2x'(d(T) - \mathbf{s}(T)) + e(T) + \mathbf{h}(T) \leq 1\right\}, \tag{2.1}$$

where the parameters can be found from the differential equations

$$\begin{gathered}
\dot{P}(t) = -P(t)\mathbf{A}(t) - \mathbf{A}'(t)P(t) + G'(t)CG(t) - P(t)b(t)C_1b'(t)P(t),\\
\dot{d}(t) = P(t)\mathbf{b}(t)y(t) - \mathbf{A}'(t)d(t) + G'(t)Cy(t) - P(t)b(t)C_1b'(t)d(t),\\
\dot{e}(t) = 2y'(t)\mathbf{b}'(t)d(t) + |y(t)|_C^2 - |b'(t)d(t)|_{C_1}^2, \quad \text{or}\\
\dot{P}(t) = -P(t)A(t) - A'(t)P(t) - P(t)b(t)b'(t)P(t)\\
+(cb'(t)P(t) + G(t))'C(cb'(t)P(t) + G(t)),\\
\dot{d}(t) = -A'(t)d(t) + (cb'(t)P(t) + G(t))'C(y(t) + cb'(t)d(t)) - P(t)b(t)b'(t)d(t),\\
\dot{e}(t) = |y(t) + cb'(t)d(t)|_C^2 - d'(t)b(t)b'(t)d(t),\\
P(0) = 0, \quad d(0) = 0, \quad e(0) = 0.
\end{gathered} \tag{2.2}$$

The value on the left-hand side of inequality in (2.1) equals

$$\min_{w(\cdot)} J_T(w,y) + 2x'\mathbf{s}(T) + \mathbf{h}(T)$$

under the condition $x(T) = x$. On the other hand, we can rewrite inequality in (2.1) as

$$\begin{gathered} \mathbb{X}_T(y) = \left\{x \in \mathbb{R}^n : |x - x_1(T)|^2_{P(T)} + h(T) + 2x'\mathbf{s}(T) + \mathbf{h}(T) \le 1\right\}, \\ x_1(T) = P^-(T)d(T), \quad h(T) = e(T) - d'(T)P^-(T)d(T), \\ d(T) \in \operatorname{im} P(T), \end{gathered} \tag{2.3}$$

where P^- is the pseudoinverse matrix.

For further, let us emphasize the parameters of IS in another form. First, we minimize the functional

$$J_T(w,y) = \left\langle |y(\cdot) - G(\cdot)x(\cdot)|^2_C + |w(\cdot)|^2 \right\rangle, \quad C = (cc')^{-1}, \tag{2.4}$$

with respect to $w(\cdot)$ under constraints (1.5), (1.6), and $x(T) = x$. For this purpose, we write the solution

$$x(t) = \mathbf{X}(t,T)x - \langle \mathbf{X}(t,\cdot)\left(\mathbf{b}(\cdot)y(\cdot) + b(\cdot)D_1 w(\cdot)\right)\rangle_t\,,$$

where

$$\partial \mathbf{X}(u,t)/\partial u = \mathbf{A}(u)\mathbf{X}(u,t), \quad \mathbf{X}(t,t) = I_n,$$

that is $\mathbf{X}(u,t)$ is a fundamental matrix for equation (1.6). Introduce linear integral operators (see [15])

$$Y_t y = \langle \mathbf{X}(t,\cdot)\mathbf{b}(\cdot)y(\cdot)\rangle_t\,, \quad W_t w = \langle \mathbf{X}(t,\cdot)b(\cdot)D_1 w(\cdot)\rangle_t\,,$$

and obtain functional (2.4) in the form

$$J_T(w,y) = \|y(\cdot) - G(\cdot)\left(\mathbf{X}(\cdot,T)x - Y_\cdot y - W_\cdot w\right)\|^2_C + \|w(\cdot)\|^2. \tag{2.5}$$

In formula (2.5), the quadratic terms with $w(\cdot)$ may be written as

$$\begin{gathered} \left\langle w'(\cdot)\mathbf{K}w(\cdot)\right\rangle = \|w(\cdot)\|^2 + \|G(\cdot)W_\cdot w\|^2_C, \quad \text{where} \\ \mathbf{K}w(t) = w(t) + D_1'b'(t)\left\langle \mathbf{X}'(\cdot,t)G'(\cdot)CG(\cdot)W_\cdot w\right\rangle^t, \quad \text{or} \\ \mathbf{K} = id_{l-m} + W_\cdot^* G'(\cdot)CG(\cdot)W_\cdot = \mathbf{S}_\cdot^*\mathbf{S}_\cdot, \quad \mathbf{S}_t w = c'CG(t)W_t w + D_1 w(t). \end{gathered} \tag{2.6}$$

Here $\mathbf{K}$ is a linear Volterra-type (see [6]), self-adjoint, and coercive operator,

$$\mathbf{K} : L_2^{l-m}[0,T] \to L_2^{l-m}[0,T].$$

Note that the conjugate operator $W_\cdot^*$ does not depend on T in (2.6) because

$$W_t^* f = D_1'b'(t)\left\langle \mathbf{X}'(\cdot,t)f(\cdot)\right\rangle^t;$$

id_{l-m} is the identical operator in $L_2^{l-m}[0,T]$. The minimum of (2.5) with respect to $w(\cdot)$ is reached at the function

$$w_0(\cdot) = -\mathbf{K}^{-1}W_\cdot^* G'(\cdot)C\mathbf{f}_\cdot(x,y), \quad \mathbf{f}_\cdot(x,y) = y(\cdot) - G(\cdot)(\mathbf{X}(\cdot,T)x - Y_\cdot y). \tag{2.7}$$

After substitution of (2.7) into (2.5) we obtain

$$\begin{aligned} \min_w J_T(w,y) &= \left\langle \mathbf{f}_\cdot'(x,y)C\mathbf{f}_\cdot(x,y)\right\rangle - \left\langle \mathbf{f}_\cdot'(x,y)CG(\cdot)W_\cdot\mathbf{K}^{-1}W_\cdot^*G'(\cdot)C\mathbf{f}_\cdot(x,y)\right\rangle \\ &= \left\langle \mathbf{f}_\cdot'(x,y)\varkappa\mathbf{f}_\cdot(x,y)\right\rangle \quad \text{where} \quad \varkappa = C - CG(\cdot)W_\cdot\mathbf{K}^{-1}W_\cdot^*G'(\cdot)C. \end{aligned}$$

Lemma 1. *The operator*

$$\varkappa = C^{1/2}\big(id_m - C^{1/2}G(\cdot)W_{\cdot}\mathbf{K}^{-1}W_{\cdot}^{*}G'(\cdot)C^{1/2}\big)C^{1/2}$$

is reversible and its reverse has the form

$$\varkappa^{-1} = cc' + G(\cdot)W_{\cdot}W_{\cdot}^{*}G'(\cdot). \tag{2.8}$$

Proof. Let $\mathcal{T} = C^{1/2}G(\cdot)W_{\cdot}$, then

$$\mathbf{K} = id_{l-m} + \mathcal{T}^{*}\mathcal{T}.$$

Therefore,

$$id_m - \mathcal{T}(id_{l-m} + \mathcal{T}^{*}\mathcal{T})^{-1}\mathcal{T}^{*} = (id_m + \mathcal{T}\mathcal{T}^{*})^{-1} = C^{-1/2}\left(cc' + G(\cdot)W_{\cdot}W_{\cdot}^{*}G'(\cdot)\right)^{-1}C^{-1/2}.$$

Multiplying this equality by $C^{1/2}$ from the left and the right, we obtain (2.8). ☐

From (2.8), we obtain

$$\varkappa = C^{1/2}\left(id_m + \mathcal{T}\mathcal{T}^{*}\right)^{-1}C^{1/2}.$$

Then

$$\mathbf{K}^{-1} = (id_{l-m} + \mathcal{T}^{*}\mathcal{T})^{-1} = id_{l-m} - \mathcal{T}^{*}(id_m + \mathcal{T}\mathcal{T}^{*})^{-1}\mathcal{T} = id_{l-m} - \mathcal{T}^{*}C^{-1/2}\varkappa C^{-1/2}\mathcal{T}.$$

Lemma 2. *The operators* $\mathbf{K}^{-1}$ *in* (2.6) *and* $\varkappa$ *in* (2.8) *can be calculated via differential equations. They are defined by the formulas*:

$$\begin{gathered}
\varkappa f(t) = C(f(t) - G(t)\mathbf{q}(t)),\\
\dot{\mathbf{q}}(t) = \mathbf{A}(t)\mathbf{q}(t) - b(t)C_1b'(t)\mathbf{p}(t), \quad \mathbf{q}(T) = 0,\\
\dot{\mathbf{p}}(t) = -\mathbf{A}'(t)\mathbf{p}(t) + G'(t)C(f(t) - G(t)\mathbf{q}(t)), \quad \mathbf{p}(0) = 0; \quad \mathbf{p}(t) = -P(t)\mathbf{q}(t) + z(t),\\
\dot{z}(t) = -\left(\mathbf{A}(t) + b(t)C_1b'(t)P(t)\right)'z(t) + G'(t)Cf(t), \quad z(0) = 0;\\
\mathbf{K}^{-1}w(t) = w(t) - D_1'b'(t)\mathbf{q}(t),\\
\dot{\mathbf{q}}(t) = -\mathbf{A}(t)\mathbf{q}(t) + G'(t)CG(t)\mathbf{p}(t), \quad \mathbf{q}(0) = 0,\\
\dot{\mathbf{p}}(t) = \mathbf{A}(t)\mathbf{p}(t) - b(t)D_1(w(t) - D_1'b(t)\mathbf{q}(t)), \quad \mathbf{p}(T) = 0; \quad \mathbf{q}(t) = P(t)\mathbf{p}(t) - z(t),\\
\dot{z}(t) = -\left(\mathbf{A}(t) + b(t)C_1b'(t)P(t)\right)'z(t) - P(t)b(t)D_1w(t), \quad z(0) = 0,
\end{gathered}$$

where $\mathbf{p}(t)$, $\mathbf{q}(t)$ *are auxiliary variables.*

Proof. We consider the equation $\varkappa^{-1}\lambda(t) = f(t)$ and seek its solution $\lambda(\cdot)$ in the form

$$\lambda(\cdot) = C(f(\cdot) - G(\cdot)\mathbf{q}(\cdot)).$$

After substitution in the equation we have the integral relation for $\mathbf{q}(\cdot)$:

$$-\mathbf{q}(\cdot) + WW^{*}G'(\cdot)C(f(\cdot) - G(\cdot)\mathbf{q}(\cdot)) = 0.$$

Introducing the variable

$$\mathbf{p}(t) = \left\langle \mathbf{X}'(\cdot,t)G'(\cdot)C(f(\cdot) - G(\cdot)\mathbf{q}(\cdot))\right\rangle^{t},$$

we obtain the differential equations for $\mathbf{q}(\cdot)$ and $\mathbf{p}(\cdot)$ after differentiation. The arising boundary value problem is solved by the substitution

$$\mathbf{p}(\cdot) = -P(\cdot)\mathbf{q}(\cdot) + z(\cdot),$$

where $P(\cdot)$ is the solution of matrix Riccati equation in (2.2). Similarly, we consider the equation $\mathbf{K}\lambda(t) = w(t)$ and seek a solution $\lambda(\cdot)$ in the form

$$\lambda(\cdot) = w(\cdot) - D_1' b'(\cdot)\mathbf{q}(\cdot).$$

After substitution in the equation, we have the integral relation for $\mathbf{q}(t)$:

$$\mathbf{q}(t) = \left\langle \mathbf{X}'(\cdot,t)G'(\cdot)CG(\cdot)\mathbf{p}(\cdot)\right\rangle^t, \quad \mathbf{p}(\cdot) = W(w(\cdot) - D_1' b'(\cdot)\mathbf{q}(\cdot)).$$

We obtain the corresponding differential equations for $\mathbf{q}(\cdot)$ and $\mathbf{p}(\cdot)$ after differentiation. The arising boundary value problem is solved by the substitution

$$z(\cdot) = P(\cdot)\mathbf{p}(\cdot) - \mathbf{q}(\cdot).$$

□

Corollary 1. *The following equality is true:*

$$\varkappa G(\cdot)\mathbf{X}(\cdot,T) = CG(\cdot)\mathcal{X}(\cdot,T),$$

where $\mathcal{X}(t,T)$ is the fundamental matrix of the differential equation

$$\dot{x}(t) = \big(\mathbf{A}(t) + b(t)C_1 b'(t)P(t)\big)\, x(t).$$

Proof. We have

$$z(t) = \left\langle \mathcal{X}'(\cdot,t)G'(\cdot)CG(\cdot)\mathbf{X}(\cdot,T)\right\rangle^t = P(t)\mathbf{X}(t,T).$$

Therefore,

$$p(t) = P(t)(\mathbf{X}(t,T) - q(t)), \quad q(t) = \left\langle \mathcal{X}(t,\cdot)b(\cdot)C_1 b'(\cdot)P(\cdot)\mathbf{X}(\cdot,T)\right\rangle_t.$$

Thus,

$$\varkappa G(\cdot)\mathbf{X}(\cdot,T) = CG(t)\left(\mathbf{X}(t,T) - \left\langle \mathcal{X}(t,\cdot)b(\cdot)C_1 b'(\cdot)P(\cdot)\mathbf{X}(\cdot,T)\right\rangle_t\right) = CG(t)\mathcal{X}(t,T).$$

□

We can compare the functional relation for $\min_w J_T(w,y)$, equations (2.2), (2.3), and express the parameters $P(T)$, $d(T)$, and $e(T)$ through the operators $\mathbf{K}$ and $\varkappa$. Indeed, we have

$$\begin{aligned} \left\langle \mathbf{X}'(\cdot,T)G'(\cdot)\varkappa G(\cdot)\mathbf{X}(\cdot,T)\right\rangle &= P(T), \\ \left\langle \mathbf{X}'(\cdot,T)G'(\cdot)\varkappa \mathbf{Y}.y\right\rangle &= d(T), \\ \left\langle \mathbf{Y}'_. y\varkappa \mathbf{Y}.y\right\rangle = e(T), &\quad \text{where} \quad \mathbf{Y}.y = y(\cdot) + G(\cdot)Y.y. \end{aligned} \tag{2.9}$$

The complete observability of system (1.6), that is the condition

$$\left\langle \mathbf{X}'(\cdot,t)G'(\cdot)G(\cdot)\mathbf{X}(\cdot,t)\right\rangle^t > 0, \quad \forall t \in (0,T], \tag{2.10}$$

implies the positivity of corresponding matrix $P(t) > 0$, $t > 0$.

Under conditions (2.10), the set (2.1) is a bounded ellipsoid and

$$x_1(t) = P^{-1}(t)d(t), \quad x_2(t) = -P^{-1}(t)\mathbf{s}(t), \quad \hat{x}(t) = x_1(t) + x_2(t).$$

Parameters x_1, x_2, $\hat{x}(t)$, and $h(t) = e(t) - d'(t)P^{-1}(t)d(t)$ from (2.3) satisfy the differential equations

$$\begin{gathered}
\dot{x_1}(t) = \mathbf{A}(t)x_1 + P^{-1}(t)G'(t)C(y(t) - G(t)x_1(t)) + \mathbf{b}(t)y(t) \\
= A(t)x_1(t) + \left(b(t)c' + P^{-1}(t)G'(t)\right) C(y(t) - G(t)x_1(t)), \\
\dot{x_2}(t) = A(t)x_2(t) + \left(b(t)c' + P^{-1}(t)G'(t)\right) C(cb'(t)\mathbf{s}(t) - G(t)x_2(t)) \\
-b(t)b'(t)\mathbf{s}(t) - P^{-1}(t)s(t), \quad \dot{\mathbf{s}}(t) = -A'(t)\mathbf{s}(t) + s(t), \\
\dot{\hat{x}}(t) = A(t)\hat{x}(t) + \left(b(t)c' + P^{-1}(t)G'(t)\right) C(y(t) + cb'(t)\mathbf{s}(t) \\
-G(t)\hat{x}(t)) - b(t)b'(t)\mathbf{s}(t) - P^{-1}(t)s(t), \\
\dot{h}(t) = |y(t) - G(t)x_1(t)|^2_C \,.
\end{gathered} \tag{2.11}$$

There is a problem with initial states $\hat{x}(0)$, $h(0)$ for these equations because of $P(t)$ near 0. Therefore, the observer may use other functional approach in Lemmas 1, 2. Anyway, the observer can build IS $\mathbb{X}_T(y)$.

3. The problem for the controller

Recall that the controller deals with system (1.4), where bars are omitted, and tries to move the initial state x_0 to the final state x_T. It can be done if the minimum of the left-hand side of the inequality in (1.4) is not less than one. This minimum is equal to

$$\begin{gathered}
|x_{0,T}|^2_{\mathcal{P}^-(0)} + 2x'_T\mathbf{s}(T) + \mathbf{h}(T) \le 1, \quad x_{0,T} = X(0,T)x_T - x_0, \\
\mathcal{P}(0) = \left\langle X(0,\cdot)b(\cdot)b'(\cdot)X'(0,\cdot)\right\rangle, \quad x_{0,T} \in \operatorname{im}\mathcal{P}(0), \\
\dot{\mathcal{P}} = A(t)\mathcal{P} + \mathcal{P}A'(t) - b(t)b'(t), \quad \mathcal{P}(T) = 0.
\end{gathered} \tag{3.1}$$

The inequality in (3.1) is necessary and sufficient for vectors x_0, x_T in order to transfer x_0 to x_T by some control. The control action with minimal L^l_2-norm is

$$v^0(\cdot) = b'(\cdot)X'(0,\cdot)\mathcal{P}^-(0)x_{0,T}, \tag{3.2}$$

but this control does not provide the maximal volume of IS $\mathbb{X}_T(y)$ or the worst signal for the observer. Let $w^0(\cdot) = D'_1v^0(\cdot)$. For any control $v(\cdot)$ that transfers x_0 to x_T and generates a signal of $y(\cdot)$ we get the equalities

$$\begin{gathered}
W_0w = \overline{x}_{0,T} - Y_0y, \quad w(\cdot) = D'_1v(\cdot), \quad \overline{x}_{0,T} = \mathbf{X}(0,T)x_T - x_0, \\
\mathbf{f}_t(x_T, y) + G(t)W_tw = cv(t).
\end{gathered} \tag{3.3}$$

As any function from $L^l_2[0,T]$ admits an orthogonal expansion $v^0(\cdot) + v(\cdot)$, $v(\cdot) \in \ker\mathcal{D}_0$, where

$$\mathcal{D}_0v = \langle X(0,\cdot)b(\cdot)v(\cdot)\rangle\,,$$

the function $v(\cdot)$ corresponds to $x_{0,T} = 0$ and generates corresponding signal $y(\cdot)$ such that

$$W_0w = -Y_0y, \quad w(\cdot) = D'_1v(\cdot), \quad \mathbf{f}_{\cdot}(0,y) = \mathbf{Y}_{\cdot}y, \quad \mathbf{f}_{\cdot}(0,y) + G(\cdot)W_{\cdot}w = cv(\cdot). \tag*{(3.3)$'$}$$

In (3.3)$'$, we set $x_0 = x_T = 0$.

The IS $\mathbb{X}_T(y)$ defined in (2.1) may be unbounded, for example, be an epigraph of a parabola. To eliminate this possibility we assume that the property of complete observability (2.10) is valid. Then $\mathbb{X}_T(y)$ is an ellipsoid with the center $\hat{x}(T) = x_1(T) + x_2(T)$. Its volume depends only on the value

$$H(T) = e(T) - |d(T) - \mathbf{s}(T)|^2_{P^{-1}(T)} = h(T) + 2\mathbf{s}'(T)P^{-1}(T)d(T) - |\mathbf{s}(T)|^2_{P^{-1}(T)}. \tag{3.4}$$

The controller wants to minimize the value $H(T)$ that gives him the maximal volume of IS $\mathbb{X}_T(y)$. As the center $\hat{x}(T)$ is the aiming point for the observer, the controller has to strive to maximization of the value $|x(T) - \hat{x}(T)|$ at the same time. Thus, a two-criteria optimization problem is obtained.

To solve this, let us fix a number α and consider the scalar maximization problem with parametric functionals:

$$\begin{gathered}\alpha(1 - H(T)) + (1-\alpha)\,|x_T - \hat{x}(T)| = \alpha - \mathcal{I}_\alpha \to \max_{v(\cdot)} \quad \text{under the constraints}\\ x_{0,T} = \mathcal{D}_0 v = \langle X(0,\cdot)b(\cdot)v(\cdot)\rangle\,, \quad \|v(\cdot)\|^2 + 2x'_T\mathbf{s}(T) + \mathbf{h}(T) \le 1;\\ \mathcal{I}_\alpha = \alpha H(T) - (1-\alpha)|x_T - \hat{x}(T)|, \quad \alpha \in [0,1].\end{gathered} \tag{3.5}$$

This is equivalent to the following problems:

$$\begin{gathered}\mathcal{I}_\alpha \to \min_{v(\cdot)} \quad \text{or} \quad \mathcal{I}_{\alpha,l} \to \min_{v(\cdot),l}, \quad \text{where}\\ \mathcal{I}_{\alpha,l} = \alpha H(T) - (1-\alpha)l'(x_T - \hat{x}(T)), \quad |l| \le 1.\end{gathered} \tag{3.6}$$

3.1. Functional approach to the solution

From (2.3), (2.7), (2.8), (2.9), and (2.10), we see that the values $H(T)$ and $\mathcal{I}_{\alpha,l}$ are quadratic forms with respect to $y(\cdot)$. Consider the terms from (2.3), (3.3), (3.4), and (3.5) in detail. With the help of (1.3) and (1.6), we have

$$\begin{gathered}y(\cdot) = G(\cdot)(X(\cdot,T)x_T - \mathcal{D}.v) + cv(\cdot), \quad \mathcal{D}_t v = \langle X(t,\cdot)b(\cdot)v(\cdot)\rangle_t\,,\\ \mathbf{Y}.y = y(\cdot) + G(\cdot)Y.y = G(\cdot)\mathbf{X}(\cdot,T)x_T + \mathbf{D}.v,\\ \mathbf{D}_t v = cv(t) - G(t)\,\langle\mathbf{X}(t,\cdot)b(\cdot)C_1v(\cdot)\rangle_t\,.\end{gathered} \tag{3.7}$$

Differentiating $X(t,T) + \langle\mathbf{X}(t,\cdot)\mathbf{b}(\cdot)G(\cdot)X(\cdot,T)\rangle_t$ with respect to t, we obtain the equality $X(t,T) + Y_tG(\cdot)X(\cdot,T) = \mathbf{X}(t,T)$. With the help of operator $\mathcal{D}.$, we can rewrite equality (3.2) as

$$v^0(\cdot) = \mathcal{D}_0^- x_{0,T}, \quad \mathcal{D}_0^- = b'(\cdot)X'(0,\cdot)\mathcal{P}^-(0), \quad x_{0,T} \in \operatorname{im}\mathcal{D}_0 = \operatorname{im}\mathcal{P}(0).$$

Here, $\mathcal{D}_0^-$ is a pseudoinverse operator in a Hilbert space (see [14]). This operator can be expressed in explicit form.

We see that $h(T) = \langle g'(\cdot)\mathcal{L}g(\cdot)\rangle \ge 0$, where $g(\cdot) = \mathbf{Y}.y$ and a symmetric and nonnegative operator $\mathcal{L}$ has the form

$$\mathcal{L} = \varkappa - \varkappa G(\cdot)\mathbf{X}(\cdot,T)P^{-1}(T)\mathbf{X}'(\cdot,T)G'(\cdot)\varkappa.$$

Let $f(\cdot) = g(\cdot) - G(\cdot)\mathbf{X}(\cdot,T)P^{-1}(T)\,\langle\mathbf{X}'(\cdot,T)G'(\cdot)\varkappa g(\cdot)\rangle$. Then

$$f(\cdot) \in \mathcal{F} = \left\{f : \left\langle\mathbf{X}'(\cdot,T)G'(\cdot)\varkappa f(\cdot)\right\rangle = 0\right\}, \tag{3.8}$$

where $\mathcal{F}$ is a subspace with finite codimension. Since $\mathcal{L}g(\cdot) = \varkappa f(\cdot)$, we obtain

$$h(T) = \left\langle g'(\cdot)\mathcal{L}g(\cdot)\right\rangle = \left\langle f'(\cdot)\varkappa f(\cdot)\right\rangle$$

and arrive at the assertion.

Proposition 1. *The value $h(T)=0$ if and only if the function $f(\cdot)\in\mathcal{F}$ from (3.8) in the expansion $g(\cdot)=f(\cdot)+G(\cdot)\mathbf{X}(\cdot,T)p$ equals zero. In turn, this is equivalent to the following equalities:*

$$\mathbf{Y}.y=G(\cdot)\mathbf{X}(\cdot,T)P^{-1}(T)\left\langle\mathbf{X}'(\cdot,T)G'(\cdot)\varkappa\mathbf{Y}.y\right\rangle,\quad\text{or}$$
$$\mathbf{D}.v=G(\cdot)\mathbf{X}(\cdot,T)P^{-1}(T)\left\langle\mathbf{X}'(\cdot,T)G'(\cdot)\varkappa\mathbf{D}.v\right\rangle.$$

Here $\mathbf{Y}.y-\mathbf{D}.v=G(\cdot)\mathbf{X}(\cdot,T)x_T$ from (3.7).

We solve problem (3.6) with constraints by Kuhn–Tucker's theorem. Note that the equality in (3.1) leads us to unique control in (3.2) and the controller cannot choose any other control action. Therefore, suppose that the strict inequality in (3.1) is fulfilled. Then

$$|x_{0,T}|^2_{\mathcal{P}^-(0)}=\|v^0(\cdot)\|^2<1-2x'_T\mathbf{s}(T)-\mathbf{h}(T).$$

Remark 2. The operator $\mathbf{Y}.$ is invertible. Indeed, if

$$\mathbf{Y}.y=y(\cdot)+G(\cdot)Y.y(\cdot)=g(\cdot),$$

then

$$y(\cdot)=g(\cdot)-G(\cdot)\mathbf{p}(\cdot),\quad\text{where}\quad\mathbf{p}(t)=Y_t(g(\cdot)-G(\cdot)\mathbf{p}(\cdot)).$$

We differentiate this equality and come to the differential equation

$$\dot{\mathbf{p}}(t)=\mathbf{A}(t)\mathbf{p}(t)-\mathbf{b}(t)(g(t)-G(t)\mathbf{p}(t)),\quad\mathbf{p}(T)=0,\quad\text{or}\quad\dot{\mathbf{p}}(t)=A(t)\mathbf{p}(t)-b(t)c'Cg(t).$$

Let $\mathbf{Y}.y^0=g^0(\cdot)$ and $\mathbf{Y}.y=g(\cdot)$, where $y(\cdot)$ is generated by the function $v(\cdot)$ from $(3.3)'$. We see that

$$\mathcal{I}_{\alpha,l}=\alpha\left\langle(g^0(\cdot)+g(\cdot))'\mathcal{L}(g^0(\cdot)+g(\cdot))\right\rangle$$
$$+(2\alpha\mathbf{s}(T)+(1-\alpha)l)'P^{-1}(T)\left\langle\mathbf{X}'(\cdot,T)G'(\cdot)\varkappa(g^0(\cdot)+g(\cdot))\right\rangle+\bar{J},$$

where $\bar{J}$ does not depend on $y(\cdot)$. Let us seek optimal $v(\cdot)$ from $(3.3)'$ in the form

$$v(\cdot)=c'C\overline{f}(\cdot)+D_1w(\cdot),\quad\text{where}\quad\overline{f}(\cdot)=\mathbf{Y}.y+G(\cdot)W.w=cv(\cdot).$$

Now, let $g(\cdot)=\mathbf{Y}.y=\mathbf{f}.(0,y)$ be equal to an expansion $f(\cdot)+G(\cdot)\mathbf{X}(\cdot,T)p$, where $f(\cdot)\in\mathcal{F}$ and $p\in\mathbb{R}^n$. Then

$$\mathcal{I}_{\alpha,l}=\alpha\left\langle f'(\cdot)\varkappa f(\cdot)\right\rangle+(2\alpha\mathbf{s}(T)+(1-\alpha)l)'p+2\alpha\left\langle f'(\cdot)\varkappa g^0(\cdot)\right\rangle+\widetilde{J},$$

where $\widetilde{J}$ does not depend on $f(\cdot)$ and p. Substituting $v(\cdot)$ into inequality (3.6), we obtain the constraint

$$\|f(\cdot)+G(\cdot)\left(\mathbf{X}(\cdot,T)p+W.w\right)\|^2_C+\|w(\cdot)\|^2+\|v^0(\cdot)\|^2+2x'_T\mathbf{s}(T)+\mathbf{h}(T)\leq1.$$

By (2.5), (2.6), (2.7), and Lemma 1, we see that the minimum of the left-hand side of the constraint with respect to $w(\cdot)$ equals

$$\|u(\cdot)\|^2_\varkappa+\|v^0(\cdot)\|^2+2x'_T\mathbf{s}(T)+\mathbf{h}(T),\quad\text{with optimal function}$$
$$w_0(\cdot)=-\mathbf{K}^{-1}W^*_.G'(\cdot)Cu(\cdot),\quad\text{where}\quad u(\cdot)=f(\cdot)+G(\cdot)\mathbf{X}(\cdot,T)p.$$

In order to minimize $\mathcal{I}_{\alpha,l}$, we compose the Lagrange function, taking into account the relations in (3.7):

$$L = \mathcal{I}_{\alpha,l} + k\|u(\cdot)\|^2_{\varkappa} - 2z'\mathcal{D}_0 v - 2q'\left(\left\langle \mathbf{X}'(\cdot,T)G'(\cdot)\varkappa u(\cdot)\right\rangle - P(T)p\right),$$

where $k \geq 0$, z, $q \in \mathbb{R}^n$ are Lagrange multipliers. We use the expansion for $v(\cdot)$, the function $w_0(\cdot)$, and the operator $\mathbf{S}_{\cdot}$ from (2.6). Therefore, we obtain

$$v(\cdot) = c'Cu(\cdot) + \mathbf{S}_{\cdot}w_0 = \mathcal{U}_{\cdot}u(\cdot), \quad \text{where} \quad \mathcal{U}_{\cdot} = \left(c' - \mathbf{S}_{\cdot}\mathbf{K}^{-1}W^*_{\cdot}G'(\cdot)\right)C.$$

After differentiation of L with respect to $u(\cdot)$ and p, we have the equations

$$\begin{gathered}
(\alpha + k)\varkappa u(\cdot) + \alpha\varkappa g^0(\cdot) - \mathcal{U}^*_{\cdot}\mathcal{D}^*_0 z - \varkappa G(\cdot)\mathbf{X}(\cdot,T)q = 0;\\
\alpha\mathbf{s}(T) + (1-\alpha)l/2 - \alpha P(T)p - \alpha\left\langle \mathbf{X}'(\cdot,T)G'(\cdot)\varkappa g^0(\cdot)\right\rangle + P(T)q = 0.
\end{gathered} \tag{3.9}$$

These equations allow us to express the function $u(\cdot)$ in terms of q and z:

$$u(\cdot) = (\alpha + k)^{-1}\left(G(\cdot)\mathbf{X}(\cdot,T)q - \alpha g^0(\cdot) + \varkappa^{-1}\mathcal{U}^*_{\cdot}\mathcal{D}^*_0 z\right).$$

Then, the substitution of $u(\cdot)$ into the equality conditions gives the equations

$$\begin{gathered}
(\alpha + k)P(T)p + \alpha\left\langle \mathbf{X}'(\cdot,T)G'(\cdot)\varkappa g^0(\cdot)\right\rangle - \left\langle \mathbf{X}'(\cdot,T)G'(\cdot)\varkappa\mathcal{U}^*_{\cdot}\mathcal{D}^*_0\right\rangle z = P(T)q;\\
\mathcal{D}_0\mathcal{U}_{\cdot}\left(\varkappa^{-1}\mathcal{U}^*_{\cdot}\mathcal{D}^*_0 z + G(\cdot)\mathbf{X}(\cdot,T)q - \alpha g^0(\cdot)\right) = 0.
\end{gathered} \tag{3.10}$$

We obtain a system of linear algebraic equations for p, q, and z; moreover,

$$\begin{gathered}
z = \left(D_0\mathcal{U}_{\cdot}\varkappa^{-1}\mathcal{U}^*_{\cdot}\mathcal{D}^*_0\right)^{-1} D_0\mathcal{U}_{\cdot}(\alpha g^0(\cdot) - G(\cdot)\mathbf{X}(\cdot,T)q),\\
q = \alpha p + P^{-1}(T)\left(\alpha\left\langle \mathbf{X}'(\cdot,T)G'(\cdot)\varkappa g^0(\cdot)\right\rangle + (\alpha-1)l/2 - \alpha\mathbf{s}(T)\right).
\end{gathered}$$

After substitution of z and q into the first equation in (3.10), we obtain the equation for p:

$$\begin{gathered}
kP(T)p - \left\langle \mathcal{X}'(\cdot,T)G'(\cdot)C\mathcal{U}^*_{\cdot}\mathcal{D}^*_0\right\rangle \left(D_0\mathcal{U}_{\cdot}\varkappa^{-1}\mathcal{U}^*_{\cdot}\mathcal{D}^*_0\right)^{-1} D_0\mathcal{U}_{\cdot}\big(\alpha g^0(\cdot)\\
-G(\cdot)\mathbf{X}(\cdot,T)\left(\alpha p - P^{-1}(T)\left(\alpha\left\langle \mathcal{X}'(\cdot,T)G'(\cdot)Cg^0(\cdot)\right\rangle + (\alpha-1)l/2 - \alpha\mathbf{s}(T)\right)\right)\big)\\
= (\alpha-1)l/2 - \alpha\mathbf{s}(T).
\end{gathered} \tag{3.11}$$

Under fixed l and α, the solution vector $p(k)$ depends on k. After substitution, we have $u(k)$ and $v(k)$. Note that some of the parameters in (3.11) can be simplified. Indeed,

$$\begin{gathered}
\varkappa^{-1}\mathcal{U}^*_{\cdot} = \left(C^{-1} + G(\cdot)W_{\cdot}W^*_{\cdot}G'(\cdot)\right)C\left(c - G(\cdot)W_{\cdot}\mathbf{K}^{-1}(D_1' + W^*_{\cdot}G'(\cdot)Cc)\right)\\
= c - G(\cdot)W_{\cdot}\mathbf{K}^{-1}D_1' - G(\cdot)W_{\cdot}\mathbf{K}^{-1}W^*_{\cdot}G'(\cdot)Cc + G(\cdot)W_{\cdot}W^*_{\cdot}G'(\cdot)Cc\\
-G(\cdot)W_{\cdot}W^*_{\cdot}G'(\cdot)CG(\cdot)W_{\cdot}\mathbf{K}^{-1}D_1' - G(\cdot)W_{\cdot}(K - id_{l-m})K^{-1}W^*_{\cdot}G'(\cdot)Cc = c - G(\cdot)W_{\cdot}D_1'.
\end{gathered}$$

Further,

$$\begin{gathered}
\mathcal{U}_{\cdot}\varkappa^{-1}\mathcal{U}^*_{\cdot} = \left(c'C - (c'CG(\cdot)W_{\cdot} + D_1)K^{-1}W^*_{\cdot}G'(\cdot)C\right)(c - G(\cdot)W_{\cdot}D_1') = c'Cc\\
-c'CG(\cdot)W_{\cdot}D_1' + (c'CG(\cdot)W_{\cdot} + D_1)K^{-1}(K - id_{l-m})D_1' = id_l - \mathbf{S}_{\cdot}\mathbf{K}^{-1}\mathbf{S}^*_{\cdot}.
\end{gathered}$$

Now, let us formulate the main conclusion for the controller.

Theorem 1. *Let* $v_{0,l}(\cdot)$ *be the solution to* (3.6), (3.9), (3.10), *and* (3.11) *for* $k = 0$ (*i.e., without the norm constraint*). *If*

$$\|v_{0,l}(\cdot)\|^2 + 2x_T'\mathbf{s}(T) + \mathbf{h}(T) + |x_{0,T}|^2_{\mathcal{P}^-(0)} \le 1,$$

then the general solution $v_l(\cdot)$ *of problem in* (3.6) *equals* $v_{0,l}(\cdot) + v^0(\cdot)$*; otherwise,* $v_l(\cdot) = v_{k,l}(\cdot) + v^0(\cdot)$*, where* $k > 0$ *can be found from the nonlinear equation*

$$\|v_{k,l}(\cdot)\|^2 + 2x_T'\mathbf{s}(T) + \mathbf{h}(T) + |x_{0,T}|^2_{\mathcal{P}^-(0)} = 1.$$

The parameters of the linear algebraic equations for $p(k)$*,* $z(k)$*, and* $q(k)$ *can be calculated in advance. Besides, the equality*

$$\min_{v(\cdot)} \mathcal{I}_\alpha = \min_{|l|\le 1} \mathcal{I}_{\alpha,l}(v_l(\cdot))$$

must be satisfied.

3.2. The solution via differential equations

There is another way for the controller. Since the inequality in (3.1) is strict, there exists a small $\beta > 0$ such that the inequality

$$\|v(\cdot)\|^2 \le 1 - 2x_T'\mathbf{s}(T) - \mathbf{h}(T) - \beta|x_0|^2 \tag{3.12}$$

is valid for $v(\cdot) = v^0(\cdot)$ from (3.2). Considering this inequality as a new constraint on disturbances and x_0, we can use equations (2.11) with zero initial data and $P(0) = \beta I_n$. We can write

$$\begin{aligned} x_{1,t}u = \langle X(t,\cdot)Pb(\cdot)Cu(\cdot)\rangle^t, \quad Pb(\cdot) = b(\cdot)c' + P^{-1}(\cdot)G'(\cdot), \\ y(t) = G(t)x_{1,t}u + u(t) = U_t u, \quad \mathbf{f}_t(x_T, y) + G(t)W_t w = f(t). \end{aligned} \tag{3.13}$$

Remark 3. The operator $U_\cdot$ is invertible. Indeed, if

$$U_\cdot u = u(\cdot) + G(\cdot)x_{1,\cdot}u = y(\cdot),$$

then

$$u(\cdot) = y(\cdot) - G(\cdot)\mathbf{p}(\cdot), \quad \text{where} \quad \mathbf{p}(t) = \langle X(t,\cdot)Pb(\cdot)C(y(\cdot) - G(\cdot)\mathbf{p}(\cdot))\rangle^t.$$

We differentiate this equality and arrive at the differential equation

$$\dot{\mathbf{p}}(t) = A(t)\mathbf{p}(t) + Pb(t)C(y(t) - G(t)\mathbf{p}(t)), \quad \mathbf{p}(T) = 0.$$

Using the orthogonal expansion $v(\cdot) = c'Cf(\cdot) + D_1w(\cdot)$, we reduce the controller's problem to the following:

$$\begin{gathered} \alpha\big(\|u(\cdot)\|_C^2 + 2\mathbf{s}'(T)x_{1,T}u - |\mathbf{s}(T)|^2_{P^{-1}(T)} + \mathbf{h}(T)\big) - (1-\alpha)l'\left(x_T - x_{1,T}u - x_2(T)\right) \to \min_{u(\cdot),w(\cdot)} \\ \text{with constraints} \quad x_{0,T} = \mathcal{D}_0 v(\cdot), \\ \|u(\cdot)\|_C^2 + 2\mathbf{s}'(T)x_{1,T}u - |\mathbf{s}(T)|^2_{P^{-1}(T)} + \mathbf{h}(T) \le 1, \\ \|w(\cdot)\|^2 + \|f(\cdot)\|_C^2 + 2x_T'\mathbf{s}(T) + \mathbf{h}(T) \le 1 - \beta|x_0|^2, \quad |l| \le 1. \end{gathered} \tag{3.14}$$

Problem (3.14) can be solved by Kuhn–Tucker's theorem as before. We take the second-to-last row in (3.14) as the inequality constraint and assume that

$$v(\cdot) = \mathcal{U}_{\cdot}(\mathbf{Y}_{\cdot}y + G(\cdot)\mathbf{X}(\cdot,T)x_T).$$

Similar to the previous Subsection 3.1 and to (2.7), we use

$$w_0(\cdot) = -\mathbf{K}^{-1}W_{\cdot}^{*}G'(\cdot)C\mathbf{f}_{\cdot}(x_T, y).$$

Compose the Lagrange function

$$L = \alpha\|u(\cdot)\|_C^2 + (2(\alpha+k)\mathbf{s}(T) + (1-\alpha)l)'\, x_{1,T}u - 2z'\mathcal{D}_0\mathcal{U}_{\cdot}\mathbf{Y}_{\cdot}U_{\cdot}u + k\|u(\cdot)|_C^2,$$

where z, $k \geq 0$ are Lagrange multipliers. Further, the solution proceeds in a similar way to that in the previous section. After differentiation of L with respect to $u(\cdot)$, we have the equation

$$(\alpha+k)Cu(\cdot) + CPb'(\cdot)X'(T,\cdot)((\alpha+k)\mathbf{s}(T) + (1-\alpha)l/2) - U_{\cdot}^{*}\mathbf{Y}_{\cdot}^{*}\mathcal{U}_{\cdot}^{*}\mathcal{D}_0^{*}z = 0. \tag{3.15}$$

Substitution of $u(\cdot)$ into the equality conditions gives a linear algebraic equation for z:

$$\begin{aligned}\mathcal{D}_0\mathcal{U}_{\cdot}\mathbf{Y}_{\cdot}U_{\cdot}\left(-Pb'(\cdot)X'(T,\cdot)((\alpha+k)\mathbf{s}(T) + (1-\alpha)l/2) + cc'U_{\cdot}^{*}\mathbf{Y}_{\cdot}^{*}\mathcal{U}_{\cdot}^{*}\mathcal{D}_0^{*}z\right)\\ = (x_{0,T} - \mathcal{D}_0\mathcal{U}_{\cdot}G(\cdot)\mathbf{X}(\cdot,T)x_T)\,(\alpha+k).\end{aligned} \tag{3.16}$$

Now we formulate the conclusion.

Theorem 2. *Choose a small $\beta > 0$ such that inequality (3.12) is valid for $v(\cdot) = v^0(\cdot)$ and solve the problem for the observer with this constraint. Then the controller can use the function $u(\cdot)$ from (3.13) in order to solve his problem (3.14) via (3.15) and (3.16). The optimal controller's function is*

$$v_l(\cdot) = \mathcal{U}_{\cdot}(\mathbf{Y}_{\cdot}U_{\cdot}u_l + G(\cdot)\mathbf{X}(\cdot,T)x_T)),$$

where $u_l(\cdot) = u_{0,l}(\cdot)$ for $k = 0$ provided that

$$\|u_{0,l}(\cdot)\|_C^2 + 2\mathbf{s}'(T)x_{1,T}u_{0,l} - |\mathbf{s}(T)|_{P^{-1}(T)}^2 + \mathbf{h}(T) \leq 1.$$

Otherwise, $u_l(\cdot) = u_{k,l}(\cdot)$, where $k > 0$ can be found from the nonlinear equation

$$\|u_{k,l}(\cdot)\|_C^2 + 2\mathbf{s}'(T)x_{1,T}u_{k,l} - |\mathbf{s}(T)|_{P^{-1}(T)}^2 + \mathbf{h}(T) = 1.$$

The parameters of the linear algebraic equation for $z(k)$ can be calculated in advance. Besides, the equality

$$\min_{v(\cdot)} \mathcal{I}_{\alpha} = \min_{|l|\leq 1} \mathcal{I}_{\alpha,l}(v_l(\cdot))$$

must be satisfied.

4. Numerical algorithm for the controller

Despite the fact that coefficients of the algebraic equations in the previous section can be calculated in advance, we may sometimes give a simpler solution for the controller. Here we obtain a piecewise-constant optimal control. For simplicity, let $s(\cdot) = 0$, $r(\cdot) = 0$ in (1.2)–(1.4). Choose a grid $0 = t_0 < t_1 < \cdots < t_N = T$, $t_k - t_{k-1} = \delta$, and suppose $v(t) \equiv v_k$ on $[t_{k-1}, t_k]$. Then we arrive at the following discrete-time equations as in [2]:

$$\begin{gathered} x_k = A_k x_{k-1} + b_k v_k, \quad y_k = G_k x_{k-1} + c_k v_k, \quad k \in 1:N, \quad \text{where} \\ A_k = X(t_k, t_{k-1}), \quad b_k = \langle X(\cdot, t_{k-1}) b(\cdot) \rangle_{t_{k-1}}^{t_k}, \quad G_k = \langle G(\cdot) X(\cdot, t_{k-1}) \rangle_{t_{k-1}}^{t_k}, \\ y_k = \langle y(\cdot) \rangle_{t_{k-1}}^{t_k}, \quad c_k = ch + \left\langle G(\bullet) \langle X(\cdot, t_{k-1}) b(\cdot) \rangle_{t_{k-1}}^{\bullet} \right\rangle_{t_{k-1}}^{t_k}. \end{gathered} \tag{4.1}$$

In this section, we introduce the notation $\mathbf{x} = [x_0; \ldots; x_{N-1}] \in \mathbb{R}^{nN}$, $\mathbf{v} = [v_1; \ldots; v_N] \in \mathbb{R}^{lN}$, $\mathbf{y} = [y_1; \ldots; y_N] \in \mathbb{R}^{mN}$, and others below. Then

$$\mathbf{x} = \mathbf{A} x_T - \mathbf{B}\mathbf{v}, \quad \mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{c}\mathbf{v}$$

for appropriate matrices. We set

$$\mathbf{C} = (\mathbf{c}\mathbf{c}')^{-1}, \quad \mathbf{C}_1 = I_{lN} - \mathbf{c}'\mathbf{C}\mathbf{c}.$$

Therefore,

$$\mathbf{v} = \mathbf{c}'\mathbf{C}(\mathbf{y} - \mathbf{G}\mathbf{x}) + \mathbf{D}_1\mathbf{w}, \quad \text{where} \quad \mathbf{C}_1 = \mathbf{D}_1\mathbf{D}_1', \quad \mathbf{D}_1'\mathbf{D}_1 = I_{mN}, \quad \mathbf{w} = \mathbf{D}_1'\mathbf{v}.$$

After substitution, we have

$$\mathbf{x} = -\mathbf{b}\mathbf{B}(\mathbf{c}'\mathbf{C}\mathbf{y} + \mathbf{D}_1\mathbf{w}) + \mathbf{b}\mathbf{A}x_T, \quad \mathbf{b} = (I_{nN} - \mathbf{B}\mathbf{c}'\mathbf{C}\mathbf{G})^{-1}.$$

Finally, we come to the inequality

$$\begin{gathered} |\mathbf{v}|^2 = |\mathbf{y} - \mathbf{G}\mathbf{x}|_{\mathbf{C}}^2 + |\mathbf{w}|^2 = |\mathbf{Y}\mathbf{y} - \mathbf{T}x_T + \mathbf{W}\mathbf{w}|_{\mathbf{C}}^2 + |\mathbf{w}|^2 \le 1/\delta, \quad \text{where} \\ \mathbf{Y} = I_{mN} + \mathbf{G}\mathbf{b}\mathbf{B}\mathbf{c}'\mathbf{C}, \quad \mathbf{T} = \mathbf{G}\mathbf{b}\mathbf{A}, \quad \mathbf{W} = \mathbf{G}\mathbf{b}\mathbf{B}\mathbf{D}_1. \end{gathered} \tag{4.2}$$

The minimum in (4.2) with respect to $\mathbf{w}$ equals

$$\begin{gathered} |\mathbf{Y}\mathbf{y} - \mathbf{T}x_T|_{\varkappa}^2, \quad \text{where} \quad \varkappa = \mathbf{C} - \mathbf{C}\mathbf{W}\mathbf{K}^{-1}\mathbf{W}'\mathbf{C}, \\ \mathbf{K} = I_{mN} + \mathbf{W}'\mathbf{C}\mathbf{W}, \quad \mathbf{w}_0 = -\mathbf{K}^{-1}\mathbf{W}'\mathbf{C}\left(\mathbf{Y}\mathbf{y} - \mathbf{T}x_T\right). \end{gathered}$$

Thus, we obtain the discrete-time IS in the form

$$\begin{gathered} \mathbb{X}_T(y) = \left\{ x \in \mathbb{R}^n : |x - \hat{x}|_P^2 + h \le 1/\delta \right\}, \quad P = \mathbf{T}'\varkappa\mathbf{T}, \quad \hat{x} = P^{-1}d, \\ d = \mathbf{T}'\varkappa\mathbf{Y}\mathbf{y}, \quad h = \mathbf{y}'\mathbf{Y}'\varkappa\mathbf{Y}\mathbf{y} - d'P^{-1}d. \end{gathered}$$

The transition from x_0 to x_T is performed by equation

$$x_T = \bar{A}x_0 + \mathbf{D}\mathbf{v},$$

where $\bar{A}$ and $\mathbf{D}$ are appropriate matrices composed from (4.1). For fixed x_0 and x_T, we define

$$\mathbf{v}^0 = \mathbf{D}^-(x_T - \bar{A}x_0)$$

and try to find $\mathbf{v}$ such that

$$\begin{gathered} \mathbf{x} = -\mathbf{B}\mathbf{v}, \quad \mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{c}\mathbf{v}, \quad \mathbf{D}\mathbf{v} = 0, \quad |\mathbf{v}|^2 + |\mathbf{v}^0|^2 \le 1/\delta, \\ \mathcal{I}_{\alpha,l} = \alpha(\mathbf{y} + \mathbf{y}^0)'\mathbf{Y}'\mathcal{L}\mathbf{Y}((\mathbf{y} + \mathbf{y}^0) - (1 - \alpha)l(x_T - P^{-1}\mathbf{T}'\varkappa\mathbf{Y}(\mathbf{y} + \mathbf{y}^0)) \to \min_{\mathbf{y}}, \\ \text{where} \quad \mathcal{L} = \varkappa - \varkappa\mathbf{T}P^{-1}\mathbf{T}'\varkappa. \end{gathered}$$

Here $\mathbf{y}^0$ is generated by x_T and $\mathbf{v}^0$. Using

$$\mathbf{w}_0 = -\mathbf{K}^{-1}\mathbf{W}'\mathbf{C}(\mathbf{Y}\mathbf{y} - \mathbf{T}x_T),$$

we can get

$$\mathbf{v} = \mathbf{V}(\mathbf{Y}\mathbf{y} - \mathbf{T}x_T), \quad \mathbf{V} = \mathbf{c}'\mathbf{C} - (\mathbf{c}'\mathbf{C}\mathbf{W} + \mathbf{D}_1)\mathbf{K}^{-1}\mathbf{W}'\mathbf{C}, \quad \mathbf{V}'\mathbf{V} = \varkappa.$$

Denote $\mathbf{Y}\mathbf{y}$ by $\mathbf{g}$ and $\mathbf{Y}\mathbf{y}^0$ by $\mathbf{g}^0$. Now, let $\mathbf{g}$ be equal to an expansion $f + \mathbf{T}p$, where

$$f \in \mathcal{F} = \{f : \mathbf{T}'\varkappa f = 0\}, \quad p \in \mathbb{R}^n.$$

Then

$$\begin{aligned} \mathcal{I}_{\alpha,l} &= \alpha(\mathbf{g}'\varkappa\mathbf{g} - p'Pp + 2(\mathbf{g} - \mathbf{T}p)'\varkappa\mathbf{g}^0) + (1-\alpha)l'p + \bar{J}_{\alpha,l}, \\ \bar{J}_{\alpha,l} &= \alpha\mathbf{g}^{0'}\mathcal{L}\mathbf{g}^0 - (1-\alpha)l'(x_T - P^{-1}\mathbf{T}'\varkappa\mathbf{g}^0). \end{aligned} \tag{4.3}$$

As usual, we compose the Lagrange function and obtain algebraic equations after differentiation:

$$\begin{gathered} (\alpha + k)\mathbf{g} = \varkappa^{-1}\mathbf{V}'\mathbf{D}'z + \mathbf{T}j - \alpha\mathbf{g}^0, \\ -\alpha Pp + l(1-\alpha)/2 - \alpha\mathbf{T}'\varkappa\mathbf{g}^0 + Pj = 0, \\ (\alpha + k)Pp = \mathbf{T}'\mathbf{V}'\mathbf{D}'z + Pj - \alpha\mathbf{T}'\varkappa\mathbf{g}^0, \\ \mathbf{D}\mathbf{V}(\varkappa^{-1}\mathbf{V}'\mathbf{D}'z + \mathbf{T}j - \alpha\mathbf{g}^0) = 0. \end{gathered}$$

Here k, z, j are Lagrange multipliers. The selection of z and j is as follows:

$$\begin{gathered} z = (\mathbf{T}'\mathbf{V}'\mathbf{D}')^{-1}(l(1-\alpha)/2 + kPp), \\ j = \alpha P + P^{-1}(l(\alpha - 1)/2 + \alpha\mathbf{T}'\varkappa\mathbf{g}^0), \end{gathered}$$

and substituting them into the equations above gives us the equation for p:

$$\begin{gathered} \mathbf{D}\mathbf{V}\varkappa^{-1}\mathbf{V}'\mathbf{D}'(\mathbf{T}'\mathbf{V}'\mathbf{D}')^{-1}((1-\alpha)/2 + kPp) \\ +\mathbf{D}\mathbf{V}\mathbf{T}(\alpha p + P^{-1}(l(\alpha - 1)/2 + \alpha\mathbf{T}'\varkappa\mathbf{g}^0)) - \alpha\mathbf{D}\mathbf{V}\mathbf{g}^0 = 0. \end{gathered}$$

For a fixed α, the solution vector p depends on k and l. After substitution we have $z(k,l)$, $j(l)$, and $g(k,l)$. Finally, we have to minimize the expression with respect to l.

5. Test numerical example

Consider the movement of a material particle along the straight line subjected to perturbations $v^1(t)$ and $v^2(t)$:

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = v^1(t) + v^2(t), \quad t \in [0, T].$$

Let the perturbation v^2 influence the observation equation as well:

$$y(t) = x^1(t) + v^2(t).$$

These equations lead to a system of the form (1.6):

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = -x^1 + y(t) + v^1(t).$$

The vector function $v(t)$ is constrained here by the integrated restriction (1.2) with $r = 0$ and $s = 0$. Constant matrices have the form

$$G = [1\ 0], \quad c = [0\ 1], \quad C = 1, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Besides, we have

$$w(t) = D_1' v(t) = v^1(t), \quad \mathbf{X}(t,T) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

Equations (2.2) and (2.11) have the form

$$\begin{gathered}
\dot{P}(t) = -P(t)\mathbf{A} - \mathbf{A}'P(t) + G'CG - P(t)bC_1b'P(t), \\
\dot{d}(t) = P(t)\mathbf{b}y(t) - \mathbf{A}'d(t) + G'Cy(t) - P(t)bC_1b'd(t), \\
\dot{e}(t) = 2y'(t)\mathbf{b}'d(t) + |y(t)|_C^2 - |b'(t)d(t)|_{C_1}^2, \\
P(0) = 0, \quad d(0) = 0, \quad e(0) = 0, \quad \hat{x}(t) = P^{-1}(t)d(t), \\
\dot{\hat{x}}(t) = A\hat{x}(t) + \left(bc' + P^{-1}(t)G'\right) C(y(t) - G\hat{x}(t)), \\
\dot{h}(t) = |y(t) - G(t)\hat{x}(t)|_C^2, \quad h(t) = e(t) - d'(t)P^{-1}(t)d(t).
\end{gathered}$$

In (3.1) and (3.2), we have

$$|x_{0,T}|^2_{\mathcal{P}^{-1}(0)} \le 1, \quad \mathcal{P}(0) = \begin{bmatrix} T^3/3 & -T^2/2 \\ -T^2/2 & T \end{bmatrix}, \quad v^0(\cdot) = b'X'(0,\cdot)\mathcal{P}^{-1}(0)x_{0,T}.$$

Now, we turn to system (4.1) with parameters $\delta = 0.1$, $T = 6$, and $N = 60$ and obtain (in the notation of Section 4)

$$G = [\delta\ \delta^2/2], \quad b = \begin{bmatrix} \delta^2 & \delta^2 \\ \delta & \delta \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \delta \\ 0 & 0 \end{bmatrix}, \quad c = [\delta^3/3\ \delta + \delta^3/3], \quad \mathbf{C} = (cc')^{-1}I_N.$$

The constraints on $\mathbf{v}$ have the form

$$|\mathbf{v}|^2 \le 1/\delta.$$

Suppose that $x_0 = [1; 0]$, $x_T = [8; 2]$. Then the control action

$$\mathbf{v}^0 = \mathbf{D}^-(x_T - \bar{A}x_0)$$

with minimal squared norm $|\mathbf{v}^0|^2 = 3.6112$ gives the trajectory in Fig. 1.

In this figure, the large ellipse is the reachable set from x_0, and the small ellipse is the IS. We see that the final point x_T is too close to the center of the IS. This is bad for the observer. Consider the parametric functional $\mathcal{I}_\alpha = \alpha(1/\delta - h) + (1-\alpha)|x_T - \hat{x}|$ and solve a problem of the form (3.6) with $\mathcal{I}_{\alpha,l}$ from (4.3). The Pareto-optimal set is shown in Fig. 2. The controller may choose α^0, for example, by maximization of $(1/\delta - h)(\alpha) + |x_T - \hat{x}|(\alpha) \to \max_\alpha$, where $(1/\delta - h)(\alpha)$ and $|x_T - \hat{x}|(\alpha)$ are optimal values from the scalar problem. If we do so, $\alpha^0 = 0.47$. For this α^0, we have the trajectory in Fig. 3. Here, the large ellipse is the reachable set from x_0, and the small ellipse is the IS. The trajectory differs from that in Fig. 1, and x_T lies practically on the border of the IS.

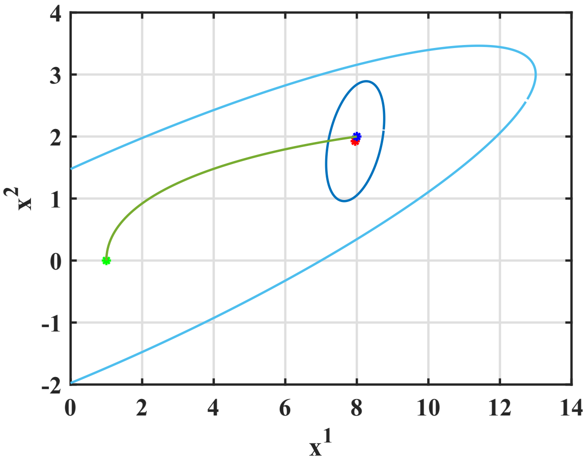

Figure 1. Trajectory with minimal square norm.

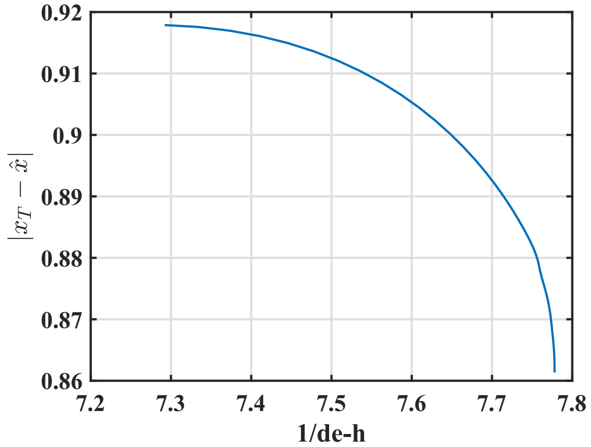

Figure 2. Pareto-optimal set.

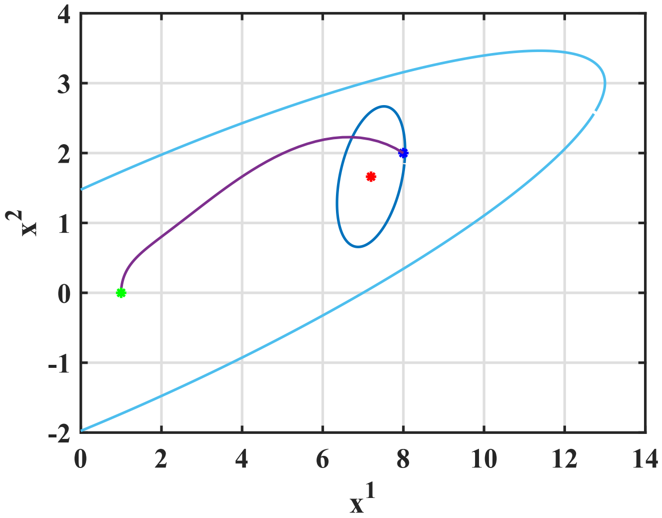

Figure 3. Optimal trajectory for controller.

Conclusion

- In this paper, the controller's task is to select a control that maximizes the information set (IS)'s volume while maximizing the IS center's distance from the final point x_T. This is necessary because the IS center can be chosen as a target point for the observer. The problem is solved by scalarization of the criteria and then by application of the Kuhn–Tucker theorem, which leads to the construction of constitutive relations for optimal control.

- In this work and in [2], we consider more general linear systems and more general constraints than in [1, 4].

- Here, we simplify constraints by a replacement of variables. This operation revealed the important fact that the IS may not be an ellipsoid at all under the given restrictions. That is why we must require complete observability (2.10) of the system.

- In previous papers, only scalar functionals were considered for the controller. In this work, a two-criterion control observation problem was investigated. As an example, the Pareto-optimal set was constructed and an optimal point on it was selected.

- In previous papers, the problems are solved by reduction to discrete models. Here, we consider the functional approach to the solution in detail. Lemmas 1 and 2 describe properties of the main operators that are used in the solution of the controller's problem.

• The solution to the controller's problem is reduced to solving a set of linear algebraic equations and to solving one nonlinear scalar equation, as was pointed out in Theorem 1.

• Since the inequality in (3.1) is strict, we can use the solution for controller via differential equations as described in Theorem 2.

• Finally, we consider a simplified solution via reduction to discrete models. Unlike in the previous paper, we describe a detailed algorithm for the solution.

• We consider the same example as in [2]. Here, however, we provide three figures that illustrate the solution. Besides, we determine the Pareto-optimal set and an optimal point on it.

• The problem can be generalized further by considering the signal as part of the state vector.

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