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COMMUTATIVE WEAKLY INVO–CLEAN GROUP RINGS

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Abstract: A ring R is called *weakly invo–clean* if any its element is the sum or the difference of an involution and an idempotent. For each commutative unital ring R and each abelian group G, we find only in terms of R, G and their sections a necessary and sufficient condition when the group ring $R[G]$ is weakly invo-clean. Our established result parallels to that due to Danchev-McGovern published in J. Algebra (2015) and proved for weakly nil-clean rings.

Keywords: Invo-clean rings, Weakly invo-clean rings, Group rings.

Introduction and conventions

Throughout the current paper, we shall assume that all rings R are associative, containing the identity element 1 which differs from the zero element 0. Our standard terminology and notation are in agreement with [9] and [10], while the specific notions and notations will be stated explicitly below. As usual, $J(R)$ denotes the Jacobson radical of a ring R and G is a multiplicative group. Both objects R and G forming the symbol $R[G]$ will stand for the group ring of G over R.

The next concept appeared in [1], [2] and [3], respectively.

Definition 1. *A ring R is said to be invo-clean if, for every $r \in R$, there exist an involution v and an idempotent e such that $r = v + e$. If $r = v + e$ or $r = v - e$, the ring is called weakly invo-clean.*

The next necessary and sufficient condition for a commutative ring R to be invo-clean was established in [1, 2], namely: *A ring R is invo-clean if, and only if, $R \cong R_1 \times R_2$, where R_1 is a nil-clean ring with $z^2 = 2z$ for all $z \in J(R_1)$, and R_2 is a ring of characteristic 3 whose elements satisfy the equation $x^3 = x$.*

Let us recall that a ring is *nil-clean* if every its element is a sum of a nilpotent and an idempotent, and it is *weakly nil-clean* if every its element is a sum or a difference of a nilpotent and an idempotent (see, for more details, [6]).

A criterion for an arbitrary commutative group ring to be nil-clean was recently obtained in [8]. Specifically, the following holds: *A commutative ring $R[G]$ is nil-clean if, and only if, the ring R is nil-clean and the group G is a 2-group.* This was generalized in [6, Theorem 2.1] by finding a suitable criterion when $R[G]$ is weakly nil-clean.

Some other related results in this subject can be found by the interested reader in [4] too.

So, the aim of this brief article is to obtain a paralleling result for the class of weakly invo-clean rings. This is successfully done below in our main Theorem 1.

1. The characterization result and a problem

We begin here with the following key formula from [7]: Suppose that R is a commutative ring and G is an abelian group. Then

$$J(R[G]) = J(R)[G] + \langle r(g-1) \mid g \in G_p,\ pr \in J(R)\rangle,$$

where G_p designates the p-primary component of G.

The next technicality already was mentioned above, but for the sake of completeness and reader's convenience, we will state it once again.

Lemma 1. [1, 2] *Let R be a commutative ring. Then the following two points hold*:
(i) *If $2 \in J(R)$, then R is invo-clean $\iff$ R is nil-clean and $z^2 = 2z$ for each $z \in J(R)$.*

(ii) *If $\mathrm{char}(R) = 3$, then R is invo-clean $\iff$ $x^3 = x$ for all $x \in R$.*

We also need the following two technical claims.

Lemma 2. *The direct product $K \times L$ of two rings K, L is invo-clean $\iff$ both K and L are invo-clean rings.*

P r o o f. It is straightforward by using of results from [1] and [2]. □

Lemma 3. *A ring R is weakly invo-clean $\iff$ either R is invo-clean or R can be decomposed as $R = K \times \mathbb{Z}_5$, where $K = \{0\}$ or K is invo-clean.*

P r o o f. It is straightforward by the utilization of results from [2] and [3]. □

We are now ready to proceed by proving the following preliminary statement (see [5] as well).

Proposition 1. *Suppose R is a non-zero commutative ring and G is an abelian group. Then $R[G]$ is invo-clean if, and only if, R is invo-clean having the decomposition $R = R_1 \times R_2$ such that precisely one of the next three items holds*:

(0) $G = \{1\}$

or

(1) $|G| > 2$, $G^2 = \{1\}$, *$R_1 = \{0\}$ or R_1 is a ring of* $\mathrm{char}(R_1) = 2$, *and $R_2 = \{0\}$ or R_2 is a ring of* $\mathrm{char}(R_2) = 3$

or

(2) $|G| = 2$, $2r_1^2 = 2r_1$ *for all $r_1 \in R_1$ (in addition $4 = 0$ in R_1), and $R_2 = \{0\}$ or R_2 is a ring of* $\mathrm{char}(R_2) = 3$.

P r o o f. If G is the trivial identity group, there is nothing to do, so we shall assume hereafter that G is non-identity.

"**Necessity.**" Since there is an epimorphism $R[G] \to R$, and an epimorphic image of an invo-clean ring is obviously an invo-clean ring (see, e.g., [1]), it follows at once that R is again an invo-clean ring. According to the criterion for invo-cleanness alluded to above, one writes that $R = R_1 \times R_2$, where R_1 is a nil-clean ring with $a^2 = 2a$ for all $a \in J(R_1)$ and R_2 is a ring whose elements satisfy the equation $x^3 = x$. Therefore, it must be that $R[G] \cong R_1[G] \times R_2[G]$, where it is not too hard to verify by Lemma 2 that both $R_1[G]$ and $R_2[G]$ are invo-clean rings.

First, we shall deal with the second direct factor $R_2[G]$ being invo-clean. Since $\mathrm{char}(R_2) = 3$, it follows immediately that $\mathrm{char}(R_2[G]) = 3$ too. Thus an application of Lemma 1 (ii) (which is an assemble of facts from [1, 2]) allows us to deduce that all elements in $R_2[G]$ also satisfy the equation $y^3 = y$. So, given $g \in G \subseteq R[G]$, it follows that $g^3 = g$, that is, $g^2 = 1$.

Next, we shall treat the invo-cleanness of the group ring $R_1[G]$. Since $\mathrm{char}(R_1)$ is a power of 2 (see [1]), it follows the same for $R_1[G]$. Consequently, utilizing once again Lemma 1 (i) (being an assortment of results from [1, 2]), we infer that $R_1[G]$ should be nil-clean, so that $z^2 = 2z$ for all $z \in J(R_1[G])$. That is why, invoking the criterion from [8], listed above, we have that G is a 2-group. We claim that even $G^2 = 1$. In fact, for an arbitrary $g \in G$, we derive with the aid of the aforementioned formula from [7] that $1 - g \in J(R_1[G])$, because $2 \in J(R_1)$. Hence $(1 - g)^2 = 2(1 - g)$ which forces that $1 - 2g + g^2 = 2 - 2g$ and that $g^2 = 1$, as desired. We now assert that $\mathrm{char}(R_1) = 2$ whenever $|G| > 2$. To that purpose, there are two nonidentity elements $g \neq h$ in G with $g^2 = h^2 = 1$. Furthermore, again appealing to the formula from [7], the element $1-g+1-h = 2-g-h$ lies in $J(R_1[G])$, because $2 \in J(R_1)$. Thus $(2-g-h)^2 = 2(2-g-h)$ which yields that $2 - 2g - 2h + 2gh = 0$. Since $gh \neq 1$ as for otherwise $g = h^{-1} = h$, a contradiction, this record is in canonical form. This assures that $2 = 0$, as wanted.

However, in the case when $|G| = 2$, i.e. when $G = \{1, g \mid g^2 = 1\} = \langle g \rangle$, we can conclude that $2r^2 = 2r$ for any $r \in R_1$. Indeed, in view of the already cited formula from [7], the element $r(1-g)$ will always lie in $J(R_1[G])$, because $2 \in J(R_1)$. We therefore may write $[r(1 - g)]^2 = 2r(1 - g)$ which ensures that $2r^2 - 2r^2 g = 2r - 2rg$ is canonically written on both sides. But this means that $2r^2 = 2r$, as pursued. Substituting $r = 2$, one obtains that $4 = 0$. Notice also that $2r^2 = 2r$ for all $r \in R_1$ and $a^2 = 2a$ for all $a \in J(R_1)$ will imply that $a^2 = 0$.

"**Sufficiency.**" Foremost, assume that (1) is true. Since R_1 has characteristic 2, whence it is nil-clean, and G is a 2-group, an appeal to [8] allows us to get that $R_1[G]$ is nil-clean as well. Since $z^2 = 2z = 0$ for every $z \in J(R_1)$, it is routinely checked that $\delta^2 = 2\delta = 0$ for each $\delta \in J(R_1[G])$, exploiting the formula from [7] for $J(R_1[G])$ and the fact that $R_1[G]$ is a modular group algebra of characteristic 2. That is why, by a consultation with Lemma 1 (i), one concludes that $R_1[G]$ is invo-clean, as expected. Further, by the usage of Lemma 1 (ii) above, we derive that $R_2[G]$ is an invo-clean ring of characteristic 3. To see that, given $x \in R_2[G]$, we write $x = \sum_{g\in G} r_g g$ with $r_g \in R_2$ satisfying $r_g^3 = r_g$. Since $G^2 = 1$ will easily imply that $g^3 = g$, one obtains that

$$x^3 = (\sum_{g\in G} r_g g)^3 = \sum_{g\in G} r_g^3 g^3 = \sum_{g\in G} r_g g = x,$$

as needed. We finally conclude with the help of Lemma 2 that $R[G] \cong R_1[G] \times R_2[G]$ is invo-clean, as expected.

Let us now point (2) be fulfilled. Since $G^2 = 1$, similarly to (1), R_2 being invo-clean of characteristic 3 implies that $R_2[G]$ is invo-clean, too. In order to prove that $R_1[G]$ is invo-clean, we observe that R_1 is nil-clean with $2 \in J(R_1)$. According to [8], the group ring $R_1[G]$ is also nil-clean. What remains to show is that for any element δ of $J(R_1[G])$ the equality $\delta^2 = 2\delta$ is valid. Since in conjunction with the explicit formula quoted above for the Jacobson radical, an arbitrary element in $J(R_1[G])$ has the form $j + j'g + r(1 - g)$, where $j, j' \in J(R_1)$ and $r \in R_1$, we have that $[j + j'g + r(1 - g)]^2 \in (J(R_1)^2 + 2J(R_1))[G] + r^2(1 - g)^2$. However, using the given conditions, $z^2 = 2z = 2z^2$ and thus $z^2 = 2z = 0$ for any $z \in J(R_1)$. Consequently, one checks that $[j + j'g + r(1 - g)]^2 = r^2(1 - g)^2 = 2r^2(1 - g) = 2r(1 - g) = 2[j + j'g + r(1 - g)]$, because $2r^2 = 2r$, as required. Therefore, $R_1[G]$ is invo-clean with Lemma 1 (i) at hand. Finally, Lemma 2 gives that $R[G] \cong R_1[G] \times R_2[G]$ is invo-clean, as promised. □

It is worthwhile noticing that concrete examples of an invo-clean ring of characteristic 4, such that its elements are solutions of the equation $2r^2 = 2r$, are the rings $\mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$.

We will prove now the following reduction of weak invo-cleanness.

Proposition 2. *Suppose that R is a commutative non-zero ring and G is an abelian group. Then $R[G]$ is weakly invo-clean which is not invo-clean if, and only if, R is a weakly invo-clean ring which is not invo-clean and $G = \{1\}$.*

P r o o f. "**Necessity.**" As it is well known and easy to establish that there is a surjection $R[G] \to R$, we may apply [2] to get that R is weakly invo-clean as well. According now to Lemma 3 we obtain that R is either invo-clean, or isomorphic to $\mathbb{Z}_5$, or decomposed as $K \times \mathbb{Z}_5$, where K is non-zero invo-clean. We will consider these three possibilities separately:

Case 1: *R is invo-clean.* Since both $R[G]$ and R have equal characteristics, it follows once again with the aid of Lemma 3 that $R[G]$ must be invo-clean too, a contrary to our assumption.

Case 2: $R \cong \mathbb{Z}_5$. It follows that $R[G] \cong \mathbb{Z}_5[G]$ has to be weakly invo-clean of characteristic 5. Employing [2], one infers that $\mathbb{Z}_5[G] \cong \mathbb{Z}_5$ whence these two rings have equal cardinalities. This, however, implies by a simple comparison of elements that $G = \{1\}$.

Case 3: *$R \cong K \times \mathbb{Z}_5$ with $K \neq \{0\}$ invo-clean.* Hence $R[G] \cong K[G] \times \mathbb{Z}_5[G]$. It follows as is Case 1 that $K[G]$ is necessarily invo-clean, whereas $\mathbb{Z}_5[G]$ is weakly invo-clean. Similarly to Case 2, we detect once again that $G = \{1\}$.

"**Sufficiency.**" It is immediate, because of the fulfillment of the isomorphism $R[G] \cong R$. □

So, combining both Propositions 1 and 2, we come to our chief result. Specifically, the following assertion is true:

Theorem 1. *Let G be an abelian group and let R be a commutative non-zero ring. Then the group ring $R[G]$ is weakly invo-clean if, and only if, at most one of the following points is true:*

(1) *$G = \{1\}$ and R is weakly invo-clean.*

(2) *$G \neq \{1\}$ and $R \cong R_1 \times R_2$ is invo-clean such that either*

(2.1) *$|G| > 2$, $G^2 = \{1\}$, $R_1 = \{0\}$ or R_1 is a ring of $\mathrm{char}(R_1) = 2$, and $R_2 = \{0\}$ or R_2 is a ring of $\mathrm{char}(R_2) = 3$*

or

(2.2) *$|G| = 2$, $2r_1^2 = 2r_1$ for all $r_1 \in R_1$ (in addition $4 = 0$ in R_1), and $R_2 = \{0\}$ or R_2 is a ring of $\mathrm{char}(R_2) = 3$.*

P r o o f. If G is trivial, there is nothing to prove because of the isomorphism $R[G] \cong R$, so let us assume henceforth that G is non-trivial.

"**Necessity.**" As already observed in Proposition 2 alluded to above, if $G \neq \{1\}$, then the ring R must be invo-clean but *not* properly weakly invo-clean, i.e., it does not contain $\mathbb{Z}_5$ as a (proper) direct factor. Thus $R[G]$ has to be invo-clean too, as $\mathrm{char}(R[G]) = \mathrm{char}(R)$. We, therefore, appeal to Proposition 1 getting the listed above two items, as desired.

"**Sufficiency.**" As in the previous direction, Proposition 1 is in use to infer that $R[G]$ is invo-clean and hence weakly invo-clean, as wanted. □

In closing, we state one more intriguing problem.

Problem 1. *Find a suitable criterion only in terms of the commutative unital ring R and the abelian group G when the group ring $R[G]$ is feebly invo-clean as defined in* [3].

In that direction, similarly to Lemma 3, the question of whether or not any (commutative) feebly invo-clean ring R which is possibly *not* weakly invo-clean possesses the decomposition $R = K \times P$, where K is a weakly invo-clean ring and P is a ring whose elements satisfy the equation $x^5 = x$ such that $P \ncong \mathbb{Z}_5$, is of some interest.

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