URAL MATHEMATICAL JOURNAL, Vol. 5, No. 2, 2019, pp. 31–54

DOI: 10.15826/umj.2019.2.004

LOCAL EXTENSIONS WITH IMPERFECT RESIDUE FIELD

Akram Lbekkouri

10507 Casa-Bandoeng, 20002 Casablanca, Morocco

lbeka11@gmail.com

Abstract: The paper deals with some aspects of general local fields and tries to elucidate some obscure facts. Indeed, several questions remain open, in this domain of research, and literature is getting scarce. Broadly speaking, we present a full description of the absolute Galois group in all cases with answers on the solvability, prosolvability and procyclicity. Furthermore, we give a result that makes "some" generalization to Abhyankar's Lemma in local case. Half-way a short section, containing a view of some future research loosely discussed, presents an attempt in the development of the theory. An Annexe elucidate several important points, concerning Hilbert's theory.

Keywords: Inertia group, Abhyankar's Lemma, Imperfect residue field, Weakly unramified, Solvability, Monogenity.

Introduction

Local fields with perfect residue field (or more generally when the residue extension is assumed to be separable) were deeply studied. The general case, when dropping off the separability of the residue extension, considered for the first time in [25] still needs more work.

This condition of separability implies that the extension of the valuation rings is monogenic and plays an imminent role in the proofs of some standard results for example Hilbert formula, Herbrand property and Hasse-Arf Theorem which remain true under the less strong condition of monogeneity. Meanwhile the property of the congruence of the ramification breaks modulo the residual characteristic, (necessarily $p > 0$) does not hold if the residue extension is not separable, even by assuming the monogeneity of the respective valuation rings extension.

The residue field is only a "fair" field, and does not have to be CDV. When assuming it as local, we can characterize a large "family" of general local fields, more precisely "the higher dimensional local fields" (such fields need not be necessarily monogenic). Paršin introduced the "2-dimensional" local fields and constructed a class field theory of them, then Hyodo defined "upper" ramification breaks, as m-tuples, for a Galois extension of "m-dimensional" local fields (with finite last residue field).

The perfectness of the residue field ($\mathrm{char}\,(\overline{K}) = p > 0$), implies necessarily the separability of the residue extension. So, by assuming the less strong condition $[\overline{K} : \overline{K}^p] = p^c < \infty$, we make a step ahead to the generalization ("c" is called the degree of imperfectness). By taking $c = 1$, I. Zhukov in [26, §1] defines a good ramification theory under the hypothesis $[\overline{K} : \overline{K}^p] = p$ (i.e. $\overline{K}$ has a p basis of length 1). Especially for such fields, he proved that all weakly unramified extensions are well ramified and then monogenic. Zhukov's theory was for "2-dimensional" local fields only, then later it was generalized to "n-dimensional" local fields by V. A. Abrashkin [2].

It depends on the choice of a subfield of "1-dimensional constants" $\mathcal{K}$ in K (a field is "1-dimensional" if it is complete with respect to its discrete valuation and has a finite residue field, it is said to be "2-dimensional" if it is complete with respect to its discrete valuation and has a residue field which is itself "1-dimensional", and so on and so forth, we can define an "n-dimensional" local field).

The theory is presented by a ramification filtration on $\mathrm{gal}\,(K^{sep}/K)$, the absolute Galois group of K, by steps beginning with $\mathrm{gal}\,(\mathcal{K}^{sep}/\mathcal{K})$, the absolute Galois group of $\mathcal{K}$.

In fact, in characteristic zero he defined $\mathcal{K}$ as the set of all $x \in K$ which are algebraic over the fraction field $\mathcal{K}_0$ of $W(F)$ where $F = \cap\overline{K}^{p^i}$ and $W(F)$ is the Witt ring of F. Such $\mathcal{K}$ is the maximal for this property and is complete with a perfect residue field. Meanwhile, in characteristic $p > 0$ it is possible to fix a "base" subfield $\mathcal{B}$ in K, complete with respect to the valuation of K and having $\mathbb{F}_p$ as a residue field. That are the $\mathbb{F}_p((\tau))$ with $v_K(\tau) > 0$, if $\mathcal{K}$ is the algebraic closure in K of the completion of $\mathcal{B}(\mathcal{R}_K)$. Here $\mathcal{R}_K$ consists of Teichmüller representatives of elements of the maximal perfect subfield in $\overline{K}$.

Defining first a ramification filtration in classical way on $\mathrm{gal}\,(\mathcal{K}^{sep}/\mathcal{K})$ he introduces then a new lower filtration on $\mathrm{gal}\,(L/K)$ indexed by a special linear ordered set $\mathbb{I} \subset \mathbb{Q}^2$ (lexicographic order). Then a new Hasse-Herbrand function $\Phi : \mathbb{I} \mapsto \mathbb{I}$ is defined with all the usual properties. Therefore, a theory of upper ramification groups, in this case, is stated. He uses the method of "eliminating wild ramification" due to Epp [6] to reduce, in a canonical way, the study of completely ramified extensions to the last one of ferociously ramified extensions. For such extension the hypothesis on $[\overline{K} : \overline{K}^p]$ implies that the extension to consider is in fact ferociously ramified with $\overline{L}/\overline{K}$ generated by only one element i.e. it is monogenic (Section 5), for which L. Spriano defines a more general ramification theory what he calls "case II", see [20, §5], [21]. Particularly in this case, the question of the "passage of the ramification to the quotient" is affirmatively solved.

Lastly, Abbes and Saito, using techniques of rigid geometry, define an upper ramification filtration in the general case successfully. Till now they cannot make the two filtrations (namely the lower of Hilbert–Zariski–Samuel and their upper) corresponding in a satisfactory way.

To sum up, the assumption of the monogeneity remains the first important step to generalization without losing the trueness of large number of important results.

Section Progression:

Here are three main sections, then a section of limelight questions and a last as annexe.

In Section 1 we prove the solvability of the inertia group of any finite extension regardless of the residual extension, then we give a discussion on the solvability of the Galois group.

In Section 2 we give a full description of the absolute Galois group in all cases.

In Section 3 Theorem 9 makes some generalization of Abhyankar's Lemma in local case.

Section 4 contains a view of some future research, an attempt to develop of the theory.

Section 5 is an Annexe section destined to briefly elucidate several important points, necessary for the study, concerning Hilbert's theory.

The main results are Theorems 1, 2, 3, 4, 5 and 9, Propositions 1 and 4, Lemma 1.

Nowhere else in the realm of abstract algebra does one see such an elegant interaction of topics as in the subject of General Theory of Local Fields.

*By **local field** we mean a complete discrete-valued field (CDVF), the residue field being not necessarily perfect. We say **classical case** when the residue field is perfect or at least when the residue extension is separable, otherwise we name it as **general case**.*

1. On the solvability in finite local extensions

Here, we study the solvability of the Galois group of some local extensions with possibly imperfect residue field. Theorem 1 is a direct proof of the solvability of the inertia group in general case, then results on solvability of n-dimensional local fields are given.

1.1. On the solvability of the inertia group

Let L/K be a finite Galois extension of local fields. The residue extension $\overline{L}/\overline{K}$ is normal, see [18, Proposition I.7.20], but need not be separable. Consider D the set of all automorphisms of $\overline{L}$ unvarying all elements of $\overline{K}$, there is a natural surjective homomorphism $\varphi : G \to D$. Indeed, let $g \in G$, g preserves $\mathcal{O}_L$ as well as $\mathcal{M}_L$, therefore g induces an automorphism of $\overline{L} = \mathcal{O}_L/\mathcal{M}_L$. Since g fixes each element of K it fixes each element of $\overline{K}$ as well, for the surjectivity of φ see the same reference. So, the inertia group of L/K is $G_0 = ker(\varphi)$, also G is solvable if and only if D and G_0 are too.

Theorem 1. *Let L/K be a finite Galois extension of any local fields without any assumption on the residual extension. Then the inertia group G_0 of L/K is solvable, furthermore it is cyclic when the residual characteristic is zero.*

It is a generalization of Serre's results in [18, Proposition IV.2.7] *and its corollaries. Published in* [11]*, the proof needs some necessary retouches that can be found here.*

P r o o f. An uniformizer π of L being fixed, let us fix a set of generators of the residue field extension and their lifts $u_1, ..., u_n$ to $\mathcal{O}_L$. Put it in another way, $\mathcal{O}_L$ is generated by $\pi, u_1, ..., u_n$ as an $\mathcal{O}_K$-algebra, with $v_(\pi) = 1$, u_i being units. Consider the map:

$$\begin{array}{rccc} \varphi_1 : & G_0 & \to & \overline{K}^{\star}, \\ & g & \to & \overline{g(\pi)/\pi}. \end{array}$$

It is clear that this is a homomorphism, write $J_1 = ker(\varphi_1)$ for the kernel of this map, so $J_1 = H_1$; we use Zariski–Samuel notation [25, ch. V, § 10]. Then again consider the homomorphism:

$$\begin{array}{rccc} \varphi_2 : & J_1 & \to & \overline{K} \oplus ... \oplus \overline{K}; \ \text{(n+1) of them}, \\ & g & \to & (\overline{(g(\pi)-\pi)/\pi^2}, \overline{(g(u_1)-u_1)/\pi}, \quad ..., \overline{(g(u_n)-u_n)/\pi}), \end{array}$$

where $\overline{(g(\alpha)-\alpha)/\pi^i}$ is the class of $(g(\alpha)-\alpha)/\pi^i$ mod π. Set $J_2 = ker(\varphi_2)$. Again by considering,

$$\begin{array}{rccc} \varphi_3 : & J_2 & \to & \overline{K} \oplus ... \oplus \overline{K}; \ \text{(n+1) of them}, \\ & g & \to & (\overline{(g(\pi)-\pi)/\pi^3}, \overline{(g(u_1)-u_1)/\pi^2}, \quad ..., \overline{(g(u_n)-u_n)/\pi^2}) \end{array}$$

and so on and so forth, until the filtration stabilizes (of course, since $\mathcal{O}_L \simeq \varprojlim \mathcal{O}_L/\mathcal{M}_L^i$) and we get a trivial J_r. From this, we conclude

1. **If the residual characteristic is** $p > 0$: it is clear that J_1 has a filtration by normal subgroups J_i, where the subquotients J_i/J_{i+1} are p-elementary abelian groups as J_i/J_{i+1} injectively maps to $(1+\mathcal{M}_L^i)/(1+\mathcal{M}_L^{i+1})$ which is canonically isomorphic to $(\overline{L}, +)$ for $i \geq 1$. Furthermore, G_0/J_1 is cyclic as it injectively maps to $\mathcal{R}_L^{\star}/(1+\mathcal{M}_L) \simeq (\overline{L}^{\star}, \times)$, and to $Aut_{\overline{L}}(\mathcal{M}_L/\mathcal{M}_L^2) \simeq (\overline{L}^{\star}, \times)$ as well, and the field $\overline{L}$ is of characteristic p. (Remark the order of G_0/J_1 is prime to p if $p \geq 3$). Furthermore, worthy to note that the maximal tamely ramified subfield T of L corresponds to the subgroup J_1. Finally, J_1 is a p-group (the unique Sylow p-subgroup of G_0) it is of order $e_{wild} f_{insep}$, which then implies the solvability of G_0.

2. **When the residual characteristic is zero**: for $i \geq 1$ the subquotients J_i/J_{i+1} being isomorphic to a subgroup of $(\overline{L}, +)$ (additive), which has no finite subgroup except $\{0\}$, J_i are trivial for all $i \geq 1$ and G_0 is cyclic. □

Remark 1. J.P. Serre in [18, Corollary IV.2.5 of Proposition 7], inspired by Zariski–Samuel in [25], gives a proof of this theorem in the classical case. Unhappily his proof breaks down in the general case because he uses the $(G_n)_n$ (lower ramification subgroups Hilbert–Zariski–Samuel's filtration of G_0). Of course, in general G_0/G_1 need not be abelian, see [25, page 297, last line], the purely inseparable part of the residue extension playing main role. Indeed, Theorem 1 in the same reference claims that the group G_0/G_1 contains a normal subgroup G_1' which is reduced to the identity in separable case (see § 5.1).

1.2. Consequences

From Theorem 1 we straightforwardly deduce the corollaries:

Corollary 1. *Let K be a local field, and let L/K be a finite Galois extension. Then L/K is solvable if and only if the maximal separable subextension of $\overline{L}/\overline{K}$ is solvable.*

Corollary 2. *Consequently, in the classical case the Galois group of L/K is solvable if and only if the Galois group of $\overline{L}/\overline{K}$ is solvable.*

1.3. On n-dimensional local fields

A complete discrete-valued field K is said to have the structure of an n-dimensional local field if there is a chain of fields $K = K_n, K_{n-1}, ..., K_1, K_0$ where K_{i+1} is a complete discrete valuation field with residue field K_i and K_0 is a finite field. The field $\overline{K} = \overline{K_n} = K_{n-1}$ is said to be the first residue field of K, respectively K_0 is the last.
Recall some facts about n-dimensional local fields:

- When assuming the last residue K_0 is perfect rather than finite, we preserve most of the properties of n-dimensional local fields.
- Some authors referred to as an n-dimensional local field over a perfect field, rather than a finite field. But we consider an n-dimensional local field over an arbitrary field K_0 as well.
- Let L/K be a finite extension. If K is an n-dimensional local field, then so is L.

Since finite extensions of a finite field are cyclic, by induction (use Corollary 1) we get:

Corollary 3. *Every finite Galois extension of a "n-dimensional" local field with the residue field of the corresponding "1-dimensional" field is finite, has a solvable Galois group.*

In "Serre's sense" a field is said to be **quasi-finite** if it is perfect and $\mathrm{gal}\,(K^{sep}/K) \simeq \widehat{\mathbb{Z}}$ (K^{sep} being a separable closure of K and $\widehat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$). Every finite quotient of $\widehat{\mathbb{Z}}$ is cyclic ($\widehat{\mathbb{Z}}$ is a profinite group as the projective limit of the finite cyclic groups $\mathbb{Z}/n\mathbb{Z}$) and thus is abelian and procyclic). Some authors allow themselves to say $\widehat{\mathbb{Z}}$ is cyclic as a topological group, even if it is not countable since the natural homomorphism $\mathbb{Z} \to \widehat{\mathbb{Z}}$ has a dense image.

So, Corollary 3 can be immediately generalized (in some sense) to the case when the residue field of the "1-dimensional" field is assumed to be quasi-finite only, if we allow ourselves to generalize the notion of "high-dimensional" local fields such way (replacing the finiteness of the residue field of the "1-dimensional" field by its perfectness). Even the perfectness of the residue field is not necessary. We can only assume that $\mathrm{gal}\,(\overline{K}^{sep}/\overline{K}) \simeq \widehat{\mathbb{Z}}$, or more generally prosolvable ($\overline{K}$ being the residue field of the "1-dimensional" field K).

But we cannot say that the result remains true when $\mathrm{gal}\,(\overline{K}^{sep}/\overline{K})$ is any profinite group. Indeed, a finite quotient of a profinite group need not be solvable. For this it is easy to construct a counter-example of course, $PSL(2, \mathbb{F}_q)$ is very often simple.

Corollary 4. *Every finite Galois extension of a "n-dimensional" local field, has necessarily a solvable Galois group if the residue field $\overline{K}$ of the corresponding "1-dimensional" form $\overline{K} = k((T))$ with k being an algebraically closed field of characteristic zero.*

P r o o f. It suffices to use the Corollary of the Proposition IV.2.8 in [18]. Indeed, we have the Galois group of the algebraic closure of $\overline{K}$ which is isomorphic to $\widehat{\mathbb{Z}}$. □

Notice that if $\mathrm{gal}\,(K^{sep}/K) \simeq \widehat{\mathbb{Z}}$ (e.g. if K is quasi-finite) then for every supernatural number n, K has only one Galois extension of degree n. Since $\widehat{\mathbb{Z}}$ has a unique closed subgroup of a given index n, see Theorem 2.7.2 in [14].

2. Absolute groups

Here, we give a whole description of the absolute groups and their classification by residual characteristic in the general case. More precise facts are found in § 2.4. Then we answer questions concerning: the nature of these groups.

By absolute groups of K a CDVF, we mean the Galois group G, the inertia group G_0 and G_W the wild ramification subgroup of a separable closure K^{sep}/K.

2.1. Hilbert decomposition of the separable closure

2.1.1. Presentation

For K being any field, consider K^{sep}/K a separable closure (that is the union of all finite Galois extensions of K), it is necessarily normal and then Galois. In general $K^{sep} \subseteq K^{alg}$, nevertheless $K^{sep} = K^{alg}$ if and only if K is perfect. Now, if K is a complete discrete-valued field then its valuation extends uniquely to K^{sep} but it is no more discrete on it, actually $v((K^{sep})^\star) = \mathbb{Q}$; furthermore, K^{sep} is not complete for the discussed valuation.

The Galois group $\mathrm{gal}\,(K^{sep}/K) = \mathcal{A}ut_K(K^{sep}/K)$, called absolute Galois group of K, is a compact topological group with respect to the profinite topology. Indeed, going over all finite extensions L/K, denote by $\mathfrak{L}$ the set of all finite Galois extensions L of K contained in K^{sep}/K, then we can write,

$$K^{sep} = \bigcup_{L\in\mathfrak{L}} L; \quad \text{and } \mathrm{gal}\,(K^{sep}/K) = \varprojlim_{L\in\mathfrak{L}} \mathrm{gal}\,(L/K).$$

Now, the maximal unramified extension K^{unr} of K in K^{sep} is the union of all fields L_0 (L_0 being the maximal unramified extension of K in L and is Galois over K), we too find that K_W, the union of all fields L_w (where L_w is a tamely ramified Galois extension in L that contains every tamely ramified extension of K in L), is a tamely ramified extension of K in K^{sep}. That is we have the tower:

$$K \text{ ——— } K^{unr} \text{ ——— } K_W \text{ ——— } K^{sep}\ .$$

K^{unr}/K and K_W/K both are Galois and $G_W = \mathrm{gal}\,(K^{sep}/K_W)$ os the absolute wild ramification group (maybe trivial), which can be considered as the projective limit of a sequence of corresponding finite wild ramification p-subgroups (in all cases the ramification filtration always exists). So, G_W is prosolvable even more pronilpotent, but in general not solvable. That is the p-Sylow subgroup of $G_0 = \mathrm{gal}\,(K^{sep}/K^{unr})$ (the absolute inertia group), and a closed normal pro-p-subgroup of $G = \mathrm{gal}\,(K^{sep}/K)$. Furthermore, write $\overline{K}^{sep}$ as a separable closure of $\overline{K}$, $\overline{K}^{sep} = \mathcal{O}_{K^{unr}}/\mathcal{M}_{K^{unr}} =$

$\overline{K^{unr}}$. Indeed, the residue field of the maximal unramified extension of K is a separable closure of $\overline{K}$. Furthermore, $\operatorname{gal}(K^{unr}/K) = \operatorname{gal}(\overline{K}^{sep}/\overline{K})$, see [13, ch. II, Proposition 7.5].

Remark 2. It is well known that G_0/G_W is a torsion free abelian group, the q-Sylow subgroups of which are free $\mathbb{Z}_q$-modules of rank $dim_{\mathbb{F}_q}\Gamma/q\Gamma$ where Γ is the additive value group. The prime numbers q are necessarily different from the residue characteristic.

2.2. General description

First, let us notice some relatiohship between the unit group and the Galois group. Recall that the unit group is abelian and the absolute Galois group is not. However we know that there is some correspondence between the unit group and the Galois group of certain subextension of K^{sep}. Indeed, when K is a local field with finite residue field its unit group is isomorphic to the Galois group of a certain totally ramified abelian extension of K. For example, the extension of $\mathbb{Q}_p$ obtained by adjoining all p-power roots of unity has Galois group $\mathbb{Z}_p^\star$ over $\mathbb{Q}_p$. This generalizes to other base fields using Lubin–Tate formal groups.

If K is any complete discrete-valued field then the unit group $\mathcal{O}_K^\star$ (as well as $\mathcal{O}_K$) is compact if and only if the residue field of K is finite, so in the case of an infinite residue field the topological group $\mathcal{O}_K^\star$ could not be a Galois group since profinite groups are compact.

2.2.1. Classification by residual characteristic

Let us proceed by cases:

1. If $\operatorname{char}(\overline{K}) = 0$, all Galois extensions are tamely ramified, the inertia group of every finite extension is cyclic and its wild ramification subgroup is trivial, see the proof of Theorem 1, hence the absolute inertia group G_0 of the absolute Galois group is the profinite completion of $\mathbb{Z}$ i.e. is isomorphic to $\widehat{\mathbb{Z}}$ so it is procyclic (by the way abelian), meanwhile G_W is trivial. In consequence the absolute Galois group is a semi-direct product of the absolute inertia group by the absolute Galois group of the residue field i.e. $G \simeq \widehat{\mathbb{Z}} \rtimes \operatorname{gal}(\overline{K^{sep}}/\overline{K})$.
   Now, when the residue field $\overline{K}$ is algebraically closed $\overline{K}^{sep} = \overline{K}$, the maximal unramified extension is trivial, in consequence the absolute inertia group equals the absolute Galois group $G_0 = G$. So, we find the main result of Theorem 4 that comes.

2. If $\operatorname{char}(\overline{K}) = p > 0$, the absolute inertia subgroup G_0 of G is isomorphic to the extension of G_W by $\prod_{q \neg p} \mathbb{Z}_q$, where $\mathbb{Z}_q$ is the ring of q-adic integers with $q \neq p$. With K^{unr} being the field fixed by G_0 in K^{sep}, K^{unr}/K is a Galois extension such that $\operatorname{gal}(K^{unr}/K)$ is isomorphic to $G_{\overline{K}}$ where $G_{\overline{K}}$ is the absolute Galois group of the residue field $\overline{K}$. That is $G_0 = G/G_W \simeq \prod_{q\neq p} Z_q \rtimes G_{\overline{K}}$ with its Galois action. Indeed, for each integer q prime to p, the group of q-th roots of unity $\mu_q(\overline{K}^{sep})$ is cyclic of order q. Consider $\mathcal{Q}$ the set of all integers q prime to p ordered by divisibility, if $q' = q.m$ by means of the transition map (rising to power m) $\mu_{q'}(\overline{K}^{sep}) \to \mu_q(\overline{K}^{sep})$ we have a canonical isomorphism $G_0/G_W \simeq \varprojlim_{q\in\mathcal{Q}} \mu_q(\overline{K}^{sep})$.

The Tate twist of $\mathbb{Z}_q$ being defined by $\mathbb{Z}_q(1) = \mu_{q^\infty}(\overline{K}^{sep})$, write, $\widehat{\mathbb{Z}}' = \prod_{r\neq p} \mathbb{Z}_r$, and $\widehat{\mathbb{Z}}'(1) = \prod_{r\neq p} \mathbb{Z}_r(1)$, we have that $\widehat{\mathbb{Z}}'(1) \simeq \widehat{\mathbb{Z}}'$ the isomorphism being not canonic. Then we get $G_0/G_W \simeq \prod_{r\neq p} \mathbb{Z}_r(1)$. Since, $G/G_0 \simeq G_{\overline{K}}$ the action by conjugation of $G_{\overline{K}}$ on G/G_0 gives the natural action on $\widehat{\mathbb{Z}}'(1)$.

Furthermore, $G/G_0 \simeq G_{\overline{K}}$ and T the absolute "tame-inertia" subgroup $T \simeq \prod_{r\neq p} \mathbb{Z}_r$ is a normal subgroup of G/G_W. In other words, we have: $(G/G_W)/T \simeq G_{\overline{K}}$.

Remark 3. For q prime, $q \neq p$ and $n \in \mathbb{N}^\star$, any cyclic finite extension of K of degree q^n, if it exists, corresponds to a quotient of $\mathrm{gal}\,(K^{sep}/K)/G_W$ that looks like $\mathbb{Z}/q^n\mathbb{Z}$. Indeed, if L/K is cyclic of degree q^n, then $[K_W L : K_W]$ has degree q^m with $m \neq n$. G_W being pro-p-group, $m = 0$, so $L \subset K_W$ hence $\mathrm{gal}\,(L/K)$ is a quotient of $\mathrm{gal}\,(K^{sep}/K)/G_W$.

2.3. Pro-solvability, pro-cyclicity and solvability

2.3.1. When is the absolute Galois group prosolvable?

Remark 4. The absolute Galois group of any Henselian discrete-valued field need not be prosolvable in general. Indeed, it admits a canonical surjection onto the absolute Galois group of the residue field given by the action on the maximal unramified extension, so if the latter is not prosolvable, the former cannot be either. See the following example.

Example 1. The absolute Galois group of $\mathbb{Q}$ is a quotient of the absolute Galois group of $\mathbb{Q}((X))$. The first is not prosolvable so, neither is the last. More generally, if K is any Henselian discrete-valued field, then the maximal unramified extension K^{unr} of K has a Galois group $\mathrm{gal}\,(K^{unr}/K)$ isomorphic to the absolute Galois group of $\overline{K}$ (i.e. $\mathrm{gal}\,(K^{unr}/K) \simeq \mathrm{gal}\,(\overline{K}^{sep}/\overline{K})$), since $\mathrm{gal}\,(K^{unr}/K)$ is a quotient of $\mathrm{gal}\,(K^{sep}/K)$ then so is $\mathrm{gal}\,(\overline{K}^{sep}/\overline{K})$.

More precisely, we have the following result.

Theorem 2. *For any Henselian discrete-valued field,*

- *the absolute wild ramification group and all wild ramification subgroups are always pronilpotent. Meanwhile,*
- *its absolute Galois group is prosolvable if and only if this is true for the absolute Galois group of the residue field.*

P r o o f. Indeed, in all cases G_W (maybe trivial) is a closed normal pro-p-subgroup of the absolute Galois group $G = \mathrm{gal}\,(K^{sep}/K)$ and is then pronilpotent. See § 2.1.1.

Consider first the case of a positive residual characteristic $p > 0$.

Denote by $(v((K^{sep})^\star)^p/v((K)^\star))$ the p-free part of the abelian torsion group $(v((K^{sep})^\star)/v((K)^\star))$ (a quotient group of $\mathbb{Q}$), then we have the exact sequence see [18]:

$$1 \to (v((K^{sep})^\star)^p/v((K)^\star))^\vee \to \mathrm{gal}\,(K_W/K) \to \mathrm{gal}\,(\overline{K}^{sep}/\overline{K}) \to 1,$$

where, $(v((K^{sep})^\star)^p/v((K)^\star))^\vee$ is the dual of $(v((K^{sep})^\star)^p/v((K)^\star))$ in the sense that is the full character group of

$$(v((K^{sep})^\star)^p/v((K)^\star)) \text{ i.e.}$$
$$(v((K^{sep})^\star)^p/v((K)^\star))^\vee = Hom((v((K^{sep})^\star)^p/v((K)^\star)), \overline{K}^{sep}).$$

In consequence we have that,

$$\mathrm{gal}\,(K_W/K) \text{ is an extension of } \mathrm{gal}\,(\overline{K}^{sep}/\overline{K}) \text{ by}$$
$$(v((K^{sep})^\star)^p/v((K)^\star))^\vee \simeq \mathrm{gal}\,(K_W/K^{unr}) = G_0/G_W \simeq \textstyle\prod_{r\neq p} \mathbb{Z}_r(1).$$

It follows that all its Sylow subgroups are normal. Then the results follow.

Furthermore, $G/G_0 \simeq G_{\overline{K}}$ and $(G/G_W)/T \simeq G_{\overline{K}}$; T the absolute "tame-inertia" subgroup. See § 2.2.1. Finally, we get that $\mathrm{gal}\,(K^{sep}/K)$ is prosolvable if and only if this is true for $G_{\overline{K}}$ the absolute Galois group of the residue field.

Consider now the case when the characteristic is zero. It still holds, indeed, G_W is trivial, so $G_0 \simeq \prod_{r\neq p} \mathbb{Z}_r(1)$. □

Remark 5. It is worthy to notice that

- G_0 need not be pronilpotent. Indeed, the tame quotient can act by a non trivial outer automorphisms on the wild subgroup.
- The equivalence in Theorem 2 concerns the prosolvability only but not the solvability. Of course take for example p-adic field $\mathbb{Q}_p$ its absolute Galois group is prosolvable (but not solvable) since every finite Galois extension of it is solvable see Proposition 1, meanwhile the absolute Galois group of its residue field, $\mathbb{F}_p$, is procyclic.
- Since, any finite quotient of a pronilpotent profinite group is nilpotent. In general, the absolute Galois group of any Henselian discrete-valued field need not be pronilpotent. See Theorem 3.

Theorem 3. *Every finite normal totally ramified extension of $\mathbb{Q}_p$ for p being an odd prime number is either cyclic or nonnilpotent. Moreover if the extension is wildly ramified, then it is cyclic.*

P r o o f. Consider such extension $K/\mathbb{Q}_p$ with the Galois group G. Suppose first that G is a p-group and let $\Phi(G)$ be its Frattini subgroup. Since $G/\Phi(G)$ is an elementary abelian p-group thus the group $G/\Phi(G)$ is cyclic and therefore from a property of Frattini subgroups G is itself cyclic. Now let the group G be nilpotent, then it is the direct sum of its Sylow subgroups. Consequently $G = G_1 \times R$ where G_1 is a p-group and the order of R is prime to p. Remark that G_1 is the ramification group of $K/\mathbb{Q}_p$. Since $K^R/\mathbb{Q}_p$ is a normal totally ramified extension (K^R the fixed field by the elements of R) and its Galois group $G_{K^R/\mathbb{Q}_p} = G_{K/Q_p}/G_{K/K^R} = G/R = G_1$ is a p-group it follows from above that G_1 is cyclic. Since for $p \neq 2$ every normal totally and tamely ramified extension K/Q_p is cyclic of degree dividing $p-1$, furthermore with $M = K^{G_1}$ we get that the group $G_{M/\mathbb{Q}_p} = G_{K^{G_1}/\mathbb{Q}_p} = G/G_1 = R$ is cyclic of order prime to p. Consequently the group $G = G_1 \times R$ is cyclic. □

2.3.2. When is the absolute Galois group procyclic?

Here we prove the converse of Proposition IV.2.8 in [18].

Theorem 4. *For a complete discrete-valued field K, the absolute Galois group is isomorphic to $\widehat{\mathbb{Z}}$ if and only if the residue field $\overline{K}$ of K is algebraically closed and is of characteristic 0.*

P r o o f. If char $(\overline{K}) = p > 0$ then the structure of the inertia group is not commutative since it has non-Galois separable finite extensions (discreteness of the valuation bounds the amount of p-power roots of unity in the maximal unramified extension when char $(K) = 0$, so p^n-th root extractions of a uniformizer will be non-Galois for large n; in characteristic p one can use Artin-Schreier extensions of some tamely ramified extensions to make non-Galois extensions). So if the Galois group is commutative then char , $(\overline{K}) = 0$, so by completeness the field must be $K = \overline{K}((T))$ for a field $\overline{K}$ of characteristic 0, and then the Galois group is an extension of gal $(\overline{K}^{sep}/\overline{K})$ by $\widehat{\mathbb{Z}}$, but this $\widehat{\mathbb{Z}}$ being a closed subgroup of $\widehat{\mathbb{Z}}$ can only happen in case of equality, so can only happen when $\overline{K}^{sep} = \overline{K}$, which is to say $\overline{K}$ is algebraically closed. Nothe that the necessary condition is proved in Proposition IV.2.8 in [18]. In such case the absolute inertia subgroup equals the absolute Galois group, $G_0 = G$. □

2.3.3. When is the absolute Galois group solvable?

If a profinite group G is solvable then it is prosolvable, the converse is not true. Of course prosolvable does not mean that $G^{(n)} = \{1\}$ for some finite n (i.e. the derived length of G is finite, $G^{(n)}$ being the n-th commutator subgroup of G), but it only means that the series $G^{(n)}$ of higher commutator groups converges to $\{1\}$, i.e. every neighbourhood of $\{1\}$ contains almost all higher commutator subgroups.

For K being any CDVF, with the current notations, $G_W = \mathrm{gal}\,(K^{sep}/K_W)$ is the absolute wild ramification group, maybe trivial, otherwise, it is a free pro-p-group of infinite rank, where p is the residual characteristic. It, is prosolvable, pronilpotent, but in general not solvable. By Corollary 5, we have G/G_W is metabelian if and only if the absolute Galois group of the residue field of K, $\overline{G} = \mathrm{gal}\,(\overline{K^{sep}}/\overline{K})$ is too.

We have also the following properties.

1. If $\underline{\mathrm{char}\,(\overline{K}) = p}$ and $\underline{\mathrm{char}\,(K) = p > 0}$.

   Since a free pro-p-group is either isomorphic to $\widehat{\mathbb{Z}}$ when it is of rank 1 otherwise it is non-solvable, $G(p)$ being the biggest quotient of G which is a pro-p-group. For more details see § 2.5 is then non-solvable, neither is G; (indeed, $G(p)$ is a factor group of it). So, we get the following result.

   Let K be any CDVF of characteristic $p > 0$ the residue field being not algebraically closed i.e. $\overline{G}$ is not trivial, (with no further assumption on the residue field). Then the absolute Galois group of K is not solvable.

2. If $\underline{\mathrm{char}\,(\overline{K}) = p}$ and $\underline{\mathrm{char}\,(K) = 0}$.
   Then G_W the absolute wild ramification group is not trivial and a free pro-p-group of infinite rank. Therefore, G_W is not solvable. And consequently G is too.

So, we have the recapitulative result:

Proposition 1. *With the current notations, for K any CDVF regardless of its characteristic, if* $\mathrm{char}\,(\overline{K}) = p$*, then if the absolute Galois group is not trivial it is then not solvable as an abstract group.*

Now, let us prove a nice and necessary result on profinite groups.

Proposition 2. *Let N be an abelian profinite group whose automorphisms group $\mathcal{A}ut(N)$ being abelian profinite too. Consider the profinite group (semi-direct product) $G = N \rtimes H$. Then we have:*
1. If H is metabelian then G' (derived group) is abelian (i.e. G is metabelian too). Consequently:
2. G is metabelian if and only if H is too.

P r o o f. 1. Let $K = C_{H(N)}$, the centralizer of N within H (the set of elements in H that commute with every element of N, in the semi-direct product). As the action of H on N by automorphisms is given by a homomorphism $H \to \mathcal{A}ut(N)$ the kernel of which is K so H/K embeds in $\mathcal{A}ut(N)$, and as $\mathcal{A}ut(N)$ is abelian, H/K is abelian as well. In other words, K contains H' the group generated by the commutators of H, so H' centralizes N. Furthermore, since H is metabelian then H' is commutative, knowing that, $G' = N \rtimes H'$, we get $N \rtimes H' = N \times H'$ is commutative. Hence, G' is abelian.
2. Consequently G/G' is commutative. Conversely, if G is metabelian then H is too. □

Remark 6. Two important remarks are worthy to be noticed:

1. For N a profinite group $\mathcal{A}ut(N)$ need not be profinite, see Example 4.4.6 in [14].
2. If $\mathcal{A}ut(N)$ is abelian, N need not be abelian too, even when N is a finite group. (There are nonabelian finite p-groups for each prime p such that the automorphism groups are abelian see [8].)

Now, from Proposition 2, and since $\mathcal{A}ut(\prod_{q\neq p} Z_q)$ is abelian, we get the following result.

Corollary 5. *Let K be any CDVF of characteristic $p>0$ with no assumption on $\overline{K}$, G being the absolute Galois group and G_W the absolute (wild) ramification subgroup of G. Then, G/G_W is metabelian if and only if the absolute Galois group of $\overline{K}$, $\overline{G}$ is too.*

2.4. Recapitulation

Let K being any CDVF with no special assumption on $\overline{K}$.

1. Let $\mathrm{char}\,(K)=p>0$. The absolute Galois group of K is not solvable, see Remark 7.
2. Let $\mathrm{char}\,(K)=0$ and $\mathrm{char}\,(\overline{K})=p$. We have wild ramification, so a non trivial G_W which is not solvable, neither is the absolute Galois group is not solvable as an abstract group, see Proposition 1.
3. Let $\mathrm{char}\,(K)=0$ and $\mathrm{char}\,(\overline{K})=0$. There is no wild ramification, so the subgroup G_W is trivial, and the absolute inertia group $G_0\simeq\widehat{\mathbb{Z}}$. Now since the absolute Galois group is isomorphic to a semi-direct product of G_0 by $\mathrm{gal}\,(\overline{K}^{sep}/\overline{K})$ i.e. $\mathrm{gal}\,(K^{sep}/K)\simeq\widehat{\mathbb{Z}}\rtimes\mathrm{gal}\,(\overline{K}^{sep}/\overline{K})$. We may have the three following cases:
   (a) If $\overline{K}$ is algebraically closed then, $\mathrm{gal}\,(\overline{K}^{sep}/\overline{K})$ is trivial. So, $\mathrm{gal}\,((\mathbb{K})^{sep}/\mathbb{K})\simeq\widehat{\mathbb{Z}}$ it procyclic hence abelian, see Theorem 4.
   (b) If $\overline{K}$ is not algebraically closed but can be endowed with a structure of C.D.V.F with residual characteristic $p>0$. we still have the non solvability straightforwardly with respect to Proposition 1. (Particularly if the field K is a High dimensional local field).
   (c) The only case that remains to study, is that when $\overline{K}$ is not algebraically closed and cannot be endowed with a structure of C.D.V.F with residual characteristic $p>0$. Note that, for example, if the residue field is $\mathbb{Q}$ it is clearly not solvable, whereas if the residue field is the fixed field of a single element from the absolute Galois group of $\mathbb{Q}$ then it is solvable. (For more details on the solvable profinite groups occurring as absolute Galois groups see [9].)

Question 1. A question that is staring immediately in the face is: "*Is an absolute Galois group either procyclic or else nonabelian?*"

But the answer is surprisingly simple, it is negative! See the following Example 2:

Example 2. Take the field $K=\mathbb{C}((X))((Y))$ with $\mathbb{C}$ the field of complex numbers. It is Henselian according to the discrete Y-adic valuation, (the residue field being $\mathbb{C}((X))$).But the absolute Galois group G of K is the direct product of two copies of $\widehat{\mathbb{Z}}$, $G\simeq\widehat{\mathbb{Z}}\times\widehat{\mathbb{Z}}$, hence abelian but non procyclic.

Note 1. With respect to our study, the result in [9]: "*For any commutative field, if the absolute Galois group is solvable then it is metabelian,*" turns out to be more relevant in global case then for CDVF, except in the single case when $\mathrm{char}\,(K)=0$ and $\mathrm{char}\,(\overline{K})=0$ and no structure of CDVF with residual characteristic $p>0$, can be defined on $\overline{K}$.

2.5. On the p-maximal extension

For details see § 2.2.1.

First case same characteristic. *Here let us assume that* $\operatorname{char}\overline{K} = p > 0$.

Let K be any CDVF of characteristic $p > 0$ with no special assumption on $\overline{K}$, the residue field of K. Write $G(p)$ for the biggest quotient of G which is a pro-p-group. $G(p)$ is the Galois group of the maximal p-extension $K(p)/K$ i.e. the compositum of all Galois extensions of p-power order. It is a free pro-p-group of rank > 1, see [19, Chap. II., § 2.2, Corollary 1, p. 75], (i.e. $G(p)$ is the profinite completion of a free group with respect to a system of normal subgroups the quotients of which are finite p-groups) such that $H^1(G(p))$ can be identified with $K/\wp(K)$ (where $\wp : x \mapsto x^p - x$) which is a vector space of infinite dimension over $\mathbb{F}_p$ (the field of p elements), since the powers T^n (with n ranging over $\mathbb{N}$ and prime to p, T being a prime element in the DVF) are linearly independent over $\mathbb{F}_p$.
First let us recall the following well known results:

Proposition 3. *Let $L = K((t))$ (Laurent Series field) with $\operatorname{char}(K) = p > 0$, $K(p)/K$ being the maximal p-extension (compositum of all Galois extensions of p-power order), then:*

- *If K is finite or countable then $G(p) = \operatorname{gal}(K(p)/K)$ is a free pro-p-group of countably infinite rank,*
- *If K is uncountable then $G(p) = \operatorname{gal}(K(p)/K)$ is a free pro-p-groups of uncountable rank (see* [12, Proposition 6.1.7]*).*

In other words and in classical case more precisely for any local field K with finite residue field we have:

Theorem 5. *Let K be any local field with finite residue field $\overline{K}$, let $\operatorname{char}(\overline{K}) = p$, then $G(p)$ as well as G_W (the wild ramification group) are free pro-p-groups of countably infinite rank.*

P r o o f. See Proposition 7.5.1 and [12, Theorem 7.5.10]. □

Remark 7. Since a free pro-p-group is either isomorphic to $\widehat{\mathbb{Z}}$ when it is of rank 1 otherwise it is non-solvable. Then $G(p)$, being a free pro-p-group of rank > 1, is non-solvable, neither is G as $G(p)$ is a factor group of it.

So, we get the following result:

Theorem 6. *Let K be any complete discrete-valued field of characteristic $p > 0$ with no assumption on the residue field. Then the absolute Galois group of K is not solvable.*

Remarks on the q-maximal extension with $q \neq \operatorname{char}(K)$: K being some field containing a q-th root of unity, q being an odd prime number and different from the characteristic of K. Write $G(q)$ for the Galois group of the q-maximal extension of K, and assume that $G(q)$ has a finite normal series with abelian factor groups (i.e. solvable). Then the derived subgroup $G(q)'$ of $G(q)$ is abelian, moreover, $G(q)$ has a normal abelian subgroup with a pro-cyclic factor group. Furthermore, we have the following result:

Theorem 7 [22]. *Under the current hypotheses and notations the following statements are equivalent:*

- *$G(q)$ is solvable.*

- *$G(q)$ is metabelian.*
- *$G(q)$ does not contain a free non-abelian subgroup.*

Now, G_W is a pro-p-group therefore, the absolute Galois group of K is prosolvabe if and only if $\mathrm{gal}\,(K_W/K)$ is too (a pro-p-group is pro-nilpotent but need not be solvable). See [5].

Second case: mixed characteristic. In this case, Safarevič in [15] showed that for $K/\mathbb{Q}_p$ an extension of degree n not containing the p-th roots of unity and if $K/\mathbb{Q}_p$ is finite of degree $n < +\infty$ then $G(p)$ is a free pro-p-group of rank $n+1$. Now if K contains μ_p (the group of the p-th roots of unity) then $G(p)$ is a Poincare group of dimension 2 that is a Demuskin group of rank $n+2$. See Theorem 7.5.11 in [12]. So, in both cases if $K/\mathbb{Q}_p$ is finite of degree $n < +\infty$ then the absolute Galois group G of K can be generated by $n+2$ elements. See Theorem 7.4.1 in [12].

Furthermore, we have the following:

- By local class field theory, the abelianized group $G(p)/G(p)'$ is isomorphic to the pro-p-completion of $K^\star$ hence it is isomorphic to $U_K^1 \times \mathbb{Z}_p$ which is not procyclic, of course U_K^1, the subgroup of 1-units in $K^\star$ is not procyclic, but it is free abelian for $p > 2$ and K not containing the p-th roots of unity.
- Any complete discrete-valued field of residue characteristic $p > 0$ has an (unramified) procyclic extension $K(p')$ generated over K by all the ℓ-th roots of unity for ℓ describing all natural integers not divisible by p, thanks to Hensel's Lemma. Of course, from Galois theory of finite fields, by adjoining such roots of unity at residual level is obtained from doing so over the prime subfield $\mathbb{F}_p$ of the residue field $\overline{K}$. For more details see § 3.4.

Note that the unramified extension $K(p')/K$ maybe trivial. For example if k is algebraically closed of characteristic p, then $k((t))$ has no unramified extension.

It is worthy to notice the following result:

Lemma 1. *In case if G has $G(p)$ as free pro-p-group of "$1 < rank \leq +\infty$" with ($\mathrm{char}\,(\overline{K}) = p$), we can add that G is a semi-direct product of $\mathrm{gal}\,(K^{sep}/K(p))$ by a subgroup isomorphic to $G(p)$.*

P r o o f. Indeed, according to Theorem 7.7.4 in [14] "*$G(p)$ is a free pro-p-group if and only if $G(p)$ is projective group (in the category of profinite groups)*", that is it has the lifting property for every extension, which is equivalent to say that for every surjective morphism from any profinite group $H \to G(p)$ there is a section (a right inverse of the morphism in question) $G(p) \to H$. So, if f is an epimorphism from G onto $G(p)$ by the projectivity of $G(p)$ there exists a homomorphism h from $G(p)$ to G such that fh is the identity map on $G(p)$. Hence, G is a semi-direct product $\ker(f)$ and $h(G(p))$ (which is isomorphic to $G(p)$). □

2.6. On the maximal unramified extension

Let K be any complete discrete-valued field of residue characteristic $p > 0$ with $\overline{K}$ being the residue field of K, write $\overline{K}^{sep} = \mathcal{O}_{K^{unr}}/\mathcal{M}_{K^{unr}}$, it is a separable closure of $\overline{K}$; (K^{unr} being the maximal unramified extension of K that is the composite of all unramified extensions inside an algebraic closure of K).

From [13, ch. II, § 7] in the general case that is when K is assumed to be Henselian only K^{unr} contains all roots of unity of order m not divisible by the residue characteristic, because the separable polynomial $X^m - 1$ splits over the separable closure of the residue field of K, and hence also over the maximal unramified extension K^{unr} of K, by Hensel's Lemma. Now write $K(p')$ for the (unramified) pro-cyclic extension of K generated by all the ℓ-th roots of unity for ℓ describing all natural integers not divisible by p, it contains a subextension $K(p'')$ that is generated over K by all the q-th roots of unity for q describing all the primes different from p.

The question remains to prove that $K(p') = K(p'')$.

First, notice that the question is certainly a question of residue fields, par excellence.

Consider, the largest finite field contained in $\overline{K}$, $\mathbb{F}_\ell$ (the finite field of ℓ elements) where ℓ is power of p. Since the finite field $\mathbb{F}_\ell$ consists of the $(\ell-1)$-th roots of unity and 0, the said roots of unity are contained in $\overline{K}$. Now, if a complete discrete valuation field has residue field containing $\mathbb{F}_\ell$, then K contains the $(\ell-1)$–th roots of unity (Hensel's Lemma). This is an if and only if statement.

Now, we can say that $\overline{K}(p')$ (defined as above) is included in the residue field of $K(p')/K$ and then $\mathbb{F}_\ell(p')$ is included in the residue field of $K(p')/K$ too, as well as $\overline{K}(p'')$.

(Note that $\mathbb{F}_\ell(p')$ and $\mathbb{F}_\ell(p'')$ are no more finite, but infinite fields of characteristic $p > 0$.)

In other words, we can replace K by $\mathbb{F}_\ell$.

Hence, if we prove that: $\mathbb{F}_\ell(p'')$ is an algebraic closure of $\mathbb{F}_\ell$. Then we get that $\mathbb{F}_\ell(p') = \mathbb{F}_\ell(p'')$ and consequently that $K(p') = K(p'')$.

Since to get a primitive f-th root of unity in a field is equivalent to getting primitive roots of unity of order equal to each prime-power factor of f, our question amounts to asking if for a given prime p and prime power ℓ^r (allowing $\ell = p$), does there exist a square-free n not divisible by p such that p mod n has order divisible by ℓ^r (so then adjoining a primitive n-th root of unity to $\mathbb{F}_p$ would give an extension of degree divisible by ℓ^r, and then do this for several such prime powers to get an extension of $\mathbb{F}_p$ generated by prime-order roots of unity such that its degree is divisible by whatever we want).

But if $(\mathbb{Z}/n\mathbb{Z})^\star$ is going to contain a cyclic subgroup of order ℓ^r then under the decomposition $(\mathbb{Z}/n\mathbb{Z})^\star = \prod(\mathbb{Z}/q_i\mathbb{Z})^\star$ for the prime factors q_i of the square-free n we see that one of the projections $(\mathbb{Z}/n\mathbb{Z})^\star \to (\mathbb{Z}/q_i\mathbb{Z})^\star$ is injective on that cyclic subgroup of order ℓ^r. Hence, if some such n is going to exist then even a prime n will have to exist which does the job. In other words, the question is exactly asking this:

Given a prime p and a prime power ℓ^r (allowing $\ell = p$), does there exist a prime q distinct from p such that p mod q has order divisible by ℓ^r?

Since $(\mathbb{Z}/q\mathbb{Z})^\star$ is cyclic, the only way it contains an element with order divisible by ℓ^r is if the size of this cyclic group is divisible by ℓ^r, which is to say $q = 1$ mod ℓ^r.

Lemma 2. *Let p be prime. To generate an algebraic closure of $\mathbb{F}_p$ it is enough to adjoin the q-th roots of unity for all prime q different from p.*

Here we must use Čebotarev's Theorem (see, [13, ch. VII, § 13, Theorem 13.4]). Indeed, Čebotarev density Theorem reduces the problem of classifying Galois extensions to that of describing the splitting of primes in extensions. Specifically, it implies that as a Galois extension of K, L is uniquely determined by the set of primes of K that split completely in it. A related corollary is that if almost all prime ideals of K split completely in L, then in fact $L = K$.

P r o o f of Lemma 2. By a simple application of non-abelian Čebotarev result, it is enough to settle that "For (possibly equal) primes p and ℓ and any integer $r > 0$ that there are lots of primes $q = 1$ mod ℓ^r such that p mod q has order divisible by ℓ^r", (e.g., lots of $q = 1$ mod 9 such that 5 mod q has order divisible by 9). Since $(\mathbb{Z}/q\mathbb{Z})^\star$ is cyclic with size divisible by ℓ^r, a sufficient condition for an element to have order divisible by ℓ^r is that it "not" be an ℓ-th power. So one way to ensure that p mod q has order divisible by ℓ^r is to make sure that p mod q is not an ℓ-th power.

So consider the non-abelian Galois extension $K = \mathbb{Q}(\zeta_\ell^r, p^{1/\ell})$ of $\mathbb{Q}$. We have $\mathrm{gal}\,(K/\mathbb{Q}) \to \mathrm{gal}\,(\mathbb{Q}(\zeta_{\ell^r})/\mathbb{Q}) = (\mathbb{Z}/\ell^r\mathbb{Z})^\star$ carrying a Frobenius element $Frob_q$ onto q mod ℓ^r, hence Čhebotarev provides many q such that $Frob_q$ is nontrivial but $q = 1 \bmod \ell^r$. For any such q, not only is q totally split in $\mathbb{Q}(\zeta_{\ell^r})$ but the extension give by adjoining an ℓ-th root of p is "non-trivial " over $\mathbb{F}_q = (\mathbb{Z}/q\mathbb{Z})^\star$. Hence, $X^\ell - p$ has no root in $\mathbb{F}_q$ (since if it has one root then it completely splits, as $\mathbb{F}_q$ contains a primitive ℓ -th root of 1 by design).

Applying this with a fixed p but several ℓ^rs (for different ℓs) and considering pairwise distinct q s thereby obtained, it follows that every finite extension of $\mathbb{F}_p$ is contained in an extension generated by prime-order roots of unity, that is exactly what we wish. □

Also, we have the following result:

Proposition 4. *Let K be any complete discrete-valued field of residue characteristic $p > 0$ with no more assumption on the residue field, then $K(p') = K(p'')$, namely the (unramified) procyclic extension of K generated by all the ℓ-th roots of unity for ℓ describing all natural integers not divisible by p equals the last one generated by all the q-th roots of unity for q describing all the primes different from p .*

P r o o f. The proposition follows from Lemma 2 immediately. □

Remark that if $\overline{K}$ is finite then $K^{unr} = K(p')$ (see [13, ch. II, § 7]). So, we have:

Corollary 6. *Let K be any complete discrete-valued field with a finite residue field of characteristic $p > 0$, then the maximal unramified extension of K is the extension generated over K by all the q-th roots of unity for all prime q different from p.*

Notice that Corollary 6 above is no more true if $\overline{K}$ is not finite, indeed:

Example 3. If $k = \mathbb{F}_p(u)$ with u transcendent on $\mathbb{F}_p$ (the field of p elements) and $K = k((x))$, then $K(v)$ with $v^n = u$ is an unramified extension of K (in the sense that $e = 1$, and in the strict sense if p does not divide n). Obviously, $K(v)$ cannot be generated by a root of unity.

3. On Abhyankar's Lemma

The aim of this section, is the proof of Theorem 9 that is "some" generalization of Abhyankar's Lemma in local case, by use of the following EPP's Theorem 8 (see [6]).

First, let us recall both the Abhyankar Lemma [7] and EPP Theorem[1].

Lemma 3 (Abhyankar, [7]). *Let $L = L_1L_2$ be the compositum of two finite algebraic extension fields of K, let $\mathcal{P}$ be prime divisor of L, which is ramified in L_i/K of order e_i $(i = 1, 2)$; then if $e_2|e_1$ and $\mathcal{P}$ is tame in L_2/K, then $\mathcal{P}$ is unramified in L/L_1.*

Theorem 8. (EPP[2]) *Let L/K be any non-trivial finite extension of discretely valued fields, it is possible to eliminate wild ramification, that is to ensure that $e_{k'L/k'K} = 1$ for some finite extension k'/k, where k is a "constant[3] subfield".*

Now, our generalization of Abhyankar's Lemma in local case can be announced as follows.

Theorem 9. *Given any finite Galois extension L/K of complete discrete-valued fields with a non necessarily perfect residue field of characteristic $p > 0$. Then there exist two separable overextensions K' and M of K such that:*

[1] EPP Theorem is an existence theorem of a reduced extension but non-constructive.

[2] Worthy to note that in [10] F.V. Kuhlmann has corrected an error in the proof of Theorem 8 of EPP's article [6]. Happily the error does not hurt any of the wording of all results in the said article.

[3] A subfield k of K is said to be constant, if it is a maximal subfield of K having a perfect residue field. Note that, such k is canonical in the mixed characteristic case).

- *$K \subset K' \subsetneq M \subseteq LK'$,*
- *LK'/K' is weakly unramified, so a uniformizer in K' remains uniformizer in LK'.*
- *M/K' is unramified.*
- *LK'/M is ferociously ramified, then the Galois group $\mathrm{gal}\,(LK'/M)$ is a p-group.*

P r o o f. First, according to Theorem 8, there exists a finite extension K'/K such that LK'/K' is weakly unramified, therefore $[LK' : K'] = [\overline{LK'} : \overline{K'}]$, i.e. $e = 1$ and $f = [LK' : K']$ (where $\overline{K'}$ is the residue field of K'). This condition implies that a uniformizer of K' remains uniformizer in LK', but the residue extension can be inseparable, furthermore it is not evident that Epp's extension K'/K is separable.

Let M be the maximal unramified (i.e. etale) extension of K' that is contained in LK'. Characterization of M: The ramification index $e_{(M/K')} = 1$, the residue extension of M/K' is separable so that M/K' is unramified, but the residue extension of LK'/M is purely inseparable (if LK'/K' is not unramified, see Remark 8).

Note that, if K has characteristic zero, then we can certainly take K'/K Galois, because if K'/K is not Galois, then we can always use its Galois closure instead.

Let T be the maximal tamely ramified subextension of LK'/K'. Characterization of T (see § 5.3): $e_{(T/K')}$ is prime to p, $e_{(LK'/T)}$ is a power of p, the residue extension of T/K' is separable, and the residue extension of LK'/T is purely inseparable. Hence if $e_{(LK'/K')} = 1$, we have $T = M$ and $[LK' : M]$ is a power of p.

Indeed, more precisely, in case of K'/K is separable LK'/M is then weakly unramified and the residue field extension is purely inseparable i.e. LK'/M is ferociously ramified (if it is not trivial). $[LK' : M] = [\overline{LK'} : \overline{M}]_{insep}$. In such case the inertia group of LK'/M is the full Galois group of LK'/M, and this group is a p-group.
In case of K'/K is purely inseparable, LK'/K' being weakly unramified, then it cannot be the case that the inertia group of L/K has a prime-to-p part, as tame ramification cannot be eliminated by an inseparable extension, in other words if the tame ramification index $e_{tame} > 1$, and if K'/K is a purely inseparable extension, then LK'/K has the same tame ramification index, so it cannot be weakly unramified, this follows from the multiplicativity of the tame ramification index. So, assuming LK'/K' is weakly unramified, then it is true that LK'/M is ferociously ramified. The proposition follows. □

Remark 8. When considering the particular case of perfect residue fields with L/K tamely ramified we get $M = LK'$, that is the Abhyankar's Lemma.

Remark 9. [26, § 1] If furthermore, we assume the hypothesis $[\overline{K} : \overline{K}^p] = p$ (i.e. $\overline{K}$ has a p basis of length 1), we get that LK'/K' is well ramified and then monogenic.

Remark 10. The usefulness of Theorem 9 is alluded to in the construction of a translated weakly unramified extension that is decomposable in an unramified and a ferociously ramified extensions. Worthy to note that such extensions arise in some situations in algebraic geometry. They are almost as important as selected in algebraic setting. For example, the book [4] which considers local extensions of discrete-valued rings having $e = 1$ in the more general case, such situations are called there as with "ramification index 1".

In a similar question of ours, Abbes and Saito proved the following different Corollary, see [1, Corollary A.2, p. 31]. *However, in their result they eliminate the fierce extension and allow to get an unfiercely ramified extension. They use the term* ***unfiercely*** *ramified for the case of finite separable extensions with separable residue extensions.*

Corollary 7 (A.2)(Abbes–Saito). *Given any finite separable extension of complete discrete-valued fields L/K, there exists a tower $K \subseteq K' \subseteq LK'$, K'/K finite separable such that*
- *a uniformizing element of K remains uniformizing element in K';*
- *LK'/K' is unfiercely ramified.*

4. Questions in the limelight in the general case

In this section some important and still open questions, that can make a fruitful subject of research, are given:

- How to completely specify the extensions having $e_{wild} > 1$ and $f_{insep} > 1$ for which there exists a normal subgroup that can "separate" ferocious from wild ramification? Note that some steps have been already done by L. Spriano, but the question is still very far from being entirely solved.

- In his paper [17, Corollary 1.3.4, p. 790] (in the equal characteristic case) and also in [16, Theorem 2, p. 568] (in the mixed characteristic case), Saito considered the following natural injective map the refined Swan conductor homomorphism ("rsw" initially defined by Kato) from the graded quotients piece of the Abbes–Saito filtration into the differentials. To be more precise, in the general case we have

$$rsw : Hom(G^r_{K,log}/G^{r+}_{K,log}, \mathbb{F}_p) \to \Omega^1_{\mathcal{O}_K}(log) \otimes_{\mathcal{O}_K} \pi_K^{-r}\overline{K}^{sep}, \tag{4.1}$$

  where K is a complete field with respect to a discrete-valuation, the residue field $\overline{K}$ being not necessarily perfect, $\overline{K}^{sep}$ a separable closure of it, G_K is the absolute Galois group, $r \in \mathbb{Q}_{>0}$, $\mathcal{O}_K$ is the ring of integers of K, π_K is uniformizer and $\Omega^1_K(log)$ is the logarithmic differential.

  It is likely that the said map is also surjective, if the residue field is perfect. Of course, when the residue field $\overline{K}$ is perfect, the right hand side (target) is just a one-dimensional vector space over the separable closure $\overline{K}^{sep}$. But there is no canonical basis. So, (4.1) reduces to

$$rsw : Hom(G^r_{K,log}/G^{r+}_{K,log}, \mathbb{F}_p) \to \pi_K^{-r} \otimes \overline{K}^{sep}.$$

  We cannot say that, the right hand side $\pi_K^{-r} \otimes \overline{K}^{sep}$ is exactly the residual ring $\mathcal{M}^r_{K^{sep}}/\mathcal{M}^{r+1}_{K^{sep}}$ where $\mathcal{M}^r_{K^{sep}} = \{x \in K^{sep}, v_{K^{sep}}(x) \geq r\}$, and $\mathcal{M}^{(r+1)}_{K^{sep}} = \{x \in K^{sep}, v_{K^{sep}}(x) \geq r+1\}$ with $v_{K^{sep}}$ the extension of the normalized valuation v_K to K^{sep}) since K^{sep} is not discretely valued. It is more correct to write $\pi_K^{-r} \otimes \overline{K}^{sep}$ differently as $\mathcal{M}^r_{K^{sep}}/\bigcup_{\epsilon>0}\mathcal{M}^{r+\epsilon}_{K^{sep}}$. For a proof of this result in perfect residue field case and for $r \in \mathbb{Z}_{>0}$, it is used to make working some means of local class field theory, then the case $r \in \mathbb{Q}_{>0}$ follows from certain base change result. The p-adic differential modules being out of the frame of this study, this question will appear in a next work.

  I think we can conjecture that this map remains surjective, even when dropping the hypothesis of the perfectness of the residue field. I have been told that some experts have pinned down the exact image of the abelian part. I think if we can run a base change argument to reach the rest of differential forms on the target, as in the case of perfect residue field, the problem will be solved. Probably, one needs to avoid the case when p is absolutely unramified in a mixed characteristic field.

5. Annexe on Hilbert's theory in the general case

The transition from the classical to the general case, requires a recall of special notions. So, let us consult our notes on Zariski–Samuel filtration as well as on Abbes–Saito ramification filtration, where some subtle and essential differences between the general and the classical cases appear. Furthermore, some important remarks and some original examples and counterexamples are given.

5.1. Hilbert–Zariski–Samuel filtration

Let L/K be any finite Galois extension of local fields with no special assumption on the residual extension and G is its Galois group.

Indeed, following Hilbert's way, in [25, ch. V] Zariski and Samuel define their lower ramification subgroups filtration as follows.

Then for any positive integer $n \geq 1$, they define the n-th ramification group G_n as the subset of G consisting of all automorphisms $\sigma \in G$ such that $\sigma(x) \equiv x$ modulo $\mathcal{M}_L^{n+1}$ for every $x \in \mathcal{O}_L$. G_n is the kernel of the action on $\mathcal{O}_L/\mathcal{M}_L^n$. They establish that G_n are invariant subgroups of G, and the quotients G_n/G_{n+1} are abelian for $n \geq 1$ [25, Lemma 1, p. 295]. Meanwhile, $G_0/G_1 (= G_T/G_{V_2}$ in Zariski–Samuel notation) need not be abelian in general case [25, ch. V, § 10, p. 297]. Indeed, there are extensions where $f_{insep} > 1$ and $e_{wild} > 1$, for which there does not exist a normal subgroup which can "separate" ferocious from wild ramification [20, § 1,page 1273]. So a second filtration H_n was necessary. By use of the homomorphism, we have

$$\begin{aligned} \lambda: \quad & G_0 \to \overline{L}^{\star}, \\ & \sigma \mapsto \lambda(\sigma) = \overline{(\sigma(\pi)/\pi)} = u_\sigma, \end{aligned}$$

H_1 is defined as the kernel of λ, that is the subgroup of all automorphisms σ in G_0 such that $u_\sigma \equiv 1$ modulo $\mathcal{M}_L$; that is such that $\sigma(\pi) - \pi \in \mathcal{M}_L^2$.

Likewise, H_i (for $i > 1$) is defined to be the kernel of the homomorphism

$$\begin{aligned} \lambda_i: \quad & G_i \to (\overline{L}, +), \\ & \sigma \mapsto \lambda_i(\sigma) = y_\sigma, \end{aligned}$$

that is the subgroup of all automorphisms σ in G_i such that $y_\sigma \equiv 0$ modulo $\mathcal{M}_L$, where y_σ is the integer $y_\sigma \in \mathcal{O}_L$ satisfying $\sigma(\pi) - \pi = y_\sigma \pi^i$ (i.e. $\sigma(\pi) - \pi \in \mathcal{M}_L^i$).

We have $G_i \supseteq H_i$ for every $i \geq 1$ (the equality occurs when the residue fields extension is separable, see [25, ch. V, § 10, p. 296]). So, $\sigma \in H_i$ implies that $\sigma(x) \equiv x \mod \mathcal{M}_L^{i+1}$ for every $x \in \mathcal{O}_L$. H_i is then the kernel of the action on $\mathcal{M}_L/\mathcal{M}_L^i$ for $i \geq 1$.

Intertwining both two filtrations of the Galois group with ramification groups, they used to define a unique filtration $G_{(n,i)}$ such that $G_n = G_{(n+1,0)}$ and $H_n = G_{(n,1)}$, as follows: for $n, i \in \mathbb{N}$ the (n,i)-ramification group $G_{(n,i)}$ of $G = \mathrm{gal}\,(L/K)$ is the subgroup of those K-automorphisms of L that induce the identity on $\mathcal{M}_L^i/\mathcal{M}_L^{n+i}$, i.e.

$$G_{(n,i)} = \{\sigma \in G; v_L(\sigma(x) - x) \geq i + n \forall x \in \mathcal{M}_L^i\} = \{\sigma \in G; \forall x \in \mathcal{M}_L^i; x - \sigma(x) \in \mathcal{M}_L^{n+i}\}.$$

Since $G_{(n,i)}$ is the kernel of the homomorphism $G \longrightarrow \mathcal{A}ut(\mathcal{M}_L^i/\mathcal{M}_L^{n+i})$ it is then a normal subgroup of G. Then we get, in the Zariski-Samuel filtration,

$$G_n = G_{(n+1,0)} \text{ and } H_n = G_{(n,1)}.$$

The $G_{(n,i)}$ with $i > 0$ makes sense, in the non-classical case only. Now, in the classical sense, *the G_n meet the usual ramification groups, see [18]. Explicitly, for $n \geq -1$ the n-th ("lower") ramification subgroup is defined as $G_n = \{\sigma \in G; i_G(\sigma) \geq n+1\}$.*

Consequences for the classical case:

The usual ramification subgroups in the classical case, are $(H_n = G_n)_{n\geq 1}$, and G_1 is called ramification group. From this Serre in [18] obtained the upper filtration by use of the Hasse-Herbrand functions ϕ defined by:

$$\phi_{L/K}(x) = \int_0^x \frac{dt}{|G_0 : G_t|},$$

and its inverse ψ (remember that φ and ϕ are only defined in case when the residue extension is separable). The upper $(G^n)_{n\geq 1}$ is related to the lower filtration by the formula $(G_n = G^{\phi(n)})_{n\geq 1}$ and $(G^n = G_{\psi(n)})_{n\geq 1}$. Note that the upper one behaves well under quotient subgroups; meanwhile, the lower one behaves well when taking subgroups.

The i such that $G_i \neq G_{i+1}$ (resp. $G^i \neq G^{i+1}$) are called lower (resp. upper) breaks.

5.2. Outline of Abbes–Saito ramification filtration

Let L/K be a finite Galois extension of local fields, then respectively we will write G_K and G_L for the absolute Galois groups of K and L. It is worthy to note that a separable closure is not complete as valued field in general.Nevertheless, a filtration on the absolute Galois group can be defined by taking inverse limit, as well as breaks.

Indeed, using techniques of rigid geometry, A. Abbes and T. Saito in [1] defined two decreasing filtrations, the first $(G_K^a)_{a\in\mathbb{Q}_{\geq 0}}$ and the second by logarithmic ramification groups $(G^a_{log,K})_{a\in\mathbb{Q}_{>0}}$ (closed normal subgroups of G_K). The filtration coincides with the classical upper numbering ramification filtration shifted by one, if the residue field of K is perfect in the sense that $G_K^{a-1} = G^a_{log,K}$ agrees with the upper numbered ramification filtration labeled by a. It is noteworthy that the filtration is left continuous and their jumps are rational.

For a real number $a > 0$, they define G^{a+} to be the topological closure of $G_K^{a+} = \overline{\cup_{b>a} G_K^b}$ and $G_K^{a-} = \cap_{b<a} G_K^b$, where b denotes a rational number. Then the following holds,

- $G_K^{a-} = G_K^a$ if $a \in \mathbb{Q}$, and $G_K^{a-} = G_K^{a+}$ if a not in $\mathbb{Q}$. It holds for the logarithmic too.
- The two filtrations by ramification groups are related as follows:

Let $j > 0$ be a rational number, then we have the following inclusions $G_K^j \supset G^j_{K,log} \supset G_K^{j+1}$, see [1, Proposition 3.15]

- G_K^1 is the absolute inertia subgroup of G_K; and G_K^{1+} the absolute wild inertia group of G_K.
- From the filtration above they define for any Galois extension L over K, the ramification filtration of the Galois group $\mathrm{gal}\,(L/K)$ by $G_K^a/(G_K^a \cap G_L)$. As a consequence, in the more general case, we have:
- $G_K^1/(G_K^1 \cap G_L)$ is the inertia subgroup of $\mathrm{gal}\,(L/K)$.
- $G_K^{1+}/(G_K^{1+} \cap G_L)$ is the wild inertia subgroup of $\mathrm{gal}\,(L/K)$.
- $\#(G_K^{1+}/(G_K^{1+} \cap G_L)) = e_{wild} f_{insep}$.
- If L/K finite unramified extension then $G_K^a = G_L^a$.
- If L/K finite tamely ramified extension with ramification index m then $G^{ma}_{log,L} = G^a_{log,K}$.

Furthermore, the logarithmic ramification filtration groups satisfy the following theorem [24, Theorem 3.7.3].

Theorem 10 [24]. *Assume that the residue field is of characteristic $p > 0$. Then the subquotients groups of the logarithmic ramification filtration $G^a_{log,K}/G^{a+}_{log,K}$ are abelian and annulated by p if $a \in \mathbb{Q}_{>0}$ and are trivial if a is irrational.*

Remark 11. It is worthy to note that we cannot make the filtration of Hilbert–Zariski–Samuel type and the last one of Abbes–Saito corresponding to each other, in a satisfactory way for example by use of some means like the well-known Hasse–Herbrand φ, ψ functions. Furthermore, the basic ramification degrees do not seem to work well as when the residue field fails to be perfect. Of course, the unramified part and the tame part are still okay, but it is not practical to separate the wild part from the residually inseparable part. Some attempts have been done, trying to describe ramification using more complex objects as ramification invariants. E.g. I.B. Zhukov used the "cutting-by-curves" method by considering the Abbes–Saito Swan conductor which is defined by looking at the generic points of the divisors. For details see [27] and [28], especially the results Theorems 2.2 and Theorems 2.4 in [27], and Remark 2.5.3 in [28]. But these notions are very far from our study.

5.3. Ramification cases

Consider a finite Galois extension L/K of local fields with Galois group $G = \mathrm{gal}\,(L/K)$, the residue extension $\overline{L}/\overline{K}$ being of characteristic $p > 0$ and not necessarily separable.

Write $K_{unr,L} = L \cap K^{unr}$ (for the maximal unramified extension of K in L i.e. the inertia field of L/K), and $G_0 = \mathrm{gal}\,(L/K_{unr,L})$ for the inertia group of L/K; so

$$G/G_0 = \mathrm{gal}\,(L/K)/\mathrm{gal}\,(L/K_{unr,L}) \simeq \mathrm{gal}\,(\overline{K}_{sep,\overline{L}}/\overline{K}),$$

where $\overline{K}_{sep,\overline{L}} = \overline{L} \cap \overline{K}^{sep}$, $\overline{K}^{sep}$ being a separable closure of the residue field $\overline{K}$.

Consider the ramification index e of the extension L/K, and f as its residue degree. Then we can write $e = e_{tame}.e_{wild}$ and $f = f_{sep}.f_{insep}$. So, we have

$$f_{sep} = \#(G/G_0) = [\overline{K}_{sep,\overline{L}} : \overline{K}] = [\overline{L} : \overline{K}]_{sep}; f_{insep} = [\overline{L} : \overline{K}_{sep,\overline{L}}] = [\overline{L} : \overline{K}]_{insep}.$$

L/K is *unramified* if f_{sep} is arbitrary and $f_{insep} = e = 1$.
L/K is *tamely ramified* if f_{sep} is arbitrary, e prime to p and $f_{insep} = 1$.
L/K is *completely ramified* if $f_{sep} = 1$, f_{insep} is arbitrary and e is a power of p.
L/K is *totally ramified* if $f_{sep} = f_{insep} = 1$ and e is arbitrary; in such case $\overline{L} = \overline{K}$.
L/K is *totally and wildly ramified* if $f_{sep} = f_{insep} = 1$ and e is a power of p.
L/K is *weakly unramified* if f_{sep}, f_{insep} are arbitrary and $e = 1$.
L/K is *ferociously ramified* or *fierce extension* if $f_{insep} > 1$ is arbitrary and $e = f_{sep} = 1$.

Note 2. "If L/K is fierce extension then it is weakly unramified, so that K contains a prime element of L."

5.4. Some well-known formulas and theorems (classical case)

$L = K(\alpha)/K$ being a finite Galois extension with Galois group G, the residue extension $\overline{L}/\overline{K}$ being separable of characteristic $p > 0$, we write f for the minimal polynomial of α.

Then we have the following useful summary of formulas and theorems, see for example [18, Ch. IV]. Meanwhile, for the general case, in [20, Examples 3.3, 3.4 and 8.1] beautiful counterexamples are given.

1. Hilbert's formula

$$v_L(\mathcal{D}_{L/K}) = \sum_{\sigma\neq I} i_G(\sigma) = \sum_{i\geq 0}(|G_i|-1) = v_L(f'(\alpha)),$$

where $\mathcal{D}_{L/K}$ is the different, $|G_i|$ the order of the i-th lower ramification group, and i_G the function:

$$\begin{aligned} i_G &: G \to \mathbb{Z}\cup\{\infty\}, \\ &\sigma \mapsto i_G(\sigma) = \inf_{x\in\mathcal{O}_L^*} v_L(\sigma(x)-x) \quad \text{for} \quad \sigma\neq 1. \end{aligned}$$

2. Herbrand's theorem

Let L/K be a finite Galois extension and L'/K a Galois subextension. Write $G = \mathrm{gal}\,(L/K)$ and $H = \mathrm{gal}\,(L'/K)$, H is then a normal subgroup of G naturally.

Theorem 11 (Herbrand). *For any $i \geq -1$ we have,*

$$(G/H)^i = G^i H/H, \quad \textit{i.e.}\ (G/H)_i = G_{\psi_{L/L'}(i)} H/H,$$

see [18, Proposition IV.3.14 and Lemma IV.3.5]. Then we straightforwardly can deduce the following result

Corollary 8. *If H is itself a ramification subgroup of G, i.e. $H = G_j$ for some j. Then*

$$(G/H)_i = \begin{cases} G_i/H, & \textit{if} \quad i\leq j, \\ \{1\}, & \textit{if} \quad i \geq j. \end{cases}$$

An important consequence of Herbrand's Theorem is that we can define upper ramification filtration $\{\mathrm{gal}\,(L/K)^i\}_i$ for an infinite Galois extension L/K as inverse limit as follows,

$$\mathrm{gal}\,(L/K)^i = \varprojlim_{L'/K\ \mathrm{finite} L'\subset L} \mathrm{gal}\,(L'/K)^i.$$

In particular, we can define an upper ramification filtration on the whole absolute Galois group as it is done in § 1.2.

3. Congruence formula The integers i such that

$$G_i \neq G_{i+1}, \quad \text{i.e. the breaks = lower ramification numbers,} \tag{5.1}$$

are congruent modulo p, see [18, Proposition IV.2.11]. This formula is no more true in every well-ramified extension, see § 1.5 that comes.

4. Hasse–Arf theorem

Theorem 12 (Hasse–Arf). *Let L/K be a finite abelian residually separable extension of any local fields. If i is such that $G_i \neq G_{i+1}$ then $\phi(i)$ is an integer.*

5.5. On the monogenic case (a step in the generalization)

L/K is said to be *monogenic* if $\mathcal{O}_L$ is generated by only one element as $\mathcal{O}_K$-algebra, the generator being not necessarily uniformizer, in general.

The Hasse–Arf theorem, Herbrand's Theorem, more generally Sen's theorem, and Hilbert's formula which are true under the strong hypothesis "$\overline{L}/\overline{K}$ separable" (see for example [18]); however remain true in the more general case when "L/K is assumed to be monogenic" see [3, 20, 23, 24] for Hasse–Arf theorem.

Except the Congruence formula (5.1) that requires necessarily the separability of $\overline{L}/\overline{K}$ see § 5 in [20].

Furthermore, from [20, Theorem 5.1] we have:

Definition 1. *A well-ramified extension L/K is defined as a finite Galois and completely ramified extension satisfying one of the three equivalent conditions:*

- *L/K is monogenic,*
- *Hilbert's formula holds,*
- *Herbrand's theorem holds for any normal subgroup.*

Remark 12. Note here the following important facts due to the monogenity.

- If L/K is monogenic then $\overline{L}/\overline{K}$ is too, but the converse is not true, see Counter-example 1.
- If $\overline{L}/\overline{K}$ is separable then L/K is monogenic, the converse is not true for a counter-example take a Galois extension of degree p such that the residue fields extension is purely inseparable, see Counter-example 1.
- In monogenic case, even by assuming that the residue extension is separable, the generator of the respective DVR need not be a uniformizer unless we are the setting of a totally ramified extension. If L/K is not totally ramified, it's very easy to give counter-examples.
- If L is the compositum of two linearly disjoint extensions L_1 and L_2 such that the residue extensions $\overline{L_1}/\overline{K}$ and $\overline{L_2}/\overline{K}$ are separable the compositum $\overline{L}/\overline{K}$ need neither be separable nor monogenic. A main example arises as follows, see Counterexample 1.

Counterexample 1. Let K be any complete discretely valued field of characteristic 0, containing a primitive p-th root of unity with residue field $\overline{K}$ of characteristic $p > 0$. Write π for a uniformizer of K and consider $L_1 = K(\sqrt[p]{\pi u})$, where u has a valuation zero, u is not a p-th power in K and does not reduce to a p-th power in $\overline{K}$, and $L_2 = K(\sqrt[p]{\pi})$. Then each is totally ramified of degree p; $\overline{L}_1/\overline{K}$ and $\overline{L}_2/\overline{K}$ are both trivial (so separable). Also $\sqrt[p]{\pi u}$ and $\sqrt[p]{\pi}$ are both roots of $f(X) = X^{2p} - \pi(1+u)X^p + \pi^2 u$ which is irreducible over K according to Schönmann criterion, so $\sqrt[p]{\pi u}$ and $\sqrt[p]{\pi}$ are linearly independent over $\mathcal{O}_K$. The compositum $L = L_1.L_2$, is an elementary abelian extension of degree p^2 since its ramification index is p and the residue field is $\overline{L} = \overline{K}(\sqrt[p]{u})$, which is inseparable of degree p over $\overline{K}$. Then we get $\overline{L}/\overline{K}$ monogenic since it is of prime degree. Prove that L/K is not monogenic.

We know that if Herbrand Theorem does not hold then the extension is not well ramified and then it is not monogenic, see [20, Lemma 5.2]. That is if there exists a normal subgroup H of G, such that $i_{G/H}(\tau) \neq 1/e_{(L/L^H)} \sum_{\sigma>\tau} i_G(\sigma)$, where $i_{G(.)}$ is the Artin ramification number.

Let $H =< \sigma >$ be the cyclic group of order p such that $\sigma(u^{1/p}) = \zeta.u^{1/p}$, where ζ is a primitive p-th root of unity. So $L^H = K(\pi^{1/p})$ with L/L^H is ferociously ramified meanwhile L^H/K is wildly ramified both of prime degree and has each a single Artin ramification number. Also char $(K) = 0$, L/L^H is ferociously ramified and $v(u) = 0$ implies that $i_G(\sigma) = s_G(\sigma)$ where $s_{G(.)}$ is the Swan ramification number.

So $i_G(\sigma) = s_G(\sigma) = v_L(\zeta - 1) = e_L/(p-1)$ for every σ. Since L^H/K is wildly ramified and $v(a) = 1$ hence $i_G(\tau) = s_G(\tau) + 1$; if τ is not the identity. That is

$$i_G(\tau) = e_{L^H}/(p-1) + 1, \quad e_{L^H} = pe, \quad e = v_K(p)$$

the absolute ramification index and $e_{L/L^H} = 1$. In this case we have

$$1/e_{(L/L^H)} \sum_{\sigma>\tau} i_G(\sigma) = 1/e_{(L/L^H)} \sum_{\sigma\in H} i_G(\sigma),$$

so, Herbrand does not hold. Then $\mathcal{O}_L$ is not monogenic over $\mathcal{O}_K$.

- The separability of a finite extension does not imply the separability of the residue extension. Indeed, it easy to construct a Counterexample 2.

Counterexample 2. Let K be CDVF with $\overline{K}$ imperfect. Regardless of the characteristic of K, consider $a \in \overline{K} \setminus \overline{K}^p$, thus $X^p - a$ is irreducible in $\overline{K}[X]$. Take

$$f(X)=X^p-bX-a$$

with $b \in \mathcal{M}_K$ ($\mathcal{M}_K$ is the maximal ideal of $\mathcal{O}_K$) requiring that

$$b \neq p^p(a/(1-p))^{(p-1)}.$$

Here f is separable and has reduction $X^p - a \in k[X]$. Here $L = K[X]/(f)$ is a degree p separable extension of K and its subring $\mathcal{O}_0 = \mathcal{O}_K[X]/(f)$ is a domain that is $\mathcal{O}_K$-finite and $\mathcal{O}_0/\pi\mathcal{O}_0 = k[X]/(X^p - a)$, is a field where an uniformiser $\pi \in \mathcal{O}_K$. To prove that $\mathcal{O}_0$ is a DVR see the proof of Lemma 4. We get $\mathcal{O}_0$=$\mathcal{O}_L$ and $\mathcal{O}_0/\mathcal{M}_0$=$l$, is the residue field of L and $e_{L/K}$=1 because the chosen uniformiser of $\mathcal{O}_K$ is an uniformiser of $\mathcal{O}_L$ too. L/K is a degree p separable extension with ramification index $e_{L/K}$=1 and the residual extension l/k is purely inseparable of degree p so L/K is not unramified. □

Assume K being CDVF with imperfect residue field $\overline{K}$. Regardless of the characteristic of K, consider any irreducible and separable polynomial f of $K[X]$ lying above $X^p - a$ with $a \in \overline{K} \setminus \overline{K}^p$. Then we have,

Lemma 4. *For θ a root of f, $L = K(\theta)/K$ is separable extension, meanwhile, its residue extension $\overline{K}(\sqrt[p]{a})/\overline{K}$ is inseparable.*

P r o o f. $X^p - a$ is separable, since it is irreducible, adjoining a root of f (which is separable since irreducible) to get $L = K(\theta) = K[X]/(f)$ gives a degree p separable extension with $\overline{K}(\sqrt[p]{a})/\overline{K}$ inside the residue field. Now, its subring $\mathcal{O}_0 = \mathcal{O}_K[X]/(f)$, is a domain that is $\mathcal{O}_K$-finite and $\mathcal{O}_0/\pi\mathcal{O}_0 = \overline{K}[X]/(X^p - a)$, is a field where $\pi \in \mathcal{O}_K$ denotes an uniformiser. Now prove that $\mathcal{O}_0$ is a DVR or equivalently that $\mathcal{O}_0$ is the integral closure of $\mathcal{O}_K$ in L. (That is true, indeed, if $\mathcal{O}$ is a DVR and f in $\mathcal{O}[X]$ has an irreducible reduction, then $\mathcal{O}[X]/f$ is again a DVR). More precisely, $\mathcal{M}_0 = \pi\mathcal{O}_0$, is a principal maximal ideal in $\mathcal{O}_0$. This is the only maximal ideal of $\mathcal{O}_0$ because any nonzero prime ideal of $\mathcal{O}_0$ intersects $\mathcal{O}_K$ in its unique nonzero prime ideal $\pi\mathcal{O}_K$ and so contains $\pi\mathcal{O}_0$. It follows that $\mathcal{O}_0$ must be DVR. Then the fundamental inequality implies the residue field is exactly $\overline{K}(\sqrt[p]{a})/\overline{K}$ and the ramification index is 1. So, you have a separable L/K with purely inseparable residue extension. □

Note 3. The hypothesis "f irreducible and separable polynomial" is necessary if $\operatorname{char}(K) > p$. Of course in such case irreducible doesn't mean separable.

Remark 13. Much more, the solvability of a finite extension does not imply the separability of the residue extension. Indeed, see the following example.

Example 4. Consider $k = \mathbb{F}_p((T_1))$, and $K = k((T_2))$, and α to be a root of the Artin–Schreier equation $f(X) = X^p - T_2^{p-1}X - T_1$ (f is obviously separable since $f' \neq 0$) and write $L = K(\alpha)$. The roots of f are $\alpha + nT_2$, with $0 \leq n \leq p-1$, thus the Galois group of L/K is solvable. Therefore, $\alpha \in \mathcal{O}_L$ (the ring of integers of L) hence is integer over $k[[T_2]]$ (the ring of integers of K), so modulo the maximal ideal we have $\alpha^p = T_1$, the residue extension is then $k(\sqrt[p]{T_1})/k$, which is purely inseparable. □

REFERENCES

1. Abbes A., Saito T. Ramification of local fields with imperfect residue fields. *Amer. J. Math.*, 2002. Vol. 124, No. 5, P. 879–920.
2. Abrashkin V. A. *Towards Explicit Description of Ramification Filtration in the 2-dimensional Case.* Prepint of Nottingham Univ., 2000. No. 00-01.
3. Borger J. A monogenic Hasse–Arf theorem. In: *Proc. of the Conf. on Ramification Theory for Arithmetic Schemes, Luminy, 1999.*
4. Bosch S., Lütkebohmert W., Raynaud M. *Néron Models.* Ergeb. Math. Grenzgeb. (3), vol. 21. Berlin, Heidelberg: Springer–Verlag, 1990. 328 p. DOI: 10.1007/978-3-642-51438-8
5. Engler A.J., Prestel A. *Valued Fields.* Springer Monogr. Math. Berlin, Heidelberg: Springer–Verlag, 2005. 208 p. DOI: 10.1007/3-540-30035-X
6. Epp H. P. Eliminating wild ramification. *Invent Math.*, 1973. Vol. 19. P. 235–249. DOI: 10.1007/BF01390208
7. Gold R., Madan M.L. Some applications of Abhyankar's Lemma. *Math. Nachr.*, 1978. Vol. 82, No. 1. P. 115–119. DOI: 10.1002/mana.19780820112
8. Jonah D., Konvisser M. Some nonabelian p-groups with abelian automorphism groups. *Arch. Math.*, 1975. Vol. 2, No. 1. P. 131—133. DOI: 10.1007/BF01229715
9. Koenigsmann J. Solvable absolute Galois groups are metabelian. *Invent. Math.*, 2001. Vol. 144. P. 1–22. DOI: 10.1007/s002220000117
10. Kuhlmann F. V. *A Correction to Epp's paper "Elimination of Wild Ramification"*, 2010. arXiv:1003.5687v1 [math.AC]
11. Lbekkouri A. On the solvability in local extensions. *An. Şt. Univ. Ovidius Constanţa*, 2014. Vol. 22. No. 2. P. 121–127. DOI: 10.2478/auom-2014-0037
12. Neukirch J., Shmidt A., Wingberg K. *Cohomology of Number Fields.* Grundlehren Math. Wiss., vol. 323. Berlin: Springer–Verlag, 2000. 720 p.
13. Neukirch J. *Algebraic Number Theory.* Berlin, Heidelberg: Springer–Verlag, 1999. 322 p. DOI: 10.1007/978-3-662-03983-0
14. Ribes L., Zalesskii P. *Profinite Groups.* Ergeb. Math. Grenzgeb. (3), vol. 40. Berlin, Heidelberg: Springer–Verlag, 2000. 483 p. DOI: 10.1007/978-3-642-01642-4
15. Safarevič I.R. On p-extensions. *Amer. Math. Soc. Transl. Ser. 2*, 1954. Vol. 4. P. 59–72.
16. Saito T. Ramification of local fields with imperfect residue fields III. *Math. Ann.*, 2012. Vol. 352. P. 567–580. DOI: 10.1007/s00208-011-0652-5
17. Saito T. Wild ramification and the characteristic cycle of an l-adic sheaf. *J. Inst. Math. Jussieu*, 2008. Vol. 8, No. 4. P. 769–829. DOI: 10.1017/S1474748008000364
18. Serre J.-P. *Local Fields.* Grad. Texts in Math., vol. 67. New York: Springer–Verlag, 1979. 241 p. DOI: 10.1007/978-1-4757-5673-9
19. Serre J.-P. *Cohomologie Galoisienne.* Lecture Notes in Math., vol. 5. Berlin Heidelberg: Springer–Verlag, 1997. 181 p. DOI: 10.1007/BFb0108758
20. Spriano L. Well ramified extensions of complete discrete valuation fields with application to the Kato Conductor. *Canad. J. Math.*, 2000. Vol. 52, No. 6. P. 1269–1309. DOI: 10.4153/CJM-2000-053-1
21. Spriano L. On ramification theory of monogenic extensions. In: *Geom. Topol. Monogr. Vol. 3: Invitation to Higher Local Fields,* eds. I. Fesenko and M. Kurihara, 2000. Part I, Sect. 18. P. 151–164.
22. Ware R. On Galois groups of maximal p-extension. *Trans. Amer. Math. Soc.*, 1992. Vol. 333, No. 2. P. 721–728. DOI: 10.2307/2154057
23. Xiao L. On ramification filtrations and p-adic differential modules, I: the equal characteristic case. *Algebra Number Theory*, 2010. Vol. 4, No. 8. P. 969–1027. DOI: 10.2140/ant.2010.4.969
24. Xiao L. On ramification filtrations and p-adic differential equations, II: mixed characteristic case. *Compos. Math.*, 2012. Vol. 148, No. 2. P. 415–463. DOI: 10.1112/S0010437X1100707X
25. Zariski O., Samuel P. *Commutative Algebra I.* Grad. Texts in Math., vol. 28. New York: Springer–Verlag, 1975. 334 p.
26. Zhukov I.B. *On Ramification Theory in the Imperfect Residue Field Case.* Prepint No. 98-02, Nottingham Univ., 1998. Accessible on arXiv:math/0201238 [math.NT]
27. Zhukov I.B. *Ramification of Surfaces Artin-Schreier Extensions*, 2002. arXiv:math/0209183 [math.AG]

28. Zhukov I.B. *Ramification of Surfaces: Sufficient Jet Order for Wild Jumps*, 2002. arXiv:math/0201071 [math.AG]