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MOMENT PROBLEMS IN WEIGHTED L^2 SPACES ON THE REAL LINE

Elias Zikkos

Khalifa University, PO. Box 127788 Abu Dhabi, United Arab Emirates

elias.zikkos@ku.ac.ae

Abstract: For a class of sets with multiple terms

$$\{\lambda_n,\mu_n\}_{n=1}^{\infty} := \{\underbrace{\lambda_1,\lambda_1,\dots,\lambda_1}_{\mu_1-times},\underbrace{\lambda_2,\lambda_2,\dots,\lambda_2}_{\mu_2-times},\dots,\underbrace{\lambda_k,\lambda_k,\dots,\lambda_k}_{\mu_k-times},\dots\},$$

having density d counting multiplicities, and a doubly-indexed sequence of non-zero complex numbers $\{d_{n,k} : n \in \mathbb{N}, k = 0,1,\dots,\mu_n - 1\}$ satisfying certain growth conditions, we consider a moment problem of the form

$$\int_{-\infty}^{\infty} e^{-2w(t)} t^k e^{\lambda_n t} f(t)\,dt = d_{n,k}, \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad k = 0,1,2,\dots,\mu_n - 1,$$

in weighted $L^2(-\infty,\infty)$ spaces. We obtain a solution f which extends analytically as an entire function, admitting a Taylor-Dirichlet series representation

$$f(z) = \sum_{n=1}^{\infty}\Big(\sum_{k=0}^{\mu_n-1} c_{n,k} z^k\Big) e^{\lambda_n z}, \quad c_{n,k} \in \mathbb{C}, \quad \forall\, z \in \mathbb{C}.$$

The proof depends on our previous work where we characterized the closed span of the exponential system $\{t^k e^{\lambda_n t} : n \in \mathbb{N},\ k = 0,1,2,\dots,\mu_n - 1\}$ in weighted $L^2(-\infty,\infty)$ spaces, and also derived a sharp upper bound for the norm of elements of a biorthogonal sequence to the exponential system. The proof also utilizes notions from Non-Harmonic Fourier series such as Bessel and Riesz–Fischer sequences.

Keywords: Moment problems, Exponential systems, Biorthogonal families, Weighted Banach spaces, Bessel and Riesz–Fischer sequences.

1. Introduction

P. Malliavin [5] considered the following in the sense of the classical Bernstein weighted polynomial approximation problem on the real line. Let $W(t)$ be a real-valued continuous function defined on the half-line $[0,+\infty)$ such that it is log-convex, that is $\log|W(e^s)|$ is a convex function on the real line. Let C_W be the weighted Banach space whose elements are the complex-valued continuous functions f defined on $[0,\infty)$, such that

$$\lim_{t\to\infty}\frac{f(t)}{W(t)} = 0,$$

equipped with the norm

$$||f||_W = \sup\left\{\frac{|f(t)|}{W(t)} : t \in [0,\infty)\right\}.$$

Suppose also that $\{\lambda_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive real numbers diverging to infinity so that $\liminf_{n\to\infty}(\lambda_{n+1}-\lambda_n) > 0$. Malliavin proved [5, Theorem 8.3] that the span of the system $\{t^{\lambda_n}\}_{n=1}^{\infty}$ is not dense in C_W if and only if there exists $\eta \in \mathbb{R}$ such that

$$\int_1^{+\infty} \frac{\log |W(e^{\sigma_\Lambda(t)-\eta})|}{t^2}\, dt < \infty, \quad \text{where} \quad \sigma_\Lambda(t) = \sum_{\lambda_n \le t} \frac{2}{\lambda_n}.$$

The question of the closure of the non-dense span of the system $\{t^{\lambda_n}\}_{n=1}^{\infty}$ was later on addressed by J. M. Anderson and K. G. Binmore [1, Theorem 3]. Provided that the λ_n are positive integers, they proved that any function in the closure extends analytically as an entire function with a gap power series expansion of the form $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$.

We note that A. Borichev [2] gave a complete characterization of the closure of polynomials in certain weighted Banach spaces on $\mathbb{R}$, when W is an even log-convex function.

Motivated by the above results, we explored in [7, 8] the properties of a class of exponential systems

$$E_\Lambda := \{t^k e^{\lambda_n t} : n \in \mathbb{N},\ k = 0, 1, 2, \ldots, \mu_n - 1\},$$

in certain weighted Banach spaces on the real line. We note that such a system is associated to a set $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ with multiple terms

$$\{\lambda_n, \mu_n\}_{n=1}^{\infty} := \{\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{\mu_1 - times}, \underbrace{\lambda_2, \lambda_2, \ldots, \lambda_2}_{\mu_2 - times}, \ldots, \underbrace{\lambda_k, \lambda_k, \ldots, \lambda_k}_{\mu_k - times}, \ldots\},$$

where

- $\{\lambda_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive real numbers diverging to infinity,
- $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of positive integers, not necessarily bounded.

We say that the set Λ is a multiplicity sequence.

In [7, 8] we assumed that the multiplicity sequence Λ belongs to a certain class denoted by $U(d, 0)$. This class and the weighted Banach spaces involved will be recalled in Section 2, while the main results from [7, 8] will be restated in Section 3.

In this paper we continue our investigations by considering a moment problem in a weighted L^2 space on the real line. Our result, Theorem 4, is proved in Section 5. Prior to that, we introduce in Section 4 some notions from Non-Harmonic Fourier Series such as Bessel and Riesz–Fischer sequences that will play a decisive role.

The following interesting result is a special case of Theorem 4.

Theorem 1. *Let*

$$w(t) = \begin{cases} t^{2m+2}, & t \ge 0, \\ 0, & t < 0, \end{cases} \quad \textit{where} \quad m \in \mathbb{N}.$$

Let $\{p_n\}_{n=1}^{\infty}$ *be the increasing sequence of prime numbers and let* $\mu_n = p_{n+1} - p_n$ *for each* $n \in \mathbb{N}$, *that is,* μ_n *is the distance between consecutive primes. Then, for any real number* $\gamma < 2$, *there exists an entire function* f *admitting a Taylor-Dirichlet series representation*

$$f(z) = \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{\mu_n - 1} c_{n,k} z^k \Big) e^{p_n z}, \quad c_{n,k} \in \mathbb{C}, \quad \forall\, z \in \mathbb{C},$$

with the series converging uniformly on compact subsets of $\mathbb{C}$, *so that*

$$\int_{-\infty}^{\infty} e^{-2w(t)} t^k e^{p_n t} f(t)\, dt = p_n^{\gamma p_n}, \quad \forall\, n \in \mathbb{N} \quad \textit{and} \quad k = 0, 1, 2, \ldots \mu_n - 1.$$

2. Notations and definitions from [7, 8]

2.1. Weighted Banach spaces

Definition 1. *We denote by $A_{\rho,\tau}$ the class of all non-negative convex functions $w(t)$ defined on the real line that satisfy the following properties:*

(i) $w(0) = 0$ *and* $w(t) \geq t^2, \quad \forall\, t \geq \tau \geq 0,$

(ii) *there is some $\rho > 0$ so that* $w(t) \leq \rho|t| \quad \forall\, t < 0,$

(iii) *for all $A > 0$ there is a positive number $t(A)$ such that* $w(t + A) \geq w(t) + t, \quad \forall\, t \geq t(A).$

Example 1. Let

$$w(t) = \begin{cases} t^{2m+2}, & t \geq 0, \\ 0, & t < 0, \end{cases} \quad \text{where} \quad m \in \mathbb{N},$$

then $w \in A_{\rho,\tau}$.

For $p \geq 1$ we denote by L^p_w the weighted Banach space of complex-valued measurable functions f defined on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |f(t)e^{-w(t)}|^p\, dt < \infty,$$

equipped with the norm

$$||f||_{L^p_w} := \left(\int_{-\infty}^{\infty} |f(t)e^{-w(t)}|^p\, dt\right)^{1/p}.$$

As usual, L^2_w is a Hilbert space when endowed with the inner product

$$\langle f, g\rangle := \int_{-\infty}^{\infty} f(t)\overline{g(t)}e^{-2w(t)}\, dt.$$

2.2. The class of multiplicity sequences $U(d, 0)$

We say that a multiplicity sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ has finite density d counting multiplicities, if

$$\lim_{n\to\infty} \frac{n_\Lambda(t)}{t} = d < \infty, \quad \text{where} \quad n_\Lambda(t) := \sum_{\lambda_n \leq t} \mu_n. \tag{2.1}$$

If $\mu_n = 1$ for all $n \in \mathbb{N}$ the above is equivalent to

$$\frac{n}{\lambda_n} \to d \quad \text{as} \quad n \to \infty.$$

Definition 2. *We denote by $L(c, d)$ the class of strictly increasing sequences $A = \{a_n\}_{n=1}^{\infty}$ having positive real terms a_n such that A has a finite density d and uniformly separated terms for some $c > 0$, that is,*

$$\frac{n}{a_n} \to d \quad \text{as} \quad n \to \infty, \quad a_{n+1} - a_n > c \quad \forall\, n \in \mathbb{N}.$$

Suppose now that a sequence $A=\{a_n\}_{n=1}^{\infty}$ belongs to the class $L(c,d)$. Then choose two positive numbers α, δ so that

$$\alpha < 1 \quad \text{and} \quad \delta \le \min\{4, c\}.$$

For each $n \in \mathbb{N}$ consider the closed segment $T_n := \{x : |x - a_n| \le a_n^{\alpha}\} \subset \mathbb{R}$. Then, choose a point in T_n that we call b_n, in an almost arbitrary way, in the sense that

$$\text{for all} \quad n \ne m \quad \text{either} \quad (I)\ b_m = b_n \quad \text{or} \quad (II)\ |b_m - b_n| \ge \delta.$$

Hence a new sequence $B = \{b_n\}_{n=1}^{\infty}$ is constructed.

We remark that the condition (I) allows for the presence of multiple terms in B. We may now rewrite $B = \{b_n\}_{n=1}^{\infty}$ in the form of a multiplicity sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$, by grouping together all those terms that have the same modulus.

Definition 3. *Fix a nonnegative constant d. We denote by $U(d,0)$ the class of all the multiplicity sequences $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ constructed in the way described above from sequences $A = \{a_n\}_{n=1}^{\infty}$ which belong to the class $L(c,d)$, for any positive constants α, δ, c, with $\alpha < 1$ and $\delta \le \min\{4, c\}$.*

Remark 1. Clearly $L(c,d)$ is a subclass of $U(d,0)$.

We now mention two important properties of a sequence $\Lambda \in U(d,0)$ [8, Section 2].

(1) Λ has the *same density* d counting multiplicities as the original sequence A from which it was constructed, that is, (2.1) holds.

(2) There exists some $\chi > 0$ independent of n, so that

$$\mu_n \le \chi \lambda_n^{\alpha} \quad \forall\, n \in \mathbb{N}. \tag{2.2}$$

We also note that since $\alpha < 1$, then $\mu_n / \lambda_n \to 0$ as $n \to \infty$, hence for every $\epsilon > 0$ there is $n(\epsilon) \in \mathbb{N}$ so that

$$\mu_n \le \epsilon \lambda_n \quad \forall\, n \ge n(\epsilon). \tag{2.3}$$

Remark 2. We use the notation $U(d,0)$ since Λ has density d and $\mu_n/\lambda_n \to 0$ as $n \to \infty$. That is, the second parameter in our notation stands for the relation between the multiplicities μ_n and their corresponding frequencies λ_n.

An interesting multiplicity sequence in the $U(1,0)$ class with unbounded multiplicities is the following.

Example 2. Let $\{p_n\}_{n=1}^{\infty}$ be the increasing sequence of prime numbers, and let $\mu_n = p_{n+1} - p_n$ for each $n \in \mathbb{N}$. Then $\Lambda = \{p_n, \mu_n\}_{n=1}^{\infty}$ belongs to the class $U(1,0)$. It can be constructed in the way described above from the set $\mathbb{N}$ of natural numbers which has density 1 (see [7, Example 1.3] and [8, Example 2.1]).

3. Our previous main results and the new one

Assuming that a multiplicity sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ belongs to the class $U(d,0)$, we obtained in [7] necessary and sufficient conditions in order for the span of E_Λ to be dense in L^p_w.

Theorem 2 [7, Theorem 1.1]. *Let $w(t)$ be a function which belongs to the class $A_{\rho,\tau}$ and suppose that $\Lambda \in U(d,0)$ for some $d>0$. Then the span of the system E_Λ is not dense in L^p_w for all $p\in[1,\infty)$, if and only if there exists $\eta\in\mathbb{R}$ such that*

$$\int_1^{+\infty} \frac{w(\sigma_\Lambda(t)-\eta)}{1+t^2}\,dt<\infty, \quad \sigma_\Lambda(t):=2\sum_{\lambda_n\le t}\frac{\mu_n}{\lambda_n}. \tag{3.1}$$

We then characterized in [8] the closure of the non-dense span of E_Λ. Moreover, in [8] we also derived an upper bound for the norm of the elements of a biorthogonal sequence

$$r_\Lambda := \{r_{n,k}:\ n\in\mathbb{N},\ k=0,1,\dots,\mu_n-1\}\subset L^2_w$$

to the system E_Λ in L^2_w, where biorthogonality means

$$\int_{-\infty}^{\infty} r_{n,k}(t)t^l e^{\lambda_j t}e^{-2w(t)}\,dt = \begin{cases} 1, & j=n, \quad l=k,\\ 0, & j=n, \quad l\in\{0,1,\dots,\mu_n-1\}\setminus\{k\},\\ 0, & j\ne n, \quad l\in\{0,1,\dots,\mu_j-1\}.\end{cases}$$

Theorem 3 [8, Theorems 2.1 and 6.1]. *Suppose that $\Lambda\in U(d,0)$ for some $d>0$, $w(t)\in A_{\rho,\tau}$ and (3.1) holds.*

Part I. *Let f be a function which belongs to the closed span of E_Λ in L^p_w for some $p\ge 1$. Then there is an entire function $g(z)$ which admits a Taylor-Dirichlet series representation*

$$g(z)=\sum_{n=1}^{\infty}\Big(\sum_{k=0}^{\mu_n-1} c_{n,k}z^k\Big)e^{\lambda_n z}, \quad c_{n,k}\in\mathbb{C}, \quad \forall\ z\in\mathbb{C},$$

with the series converging uniformly on compact subsets of $\mathbb{C}$, so that $f(x)=g(x)$ almost everywhere on the real line.

Part II. *There is a unique biorthogonal sequence r_Λ to the system E_Λ in L^2_w which belongs to its closed span, such that for every $\epsilon>0$ there is a constant $m_\epsilon>0$, independent of n and k, so that*

$$||r_{n,k}||_{L^2_w}\le m_\epsilon \exp\big\{(-2d+\epsilon)\lambda_n\log\lambda_n\big\}, \quad \forall\ n\in\mathbb{N}, \quad k=0,1,\dots,\mu_n-1. \tag{3.2}$$

Our aim in this article is to prove the following moment problem result.

Theorem 4. *Suppose that $\Lambda\in U(d,0)$ for some $d>0$, $w(t)\in A_{\rho,\tau}$ and (3.1) holds. Consider a doubly-indexed sequence of non-zero complex numbers*

$$\{d_{n,k}:\ n\in\mathbb{N},\ k=0,1,\dots,\mu_n-1\}$$

such that

$$\limsup_{n\to\infty}\frac{\log A_n}{\lambda_n\log\lambda_n}=\gamma<2d, \quad A_n=\max\{|d_{n,k}|:\ k=0,1,\dots,\mu_n-1\}. \tag{3.3}$$

Then there exists a function $f\in\overline{\text{span}}\,(E_\Lambda)$ in L^2_w that extends analytically as an entire function, admitting a Taylor-Dirichlet series representation

$$f(z)=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\mu_n-1} c_{n,k}z^k\right)e^{\lambda_n z}, \quad c_{n,k}\in\mathbb{C}, \quad \forall\ z\in\mathbb{C},$$

with the series converging uniformly on compact subsets of $\mathbb{C}$, so that

$$\int_{-\infty}^{\infty} e^{-2w(t)}t^k e^{\lambda_n t}f(t)\,dt=d_{n,k}, \quad \forall\ n\in\mathbb{N} \quad \text{and} \quad k=0,1,2,\dots\mu_n-1. \tag{3.4}$$

We point out that similar moment problems were considered in [8, Theorems 1.2 and 7.1] but the solution obtained is a continuous function on $\mathbb{R}$ rather than an entire function.

We also note that Theorem 1 follows by combining Theorem 4 with Example 1, Example 2, and

Remark 3. Suppose that Λ has a positive density d. A sufficient condition for (3.1) to hold (see the proof of [8, Theorem 2.2]) is if $w(t) \in A_{\rho,\tau}$ such that

$$t^2 \leq w(t) \leq e^{\xi t}, \quad \forall\, t \geq \tau \geq 0, \quad 0 < \xi < \frac{1}{2d}.$$

The following results are direct consequences of Theorem 4.

Corollary 1. *Let $w(t)$ be as in Example 1.*

(A) *Suppose that $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence in the $L(c,d)$ class for some $d > 0$ and consider a sequence of non-zero complex numbers $\{d_n\}_{n=1}^{\infty}$ such that*

$$\limsup_{n\to\infty} \frac{\log |d_n|}{\lambda_n \log \lambda_n} < 2d.$$

Then there exists an entire function f admitting a Dirichlet series representation

$$f(z) = \sum_{n=1}^{\infty} c_n e^{\lambda_n z}, \quad c_n \in \mathbb{C}, \quad \forall\, z \in \mathbb{C},$$

with the series converging uniformly on compact subsets of $\mathbb{C}$, so that

$$\int_{-\infty}^{\infty} e^{-2w(t)} e^{\lambda_n t} f(t)\, dt = d_n, \quad \forall\, n \in \mathbb{N}.$$

(B) *There exist entire functions f and g admitting a Dirichlet series representation*

$$f(z) = \sum_{n=1}^{\infty} c_n e^{nz}, \quad g(z) = \sum_{n=1}^{\infty} d_n e^{nz},$$

so that for all $n \in \mathbb{N}$ we have

$$\int_{-\infty}^{\infty} e^{-2w(t)} e^{nt} f(t)\, dt = n^n, \quad \int_{-\infty}^{\infty} e^{-2w(t)} e^{nt} g(t)\, dt = n!.$$

4. Bessel and Riesz–Fischer sequences

The proof of Theorem 4 depends on Theorem 3 and utilizes the following notions from Non-Harmonic Fourier Series.

Let H be a separable Hilbert space endowed with an inner product $\langle \cdot \rangle$, and consider two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ in H. We say that [6, Chapter 4, Section 2]:

(i) $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 < B||f||^2 \quad \forall\, f \in H.$$

(ii) $\{g_n\}_{n=1}^{\infty}$ is a Riesz–Fischer sequence if the moment problem $\langle f, g_n\rangle = c_n$ has at least one solution $f \in H$ for every sequence $\{c_n\}_{n=1}^{\infty}$ in the space $l^2(\mathbb{N})$.

Remark 4. It follows from [3, Proposition 2.3] that if two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ in H are biorthogonal, that is

$$\langle f_n, g_m\rangle = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases}$$

and $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence, then $\{g_n\}_{n=1}^{\infty}$ is a Riesz–Fischer sequence.

We give now a sufficient condition in order for $\{g_n\}_{n=1}^{\infty}$ to be a Riesz–Fischer sequence.

Lemma 1. *Let H be a separable Hilbert space and consider two biorthogonal sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ in H. Let $c_{n,m} = \langle f_n, f_m\rangle$ and let $C = (c_{n,m})$ be the Hermitian Gram matrix associated with $\{f_n\}_{n=1}^{\infty}$. If there is some $M > 0$ so that*

$$\sum_{n=1}^{\infty} |c_{n,m}| < M \quad \textit{for all} \quad m = 1, 2, 3, \ldots, \tag{4.1}$$

then $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are Bessel and Riesz-Fischer sequences respectively in H.

P r o o f. Relation (4.1) implies that the Gram matrix C defines a bounded linear operator on the space of sequences $l^2(\mathbb{N})$ (see [4, Lemma 3.5.3] and [6, Sec. 4.2, Lemma 1]). It then follows by [4, Lemma 3.5.1] that $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence in H. By Remark 4 we conclude that $\{g_n\}_{n=1}^{\infty}$ is a Riesz–Fischer sequence in H. □

5. Proof of Theorem 4

Clearly $\overline{\text{span}}\,(E_\Lambda)$ in L^2_w is a separable Hilbert space and let us denote this space by H_Λ. From Theorem 3 (Part II), let $\{r_{n,k}\}$ be the biorthogonal sequence to E_Λ which belongs to its closed span.

Then, define for every $n \in \mathbb{N}$ and $k = 0, 1, \ldots, \mu_n - 1$ the following:

$$U_{n,k}(t) := \lambda_n d_{n,k} r_{n,k}(t) \quad \text{and} \quad V_{n,k}(t) := \frac{t^k e^{\lambda_n t}}{\overline{\lambda_n d_{n,k}}}.$$

It easily follows that $\{U_{n,k}\}$ and $\{V_{n,k}\}$ are biorthogonal sequences in H_Λ.

We now claim that $\{U_{n,k}\}$ and $\{V_{n,k}\}$ are Bessel and Riesz–Fischer sequences respectively in H_Λ. First, since (3.2) and (3.3) hold, if we let $\epsilon = (2d - \gamma)/2$ we get

$$||U_{n,k}||_{L^2_w} \leq e^{-\epsilon\lambda_n}, \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad k = 0, 1, 2, \ldots, \mu_n - 1.$$

Then, by the Cauchy-Schwartz inequality we get

$$|\langle U_{n,k}, U_{m,j}\rangle| \leq e^{-\epsilon\lambda_n} \cdot e^{-\epsilon\lambda_m}, \quad \forall\, n, m \in \mathbb{N} \quad k = 0, 1, 2, \ldots, \mu_n - 1 \quad j = 0, 1, 2, \ldots, \mu_m - 1. \tag{5.1}$$

Next, let $c_{n,k,m,j}$ be the value of $\langle U_{n,k}, U_{m,j}\rangle$ and let C be the infinite dimensional hermitian matrix with entries the $c_{n,k,m,j}$'s, that is C is the Gram matrix associated with $\{U_{n,k}\}$. From (2.3) and (5.1) we get

$$\sum_{n=1}^{\infty} \sum_{k=0}^{\mu_n - 1} \sum_{m=1}^{\infty} \sum_{j=0}^{\mu_m - 1} |c_{n,k,m,j}| < \infty.$$

It then follows from Lemma 1 that our claim is valid.

Thus, the moment problem

$$\int_{-\infty}^{\infty} f(t)\overline{V_{n,k}(t)}e^{-2w(t)}\,dt = a_{n,k} \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad k = 0, 1, 2, \dots, \mu_n - 1,$$

has a solution in H_Λ whenever $\sum_{n=1}^{\infty}\sum_{k=0}^{\mu_n-1} |a_{n,k}|^2 < \infty$. Now, if we let

$$a_{n,k} = \frac{1}{\lambda_n} \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad k = 0, 1, \dots, \mu_n - 1,$$

then the density of Λ and relation (2.2) imply that

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\mu_n-1} |a_{n,k}|^2 = \sum_{n=1}^{\infty} \frac{\mu_n}{\lambda_n^2} < \infty.$$

Thus, $\{a_{n,k}\}$ belongs to the space $l^2(\mathbb{N})$. Hence, and recalling the definition of $V_{n,k}$, there is some function $f \in H_\Lambda$ so that

$$\int_{\infty}^{\infty} f(t)\left(\frac{t^k e^{\lambda_n t}}{d_{n,k}\lambda_n}\right) e^{-2w(t)}\,dt = \frac{1}{\lambda_n}, \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad k = 0, 1, 2, \dots, \mu_n - 1.$$

Clearly now (3.4) holds.

Finally, since $f \in H_\Lambda$ it follows from Theorem 3 (Part I) that f extends analytically as an entire function admitting a Taylor–Dirichlet series representation. Our proof is now complete.

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