THE DYNAMIC DEFORMATION
OF THREE-COMPONENT POROUS MEDIA

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Abstract: A mathematical model of the dynamic deformation of three-component elastic media saturated with liquid and gas, given by elastic moduli and coefficients characterizing the porosity and compressibility of the liquid and gas, is considered. Formulas for determining the propagation velocity of monochromatic waves in ternary porous media are obtained. The existence of three longitudinal waves depends on the discriminant of a cubic equation and the velocity ratio.

Keywords: Elasticity, Medium, Fluid, Stress, Deformation, Displacement.

1. Introduction

There are a number of papers [1–3, 8, 10] devoted to the propagation of elastic waves in two-component porous media. Among these studies, papers of M. A. Biot [1–3] should be noted. He created the theory of elasticity and consolidation of a porous medium. This theory studies settlement under the influence of a load of a porous medium containing a viscous fluid.

Phase states, laws of thermodynamics of porous systems, and attempts to solve wave problems in porous materials and moist soils were considered by Ya. I. Phrenkel [6], J. V. Reznichenko [13], and Kh. A. Rakhmatulin [12]. The studies of these authors played a huge role in creating the classic Biot–Phrenkel model.

When solving a considerable number of applied problems arising in various areas of human activity (soil, porous sintered composition materials, building materials, etc.), one has to deal with a three-component media. The complexity of describing the effects of the interaction of components, heat transfer, and other related processes has led to the fact that until now the generally accepted models (elastic medium—liquid—gas) have not been fully developed. Therefore, a mathematical three-component model that takes into account the porosity of the medium is of apparent interest.

The paper considers the ratio of the velocities of acceleration waves in a three-component porous medium to the propagation velocities of the wave surface of the porous medium in the longitudinal and transverse directions. The interpenetrating motion of the elastic component, liquid and gas is perceived as the motion of liquid, and gas in a deformable porous medium. It is supposed that the pore size is small compared to the distance at which the kinematic and dynamic characteristics of the motion change significantly. This allows us to assume that all three media are continuous and that at each point in space there are three displacement vectors.

It is proved that, in such a medium, in the general case, three waves propagate, whose velocities essentially depend on the direction of propagation of the wave surface. Graphs of the dependence of the velocity ratio on the porosity of the medium are constructed.

2. Main results

Consider a system of equations determining the dynamic behavior of a three-component medium saturated with liquid and gas in the motion of the components [9]:

- complete stress tensor in the skeleton in the presence of liquid and gas in pores

$$T_{ij} = \lambda u_{k,k}^{(1)}\delta_{ij} + \mu(u_{i,j}^{(1)} + u_{j,i}^{(1)}) + \underline{m}R_0^{(2)}u_{k,k}^{(2)}\delta_{ij} + \underline{m}R_0^{(3)}u_{k,k}^{(3)}\delta_{ij}; \tag{2.1}$$

- forces acting on the liquid and gas per unit area of the cross section of the porous medium:

$$\begin{aligned}
N &= \underline{m}R_0^{(2)}u_{k,k}^{(1)} + mR_0^{(2)}u_{k,k}^{(2)} + \underline{m}R_0^{(2)}u_{k,k}^{(3)}, \\
P &= \underline{m}R_0^{(3)}u_{k,k}^{(1)} + \underline{m}R_0^{(3)}u_{k,k}^{(2)} + mR_0^{(3)}u_{k,k}^{(3)};
\end{aligned} \tag{2.2}$$

- equations of motion of the porous media

$$\begin{aligned}
\rho_{11}\ddot{u}_i^{(1)} + \rho_{12}\ddot{u}_i^{(2)} + \rho_{13}\ddot{u}_i^{(3)} &= T_{ij,j}, \\
\rho_{21}\ddot{u}_i^{(1)} + \rho_{22}\ddot{u}_i^{(2)} + \rho_{23}\ddot{u}_i^{(3)} &= N_{,i}, \\
\rho_{31}\ddot{u}_i^{(1)} + \rho_{32}\ddot{u}_i^{(2)} + \rho_{33}\ddot{u}_i^{(3)} &= P_{,i}.
\end{aligned} \tag{2.3}$$

Here λ and μ are the Lamé coefficients; $u_i^{(\alpha)}$ are the component displacements, where $\alpha = \overline{1,3}$ stands for the medium: 1 for the rigid component, 2 for the liquid, and 3 for the gas; the dots above the letters indicate the time derivatives; indices after the comma below the letter stand for the derivatives of the corresponding coordinates; δ_{ij} is the Kronecker symbol; ρ_{11}, ρ_{22}, and ρ_{33} are effective densities of the rigid component, liquid, and gas, respectively; $\rho_{11} < 0$, $\rho_{12} < 0$, and $\rho_{13} < 0$ are the coefficients of dynamic coupling of the skeleton, liquid, and gas, respectively; $R_0^{(2)}$ and $R_0^{(3)}$ are compressibility moduli of the components saturated with liquid and gas, respectively; $0 \le m \le 1$ is the porosity of a medium, $\underline{m} = 1 - m$; and $i,j,k = \overline{1,3}$. Suppose that $\rho_{ij} = \rho_{ji}$.

Hereinafter, the repeated indices assume a summation of one to three.

An acceleration wave in a three-component porous media saturated with a liquid and gas is an isolated surface on which the stress, the forces acting on the liquid and gas, and the propagation velocities of the components are continuous while some of their partial derivatives have discontinuities.

Differentiating relations (2.1) and (2.2) in t, we obtain

$$\begin{aligned}
\dot{T}_{ij} &= \lambda v_{k,k}^{(1)}\delta_{ij} + \mu(v_{i,j}^{(1)} + v_{j,i}^{(1)}) + \underline{m}R_0^{(2)}v_{k,k}^{(2)}\delta_{ij} + \underline{m}R_0^{(3)}v_{k,k}^{(3)}\delta_{ij}, \\
\dot{N} &= \underline{m}R_0^{(2)}v_{k,k}^{(1)} + mR_0^{(2)}v_{k,k}^{(2)} + \underline{m}R_0^{(2)}v_{k,k}^{(3)}, \\
\dot{P} &= \underline{m}R_0^{(3)}v_{k,k}^{(1)} + \underline{m}R_0^{(3)}v_{k,k}^{(2)} + mR_0^{(3)}v_{k,k}^{(3)}.
\end{aligned} \tag{2.4}$$

Let us write equations (2.3) and relations (2.4) in discontinuities [5, 7, 11, 14]:

$$\begin{aligned}
\lambda[v_{k,k}^{(1)}]\delta_{ij} + \mu([v_{i,j}^{(1)}] + [v_{j,i}^{(1)}]) + \underline{m}R_0^{(2)}[v_{k,k}^{(2)}]\delta_{ij} + \underline{m}R_0^{(3)}[v_{k,k}^{(3)}]\delta_{ij} &= [\dot{T}_{ij}], \\
\underline{m}R_0^{(2)}[v_{k,k}^{(1)}] + mR_0^{(2)}[v_{k,k}^{(2)}] + \underline{m}R_0^{(2)}[v_{k,k}^{(3)}] &= [\dot{N}], \\
\underline{m}R_0^{(3)}[v_{k,k}^{(1)}] + \underline{m}R_0^{(3)}[v_{k,k}^{(2)}] + mR_0^{(3)}[v_{k,k}^{(3)}] &= [\dot{P}], \\
\rho_{11}[\dot{v}_i^{(1)}] + \rho_{12}[\dot{v}_i^{(2)}] + \rho_{13}[\dot{v}_i^{(3)}] &= [T_{ij,j}], \\
\rho_{21}[\dot{v}_i^{(1)}] + \rho_{22}[\dot{v}_i^{(2)}] + \rho_{23}[\dot{v}_i^{(3)}] &= [N_{,i}], \\
\rho_{31}[\dot{v}_i^{(1)}] + \rho_{32}[\dot{v}_i^{(2)}] + \rho_{33}[\dot{v}_i^{(3)}] &= [P_{,i}],
\end{aligned} \tag{2.5}$$

where $[\cdot]$ denotes the difference in the values of a function on different sides of the discontinuity surface.

We apply kinematic and geometric consistency conditions of first-order to relations (2.5) on the discontinuity surface:

$$[T_{ik,k}] = s_{ik}\nu_k, [\dot{T}_{ik}] = -s_{ik}G, [N_{,k}] = \eta\nu_k, [\dot{N}] = -\eta G,$$
$$[P_{,i}] = \gamma\nu_i, [\dot{P}] = -\gamma G, [v_{i,k}^{(\alpha)}] = \lambda_i^{(\alpha)}\nu_k, [\dot{v}_i^{(\alpha)}] = -\lambda_i^{(\alpha)}G. \tag{2.6}$$

Here s_{ik}, η, γ, and $\lambda_i^{(\alpha)}$ are values characterizing jumps of the first derivatives of stresses, forces acting on the liquid and gas, and the propagation velocities of the components; ν_i are the components of the unit normal to the wave surface; and G is the propagation velocity of the wave surface of the porous medium.

Using conditions (2.6), we write formulas (2.5) in the form

$$\lambda\lambda_k^{(1)}\nu_k\delta_{ij} + \mu(\lambda_i^{(1)}\nu_j + \lambda_j^{(1)}\nu_i) + \underline{m}R_0^{(2)}\lambda_k^{(2)}\nu_k\delta_{ij} + \underline{m}R_0^{(3)}\lambda_k^{(3)}\nu_k\delta_{ij} = -s_{ij}G,$$
$$\underline{m}R_0^{(2)}\lambda_k^{(1)}\nu_k + mR_0^{(2)}\lambda_k^{(2)}\nu_k + \underline{m}R_0^{(2)}\lambda_k^{(3)}\nu_k = -\eta G,$$
$$\underline{m}R_0^{(3)}\lambda_k^{(1)}\nu_k + \underline{m}R_0^{(3)}\lambda_k^{(2)}\nu_k + mR_0^{(3)}\lambda_k^{(3)}\nu_k = -\gamma G,$$
$$\rho_{11}\lambda_i^{(1)}G + \rho_{12}\lambda_i^{(2)}G + \rho_{13}\lambda_i^{(3)}G = -s_{ij}\nu_j, \tag{2.7}$$
$$\rho_{12}\lambda_i^{(1)}G + \rho_{22}\lambda_i^{(2)}G + \rho_{23}\lambda_i^{(3)}G = -\eta\nu_i,$$
$$\rho_{13}\lambda_i^{(1)}G + \rho_{23}\lambda_i^{(2)}G + \rho_{33}\lambda_i^{(3)}G = -\gamma\nu_i.$$

Excluding the values s_{ij}, η, and γ from (2.7), we get a homogeneous system for $\lambda_k^{(1)}$, $\lambda_k^{(2)}$, and $\lambda_k^{(3)}$:

$$\lambda\lambda_k^{(1)}\nu_k\nu_i + \mu(\lambda_i^{(1)} + \lambda_j^{(1)}\nu_i\nu_j) + \underline{m}R_0^{(2)}\lambda_k^{(2)}\nu_k\nu_i + \underline{m}R_0^{(3)}\lambda_k^{(3)}\nu_k\nu_i =$$
$$= \rho_{11}G^2\lambda_i^{(1)} + \rho_{12}G^2\lambda_i^{(2)} + \rho_{13}G^2\lambda_i^{(3)},$$
$$\underline{m}R_0^{(2)}\lambda_k^{(1)}\nu_k\nu_i + mR_0^{(2)}\lambda_k^{(2)}\nu_k\nu_i + \underline{m}R_0^{(2)}\lambda_k^{(3)}\nu_k\nu_i = \tag{2.8}$$
$$= \rho_{12}G^2\lambda_i^{(1)} + \rho_{22}G^2\lambda_i^{(2)} + \rho_{23}G^2\lambda_i^{(3)},$$
$$\underline{m}R_0^{(3)}\lambda_k^{(1)}\nu_k\nu_i + \underline{m}R_0^{(3)}\lambda_k^{(2)}\nu_k\nu_i + mR_0^{(3)}\lambda_k^{(3)}\nu_k\nu_i =$$
$$= \rho_{13}G^2\lambda_i^{(1)} + \rho_{23}G^2\lambda_i^{(2)} + \rho_{33}G^2\lambda_i^{(3)}.$$

Similar to [8], system (2.8) enables deriving formulas for determining the velocity of longitudinal and transverse waves in the three-component porous media.

We find propagation velocities of longitudinal waves assuming that $\lambda_k^{(\alpha)}\nu_k \neq 0$ on the wave surface. Reducing (2.8) by ν_i and summing over the repeated index i, we obtain the homogeneous system of three linear equations for $\omega_\alpha = \lambda_i^{(\alpha)}\nu_i$:

$$(\Lambda - \rho_{11}G_l^2)\omega_1 + (\underline{m}R_0^{(2)} - \rho_{12}G_l^2)\omega_2 + (\underline{m}R_0^{(3)} - \rho_{13}G_l^2)\omega_3 = 0,$$
$$(\underline{m}R_0^{(2)} - \rho_{12}G_l^2)\omega_1 + (mR_0^{(2)} - \rho_{22}G_l^2)\omega_2 + (\underline{m}R_0^{(2)} - \rho_{23}G_l^2)\omega_3 = 0, \tag{2.9}$$
$$(\underline{m}R_0^{(3)} - \rho_{13}G_l^2)\omega_1 + (\underline{m}R_0^{(3)} - \rho_{23}G_l^2)\omega_2 + (mR_0^{(3)} - \rho_{33}G_l^2)\omega_3 = 0,$$

where $\Lambda = \lambda + 2\mu$.

Define

$$\sigma_{11} = \frac{\Lambda}{M}, \quad \sigma_{22} = \frac{mR_0^{(2)}}{M}, \quad \sigma_{33} = \frac{mR_0^{(3)}}{M},$$

$$\sigma_{12} = \sigma_{21} = \sigma_{23} = \frac{\underline{m}R_0^{(2)}}{M}, \quad \sigma_{13} = \sigma_{31} = \sigma_{32} = \frac{\underline{m}R_0^{(3)}}{M},$$

$$M = \Lambda + mR_0^{(2)} + mR_0^{(3)} + 3\underline{m}R_0^{(2)} + 3\underline{m}R_0^{(3)};$$

$$\gamma_{ij} = \frac{\rho_{ij}}{\rho}, \quad \rho = \sum_i \rho_{ii} + 2 \sum_{\substack{i,j;i\neq j; \\ i<j}} \rho_{ij}. \tag{2.10}$$

Taking into account (2.10), we write system (2.9) in the dimensionless matrix form:

$$(\boldsymbol{\Sigma}z_l^2 - \boldsymbol{\Gamma})\vec{\omega} = 0, \boldsymbol{\Sigma} = \{\sigma_{ij}\}, \boldsymbol{\Gamma} = \{\gamma_{ij}\}, \vec{\omega} = \{\omega_i\}, \tag{2.11}$$

where $z_l^2 = c_l^2/G_l^2$, $c_l^2 = M/\rho$; c_l are the propagation velocities of the longitudinal waves in the porous media; G_l is the longitudinal component of the propagation velocity the wave surface in the porous medium; and z_l is the longitudinal velocity ratio.

The condition for system (2.11), homogeneous with respect to ω_1, ω_2, and ω_3, to have a non-trivial solution is that its third order determinant must be zero:

$$\left|\boldsymbol{\Sigma}z_l^2 - \boldsymbol{\Gamma}\right| = 0. \tag{2.12}$$

It is shown in what follows that condition (2.12) also defines three propagation velocities of the wave surface in the three-component porous medium.

Expanding the determinant (2.12), we obtain a cubic equation for z_l^2:

$$kz_l^6 + bz_l^4 + dz_l^2 + f = 0, \tag{2.13}$$

where

$$k = \sigma_{11}(\sigma_{22}\sigma_{33} - \sigma_{12}\sigma_{13}) + \sigma_{12}^2(\sigma_{13} - \sigma_{33}) + \sigma_{13}^2(\sigma_{12} - \sigma_{22}),$$

$$b = \gamma_{11}(\sigma_{12}\sigma_{13} - \sigma_{22}\sigma_{33}) + \gamma_{22}(\sigma_{13}^2 - \sigma_{11}\sigma_{33}) + \gamma_{33}(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) -$$

$$- \gamma_{12}(\sigma_{13}^2 + \sigma_{12}\sigma_{13} - 2\sigma_{12}\sigma_{33}) - \gamma_{13}(\sigma_{12}^2 + \sigma_{12}\sigma_{13} - 2\sigma_{13}\sigma_{22}) + \gamma_{23}\left[\sigma_{11}(\sigma_{12} + \sigma_{13}) - 2\sigma_{12}\sigma_{13}\right],$$

$$d = -\sigma_{11}(\gamma_{23}^2 - \gamma_{22}\gamma_{33}) - \sigma_{22}(\gamma_{13}^2 - \gamma_{11}\gamma_{33}) - \sigma_{33}(\gamma_{12}^2 - \gamma_{11}\gamma_{22}) -$$

$$- 2\sigma_{12}(\gamma_{12}\gamma_{33} - \gamma_{13}\gamma_{23}) - 2\sigma_{13}(\gamma_{13}\gamma_{22} - \gamma_{12}\gamma_{23}) + (\sigma_{12} + \sigma_{13})(\gamma_{12}\gamma_{13} - \gamma_{11}\gamma_{23}),$$

$$f = -(\gamma_{11}\gamma_{22}\gamma_{33} + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{11}\gamma_{23}^2 - \gamma_{22}\gamma_{13}^2 - \gamma_{33}\gamma_{12}^2).$$

We find the solution of the cubic equation (2.13) by the Cardano formulas [4]. Divide (2.13) by k and introduce a new variable

$$y = z_l^2 + \frac{b}{3k}.$$

On rearrangement, we get

$$y^3 + 3py + 2q = 0, \tag{2.14}$$

where

$$3p = \frac{d}{k} - \frac{1}{3}\left(\frac{b}{k}\right)^2, \quad 2q = 2\left(\frac{b}{3k}\right)^3 - \frac{bd}{3k^2} + \frac{f}{k}.$$

Let us calculate the discriminant $D = p^3 + q^2$. If $D < 0$, then (2.14) has three distinct real roots expressed in terms of complex values. If $D > 0$, then (2.14) has one real and two imaginary solutions. If $D = 0$, then there are three real solutions, two of which coincide.

Thus, in the considered three-component porous medium, there are three types of longitudinal waves can propagate depending on the discriminant of the cubic equation (2.14) and the velocity ratios $z_l^{(\alpha)}$.

Knowing the propagation velocities c_l of the longitudinal waves and the velocity ratios z_l, we can calculate the propagation velocity of the longitudinal wave surface in the three-component porous media by the formula $G_l^{(\alpha)} = c_l/z_l^{(\alpha)}$.

In the absence of coupling between the liquid—gas and elasticity—gas components, i.e., if $\gamma_{13} = 0$, $\gamma_{23} = 0$, $\sigma_{13} = 0$, and $\sigma_{32} = 0$, then equation (2.13) takes the form of biquadratic equation with respect to z_l^2:

$$k_1 z_l^4 + b_1 z_l^2 + d_1 = 0, \tag{2.15}$$

where

$$k_1 = \sigma_{11}\sigma_{22} - \sigma_{12}^2, \quad b_1 = 2\sigma_{12}\gamma_{12} - \sigma_{11}\gamma_{22} - \sigma_{22}\gamma_{11}, \quad d_1 = \gamma_{11}\gamma_{22} - \gamma_{12}^2.$$

Equation (2.15) coincides with the equation from [8].

Assume that $\lambda_i^{(\alpha)}\nu_i = 0$ in (2.8). Under the condition $G = G_t$, we obtain in the dimensionless form

$$(\sigma_{11}' z_t^2 - \gamma_{11})\omega_1 - \gamma_{12}\omega_2 - \gamma_{13}\omega_3 = 0,$$
$$\gamma_{12}\omega_1 + \gamma_{22}\omega_2 + \gamma_{23}\omega_3 = 0,$$
$$\gamma_{13}\omega_1 + \gamma_{23}\omega_2 + \gamma_{33}\omega_3 = 0; \tag{2.16}$$
$$\sigma_{11}' = \mu/\mathrm{M}', \quad \mathrm{M}' = \mu + mR_0^{(2)} + mR_0^{(3)} + 3\underline{m}R_0^{(2)} + 3\underline{m}R_0^{(3)},$$
$$z_t^2 = c_t^2/G_t^2, \quad c_t^2 = \mathrm{M}'/\rho.$$

For system (2.16) to have a nontrivial solution, its determinant must be zero.
Expanding the determinant

$$\begin{vmatrix} \sigma_{11}' z_t^2 - \gamma_{11} & -\gamma_{12} & -\gamma_{13} \\ \gamma_{12} & \gamma_{22} & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} \end{vmatrix},$$

we obtain an expression for determining the ratio of the propagation velocities of the transverse waves in the three-component media:

$$z_t = \sqrt{\frac{\gamma_{11}\gamma_{22}\gamma_{33} + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{11}\gamma_{23}^2 - \gamma_{22}\gamma_{13}^2 - \gamma_{33}\gamma_{12}^2}{\sigma_{11}'(\gamma_{22}\gamma_{33} - \gamma_{23}^2)}}. \tag{2.17}$$

In the absence of coupling between the liquid—gas and elastic—gas components, i.e., if $\gamma_{23} = 0$ and $\gamma_{13} = 0$, then (2.17) yields

$$z_t = \sqrt{\frac{\gamma_{11}\gamma_{22} - \gamma_{12}^2}{\sigma_{11}'\gamma_{22}}}. \tag{2.18}$$

Formula (2.18) coincides with the formula obtained in [8].

3. Calculation results

The figure, using the data in the table, shows the dependencies of the ratio of the propagation velocity of longitudinal waves in the three-component medium to the propagation velocity of the wave surface in the longitudinal direction on the medium porosity.

Table 1. Input data for calculating $z_l^{(\alpha)}$

m	σ_{11}	σ_{22}	σ_{33}	σ_{12}	σ_{13}	$\{\gamma_{ij}\}$
0.2		0.2	0.15	0.08	0.009	
0.4	0.6	0.15	0.1	0.025	0.025	$\gamma_{11} = 0.7$; $\gamma_{22} = 0.32$; $\gamma_{33} = 0.1$; $\gamma_{12} = \gamma_{13} = = \gamma_{23} = -0.02$
0.7		0.15	0.19	0.01	0.01	
0.9		0.1	0.15	0.025	0.025	
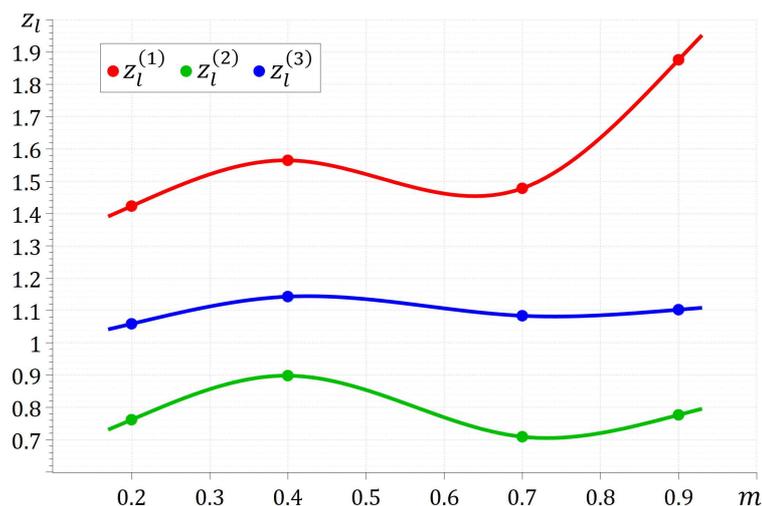
Figure 1. Velocity ratios in the three-component porous media

It is seen from the figure that the ratios $z_l^{(1)}$ and $z_l^{(2)}$ change from 1.4 to 1.9 and from 0.7 to 0.9, respectively. The ratio $z_l^{(3)}$ demonstrates a weak dependence on the porosity and is close to 1.1. Thus, in the three-component porous media, the ratios of longitudinal velocities can take values both more and less than one.

4. Conclusion

1. In the three-component porous media, three longitudinal and one transverse waves propagate whose velocities are defined by formulas (2.8) with $\lambda_k^{(\alpha)} \nu_k \neq 0$ or $\lambda_i^{(\alpha)} \nu_i = 0$.

2. In general, ratios of the longitudinal velocity components in the three-component porous medium depend on the coefficients and discriminant of a cubic equation.

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