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HAHN'S PROBLEM WITH RESPECT TO SOME PERTURBATIONS OF THE RAISING OPERATOR $X - c$

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Abstract: In this paper, we study the Hahn's problem with respect to some raising operators perturbed of the operator $X - c$, where c is an arbitrary complex number. More precisely, the two following characterizations hold: up to a normalization, the q-Hermite (resp. Charlier) polynomial is the only $H_{\alpha,q}$-classical (resp. $\mathcal{S}_\lambda$-classical) orthogonal polynomial, where $H_{\alpha,q} := X + \alpha H_q$ and $\mathcal{S}_\lambda := (X+1) - \lambda\tau_{-1}$.

Keywords: Orthogonal polynomials, Linear functional, $\mathcal{O}$-classical polynomials, Raising operators, q-Hermite polynomials, Charlier polynomials.

1. Introduction

Let $\mathcal{O}$ be a linear operator acting on the space of polynomials which sends polynomials of degree n to polynomials of degree $n + n_0$, where n_0 is a fixed integer ($n \geq 0$ if $n_0 \geq 0$ and $n \geq |n_0|$ if $n_0 < 0$). We call a sequence $\{P_n\}_{n\geq 0}$ of orthogonal polynomials $\mathcal{O}$-classical if $\{\mathcal{O}P_n\}_{n\geq 0}$ is also orthogonal.

In particular, if $\mathcal{O} = D$, the standard derivative, we recover the know family of classical orthogonal polynomials (Hermite, Laguerre, Bessel and Jacobi). This characterization is called Hahn's characterization (see [11, 18]) of the classical orthogonal polynomials. If $\mathcal{O} = H_q$, where

$$H_q f(x) = \frac{h_q f(x) - f(x)}{(q-1)x}, \quad q \neq 1, \quad h_q f(x) = f(qx),$$

we recover the so-called H_q-classical polynomials (for more details, see [12]). We can also cite [14], where the authors described the all D_ω-classical orthogonal polynomials, with

$$D_\omega f(x) := \frac{\tau_{-\omega} f(x) - f(x)}{w}, \quad \omega \neq 0, \quad \tau_{-\omega} f(x) = f(x+\omega).$$

The literature on these topics is extremely vast. See further examples in [1–5, 7, 8, 11, 12, 14].

In this paper we consider some *raising operators* related to the operator X. It is easy to see that the orthogonality is not preserved by X, then we can consider and study some perturbed operators. Here we consider the following two operators ($c = 0$ or $c = 1$):

$$H_{\alpha,q} := X + \alpha H_q \tag{1.1}$$

$$\mathcal{S}_\lambda := (X+1) - \lambda\tau_{-1}, \tag{1.2}$$

and we study the same problem, called Hahn's problem. More precisely, we find all orthogonal polynomial sequences $\{P_n\}_{n\geq 0}$ such that $\{\mathcal{O}P_n\}_{n\geq 0}$, $\mathcal{O} = H_{\alpha,q}$ or $\mathcal{S}_\lambda$, are also orthogonal. As a result, we conclude that the q-Hermite polynomial sequence is the only $H_{\alpha,q}$-classical sequence and the Charlier polynomial sequence is the only $\mathcal{S}_\lambda$-classical sequence.

The structure of the paper is the following. In Section 2, a basic background about forms of orthogonal polynomials is given. In Section 3, we show that, up to a dilatation, the q-Hermite (resp. Charlier) polynomial is the only $H_{\alpha,q}$-classical (resp. $\mathcal{S}_\lambda$-classical) orthogonal polynomial. In Section 4, we give a conclusion and describe some prospects.

2. Preliminaries

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients and $\mathbb{P}'$ be its dual space, whose elements are *forms*. We denote by $\langle u, p\rangle$ the action of $u \in \mathbb{P}'$ on $p \in \mathbb{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u. Let us define the following operations in $\mathbb{P}'$. For any form u, any polynomial f, and any $(a, b, c) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}^2$, let $Du = u'$, fu, $(x-c)^{-1}u$, $\tau_{-b}u$ and $h_a u$ be the forms defined by duality, [16]:

$$\langle fu, p\rangle := \langle u, fp\rangle, \quad \langle u', p\rangle := -\langle u, p'\rangle, \quad (fu)' = f'u + fu',$$
$$\langle h_a u, p\rangle := \langle u, p(ax)\rangle, \quad \langle \tau_{-b}u, p\rangle := \langle u, p(x-b)\rangle,$$
$$\langle (x-c)^{-1}u,\ p\rangle := \Big\langle u,\ \frac{p(x)-p(c)}{x-c}\Big\rangle, \quad p \in \mathbb{P}.$$

A form u is called *normalized* if it satisfies $(u)_0 = 1$. We assume that the forms used in this paper are normalized.

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials (MPS) with $\deg P_n = n$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathbb{P}'$, defined by $\langle u_n, P_m\rangle = \delta_{n,m}$, $n,\ m \geq 0$. Notice that u_0 is said to be the canonical functional associated with the MPS $\{P_n\}_{n\geq 0}$. The sequence $\{P_n\}_{n\geq 0}$ is called symmetric when $P_n(-x) = (-1)^n P_n(x)$, $n \geq 0$.

Let us recall the following result [17].

Lemma 1. *For any $u \in \mathbb{P}'$ and any integer $m \geq 1$, the following statements are equivalent*:

(i) $\langle u, P_{m-1}\rangle \neq 0, \quad \langle u, \quad P_n\rangle = 0,\ n \geq m.$

(ii) $\exists \lambda_\nu \in \mathbb{C}, \quad 0 \leq \nu \leq m-1, \quad \lambda_{m-1} \neq 0$ *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n\geq 0}$ of $\{P_n^{[1]}\}_{n\geq 0}$ where

$$P_n^{[1]}(x) := (n+1)^{-1}P'_{n+1}(x), \quad n \geq 0,$$

is given by

$$Du_n^{[1]} = -(n+1)u_{n+1}, \quad n \geq 0.$$

Similarly, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of $\{\tilde{P}_n\}_{n\geq 0}$, where

$$\tilde{P}_n(x) := a^{-n}P_n(ax+b)$$

with $(a, b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$, is given by

$$\tilde{u}_n = a^n(h_{a^{-1}} \circ \tau_{-b})u_n,\ n \geq 0.$$

The form u is called *regular* if we can associate with it a sequence $\{P_n\}_{n\geq 0}$ such that

$$\langle u, P_nP_m\rangle = r_n\delta_{n,m}, \quad n,\ m \geq 0, \quad r_n \neq 0, \quad n \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is then called a monic *orthogonal* polynomial sequence (MOPS) with respect to u. Note that $u=(u)_0u_0$, with $(u)_0\neq 0$. When u is regular, let F be a polynomial such that $Fu=0$. Then $F=0$, [16].

Proposition 1 [16]. *Let $\{P_n\}_{n\geq 0}$ be a MPS with $\deg P_n=n$, $n\geq 0$, and let $\{u_n\}_{n\geq 0}$ be its dual sequence. The following statements are equivalent.*

(i) *$\{P_n\}_{n\geq 0}$ is orthogonal with respect to u_0.*

(ii) $u_n=\langle u_0,P_n^2\rangle^{-1}P_nu_0,\quad n\geq 0.$

(iii) *$\{P_n\}_{n\geq 0}$ satisfies the three-term recurrence relation*

$$\begin{cases} P_0(x)=1, \quad P_1(x)=x-\beta_0, \\ P_{n+2}(x)=(x-\beta_{n+1})P_{n+1}(x)-\gamma_{n+1}P_n(x), \quad n\geq 0, \end{cases} \tag{2.1}$$

where $\beta_n=\langle u_0,xP_n^2\rangle\langle u_0,P_n^2\rangle^{-1}$, $n\geq 0$ *and* $\gamma_{n+1}=\langle u_0,P_{n+1}^2\rangle\langle u_0,P_n^2\rangle^{-1}\neq 0,\quad n\geq 0.$

If $\{P_n\}_{n\geq 0}$ is a MOPS with respect to the regular form u_0, then $\{\tilde{P}_n\}_{n\geq 0}$ is a MOPS with respect to the regular form $\tilde{u}_0=(h_{a^{-1}}\circ\tau_{-b})u_0$, and satisfies [15]

$$\begin{cases} \tilde{P}_0(x)=1, \quad \tilde{P}_1(x)=x-\tilde{\beta}_0, \\ \tilde{P}_{n+2}(x)=(x-\tilde{\beta}_{n+1})\tilde{P}_{n+1}(x)-\tilde{\gamma}_{n+1}\tilde{P}_n(x), \quad n\geq 0, \end{cases}$$

where $\tilde{\beta}_n=a^{-1}(\beta_n-b)$ and $\tilde{\gamma}_{n+1}=a^{-2}\gamma_{n+1}$.

A MOPS $\{p_n\}_{n\geq 0}$ is called D-classical, if $\{Dp_n\}_{n\geq 0}$ is also orthogonal (*Hermite, Laguerre, Bessel or Jacobi*), [10, 11]. Moreover, if $\{p_n\}_{n\geq 0}$ is orthogonal with respect to u_0, then there exists a monic polynomial ϕ with $\deg\phi\leq 2$ and a polynomial ψ with $\deg\psi=1$ such that u_0 satisfies a *Pearson's equation* (PE) [15]

$$D(\phi u_0)+\psi u_0=0.$$

Any shift leaves invariant the D-classical character. Indeed, the shifted linear functional $\tilde{u}=(h_{a^{-1}}\circ\tau_{-b})u$ fulfills the equation

$$(\widetilde{\Phi}\tilde{u})'+\widetilde{\Psi}\tilde{u}=0,$$

where (see [15, 16])

$$\widetilde{\Phi}(x)=a^{-t}\Phi(ax+b) \quad \text{and} \quad \widetilde{\Psi}(x)=a^{1-t}\Psi(ax+b).$$

3. Hahn's problem with respect to some perturbations of the raising operator $X-c$

Clearly, the orthogonality is not preserved by the operator $X-c$, which is given by

$$(X-c)\big(f(x)\big)=(x-c)f(x), \quad f\in\mathbb{P}.$$

Our goal, in this section is to describe all $\mathcal{O}$-classical orthogonal polynomials. More precisely, we find all orthogonal polynomial sequences $\{P_n\}_{n\geq 0}$ such that $\{\mathcal{O}P_n\}_{n\geq 0}$ are also orthogonal, where $\mathcal{O}=H_{\alpha,q}$ or $\mathcal{O}=\mathcal{S}_\lambda$ are the operators defined by (1.1) and (1.2). This operators are two perturbations of the operator $X-c$ where $c=0$ and $c=1$.

3.1. Orthogonal polynomials via raising operator $X - \alpha H_q$

Let us introduce the following lemma.

Lemma 2 [12]. *The following properties hold*

$$H_q(fg)(x) = f(x)(H_q g)(x) + g(x)(H_q f)(x) + (q-1)x(H_q f)(x)(H_q g)(x), \quad f, g \in \mathcal{P},$$
$$H_q(fu) = (h_{q^{-1}} f)H_q u + q^{-1}(H_{q^{-1}} f)u, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}'.$$

where

$$H_q f(x) = \frac{h_q f(x) - f(x)}{(q-1)x}, \quad q \neq 1 \quad \textit{and} \quad h_q f(x) = f(qx).$$

Now, recall the operator

$$\begin{aligned} H_{\alpha,q} : \mathbb{P} &\longrightarrow \mathbb{P}, \\ f &\longmapsto H_{\alpha,q}(f) := xf + \alpha H_q(f). \end{aligned}$$

Definition 1. *We call a sequence $\{P_n\}_{n\geq 0}$ of orthogonal polynomials $H_{\alpha,q}$-classical if there exists a sequence $\{Q_n\}_{n\geq 0}$ of orthogonal polynomials such that $H_{\alpha,q} P_n = Q_{n+1},\ n \geq 0$.*

For any MPS $\{P_n\}_{n\geq 0}$ we define the MPS $\{Q_n\}_{n\geq 0}$, given by

$$Q_{n+1}(x) := H_{\alpha,q} P_n(x),\ n \geq 0,$$

or equivalently

$$Q_{n+1}(x) := xP_n(x) + \alpha(H_q P_n)(x), \quad n \geq 0, \tag{3.1}$$

with initial value $Q_0(x) = 1$.

Our next goal is to describe all the $H_{\alpha,q}$-classical polynomial sequences. Note that, we need $\alpha \neq 0$ to ensure that $\{Q_n\}_{n\geq 0}$ is an orthogonal sequence. Indeed, if we suppose that $\alpha = 0$, the relation (3.1) becomes, for $x = 0$, $Q_{n+1}(0) = 0,\ n \geq 0$, which contradicts the orthogonality of $\{Q_n\}_{n\geq 0}$.

Clearly, the operator $H_{\alpha,q}$ raises the degree of any polynomial. Such operator is called *raising operator* [9, 13, 19]. By transposition of the operator $H_{\alpha,q}$, we get

$${}^t H_{\alpha,q} = X - \alpha H_q. \tag{3.2}$$

Denote by $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ the dual basis in $\mathbb{P}'$ corresponding to $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively. Then, according to Lemma 1 and (3.2), the relation

$$xv_{n+1} - \alpha H_q(v_{n+1}) = u_n, \quad n \geq 0, \tag{3.3}$$

holds. Assume that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are MOPS satisfying

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad \gamma_{n+1} \neq 0, \quad n \geq 0, \end{cases} \tag{3.4}$$

$$\begin{cases} Q_0(x) = 1,\ Q_1(x) = x - \rho_0, \\ Q_{n+2}(x) = (x - \rho_{n+1})Q_{n+1}(x) - \varrho_{n+1}Q_n(x), \quad \varrho_{n+1} \neq 0, \quad n \geq 0. \end{cases} \tag{3.5}$$

Next, a first result will be deduced as a consequence of the relations (3.1), (3.4) and (3.5).

Proposition 2. *The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ satisfy the following finite type relation*

$$P_n(x) + (q-1)xH_q(P_n)(x) = q^n Q_n(x), \quad n \geq 0.$$

P r o o f. Using (3.4), we obtain

$$H_q(P_{n+2})(x) = H_q\big((x-\beta_{n+1})P_{n+1}\big)(x) - \gamma_{n+1}H_q(P_n)(x), \quad n \geq 0.$$

According to the Lemma 2, we obtain for $n \geq 0$

$$H_q(P_{n+2})(x) = (x-\beta_{n+1})H_q(P_{n+1})(x) + P_{n+1}(x) + (q-1)xH_q(P_{n+1})(x) - \gamma_{n+1}H_q(P_n)(x),$$

or equivalently

$$xP_{n+2}(x) + \alpha(H_qP_{n+2})(x) = Q_{n+3}(x), \quad n \geq 0,$$

which gives us for $n \geq 0$

$$(x-\beta_{n+1})xP_{n+1}(x)+\alpha(qx-\beta_{n+1})(H_qP_{n+1})(x)-\gamma_{n+1}\big(xP_n(x)+\alpha(H_qP_n)(x)\big)+\alpha P_{n+1}(x)=Q_{n+3}(x).$$

We use (3.1) and the last equation becomes for $n \geq 0$

$$(x-\beta_{n+1})Q_{n+2}(x) + \alpha(q-1)x(H_qP_{n+1})(x) - \gamma_{n+1}Q_{n+1}(x) + \alpha P_{n+1}(x) = Q_{n+3}(x). \tag{3.6}$$

Inserting (3.5) in (3.6), we obtain

$$\alpha P_{n+1}(x) + \alpha(q-1)x(H_qP_{n+1})(x) = \big(\beta_{n+1}-\rho_{n+2}\big)Q_{n+2}(x) + \big(\gamma_{n+1} - \varrho_{n+2}\big)Q_{n+1}(x), \quad n \geq 0.$$

In fact, this result is valid for $n+1$ replaced by n. More precisely, we have for all $n \geq 0$

$$\alpha P_n(x) + \alpha(q-1)x(H_qP_n)(x) = \big(\beta_n - \rho_{n+1}\big)Q_{n+1}(x) + \big(\gamma_n - \varrho_{n+1}\big)Q_n(x),$$

with the convention $\gamma_0 = 0$. By comparing the degrees in the previous equation, we get $\beta_n = \rho_{n+1}$, $n \geq 0$ and $\alpha q^n = \gamma_n - \varrho_{n+1}$, $n \geq 0$. Hence the desired result is proven. □

Note that, for $n = 0$ the relation (3.1) gives $\rho_0 = 0$, for $n = 1$ the Proposition 2 gives

$$(x-\beta_0) + (q-1)x = qx - \rho_0 = qx,$$

then $\beta_0 = \rho_1 = 0$. Now we establish, in the next lemma, an algebraic relation between the forms u_0 and v_0.

Lemma 3. *The forms u_0 and v_0 satisfy the following relation*

$$v_0 - (q-1)H_q(xv_0) = u_0. \tag{3.7}$$

P r o o f. According to Proposition 2 we obtain

$$\big\langle v_0 - (q-1)H_q(xv_0),\, P_n\big\rangle = 0, \quad n \geq 1. \tag{3.8}$$

On the other hand,

$$\big\langle v_0 - (q-1)H_q(xv_0),\, P_0\big\rangle = 1,$$

since $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to the form v_0, where v_0 is supposed normalized. According to Lemma 1 and using (3.8), we obtain the desired result. □

Based on the last lemma, we can state the following theorem.

Theorem 1. *The form v_0 satisfies the following Pearson's equation*

$$(H_q v_0) - \frac{1}{\alpha} x v_0 = 0, \tag{3.9}$$

and then the scaled q-Hermite polynomial sequence is the only $H_{\alpha,q}$-classical sequence.

P r o o f. According to Proposition 1 (ii), the relation (3.3) can be written as follows

$$x Q_{n+1}(x) v_0 - \alpha H_q(Q_{n+1} v_0) = \lambda_n P_n(x) u_0, \quad n \geq 0, \tag{3.10}$$

where

$$\lambda_n := \langle v_0, Q_{n+1}^2 \rangle \langle u_0, P_n^2 \rangle^{-1}, \quad n \geq 0.$$

Making $n = 0$ in (3.10), we get

$$x^2 v_0 - \alpha H_q(x v_0) = -\alpha u_0, \quad \big(Q_1(x) = x, \quad \varrho_1 = -\alpha\big).$$

Substituting this relation in (3.7), we obtain

$$q H_q(x v_0) - \frac{1}{\alpha}(x^2 + \alpha) v_0 = 0.$$

Note that we have $q H_q(x v_0) = x(H_q v_0) + v_0$, then

$$(H_q v_0) - \frac{1}{\alpha} x v_0 = 0, \tag{3.11}$$

which gives

$$\big((H_q v_0) - \frac{1}{\alpha} x v_0\big)_{n+1} = 0, \quad n \geq 0,$$

and then

$$(v_0)_{n+2} = -\alpha [n]_q (v_0)_n, \quad n \geq 0.$$

Moreover, $(v_0)_1 = \rho_1 = 0$, hence $(v_0)_{2n+1} = 0,\ n \geq 0$. We can conclude that $\{Q_n\}_{n \geq 0}$ is symmetric. Using the Proposition 2, we obtain

$$Q_n(x) = q^{-n} P_n(qx), \quad n \geq 0.$$

Then we also conclude that $\{P_n\}_{n \geq 0}$ is symmetric. Moreover, the relation (3.11) corresponds to a Pearson's equation of q-Hermite linear functional, hence $Q_n(x)$ is the q-Hermite polynomial. In addition, we have $Q_n(x) = q^{-n} P_n(qx),\ n \geq 0$, then $P_n(x)$ is the scaled q-Hermite polynomial. □

3.2. Orthogonal polynomials via raising operator $(X + 1) - \lambda \tau_{-1}$

In this part, we use the following lemma.

Lemma 4 [1]. *The following properties hold*

$$D_w(fg)(x) = f(x)(D_w g)(x) + g(x)(D_w f)(x) + w(D_w f)(x)(D_w g)(x), \quad f, g \in \mathcal{P},$$
$$D_{-w}(fu) = g(D_{-w} u) + (D_{-w} g)(\tau_w u), \quad f \in \mathcal{P}, \quad u \in \mathcal{P}',$$
$$\tau_b \circ D_w = D_w \circ \tau_b \ \textit{in } \mathcal{P} \textit{ and } \mathcal{P}', \quad b \in \mathbb{C},$$

where

$$D_\omega f(x) := \frac{\tau_{-\omega} f(x) - f(x)}{w}, \quad \omega \neq 0 \quad \textit{and} \quad \tau_{-\omega} f(x) = f(x + \omega).$$

Recall the operator

$$\begin{aligned} \mathcal{S}_\lambda : \mathbb{P} &\longrightarrow \mathbb{P}, \\ f &\longmapsto \mathcal{S}_\lambda(f) = (x+1)(f) - \lambda\tau_{-1}f. \end{aligned}$$

Definition 2. *We call a sequence $\{P_n\}_{n\geq 0}$ of orthogonal polynomials $\mathcal{S}_\lambda$-classical if there exists a sequence $\{Q_n\}_{n\geq 0}$ of orthogonal polynomials such that $\mathcal{S}_\lambda P_n = Q_{n+1},\ n \geq 0$.*

For any MPS $\{P_n\}_{n\geq 0}$ we define the MPS $\{Q_n\}_{n\geq 0}$, given by

$$Q_{n+1}(x) := \mathcal{S}_\lambda P_n(x),\ n \geq 0, \tag{3.12}$$

or equivalently

$$Q_{n+1}(x) := (x+1)P_n(x) - \lambda P_n(x+1),\ n \geq 0, \tag{3.13}$$

with initial value $Q_0(x) = 1$.

Our next goal is to describe all the $\mathcal{S}_\lambda$-classical polynomial sequences. Note that, we need $\lambda \neq 0$ to ensure that $\{Q_n\}_{n\geq 0}$ is an orthogonal sequence. Indeed, if we suppose that $\lambda = 0$, the relation (3.13) becomes, for $x = -1$, $Q_{n+1}(-1) = 0,\ n \geq 0$, which contradicts the orthogonality of $\{Q_n\}_{n\geq 0}$.

Clearly, the operator $\mathcal{S}_\lambda$ raises the degree of any polynomial. Such operator is called a *raising operator* [9, 13, 19]. By transposition of the operator $\mathcal{S}_\lambda$, we get

$${}^t\mathcal{S}_\lambda = (X+1) - \lambda\tau_1. \tag{3.14}$$

Denote by $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ the dual basis in $\mathbb{P}'$ corresponding to $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively. Then, according to Lemma 1 and (3.14), the relation

$$(x+1)v_{n+1} - \lambda\tau_1 v_{n+1} = u_n, \quad n \geq 0,$$

holds. Assume that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are MOPS satisfying

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad \gamma_{n+1} \neq 0, \quad n \geq 0, \end{cases} \tag{3.15}$$

$$\begin{cases} Q_0(x) = 1, \quad Q_1(x) = x - \rho_0, \\ Q_{n+2}(x) = (x - \rho_{n+1})Q_{n+1}(x) - \varrho_{n+1}Q_n(x), \quad \varrho_{n+1} \neq 0, \quad n \geq 0. \end{cases} \tag{3.16}$$

Next, a first result will be deduced as a consequence of the relations (3.13), (3.15) and (3.16).

Proposition 3. *The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ satisfy the following finite type relation*

$$Q_n(x) = \tau_{-1}P_n(x), \quad n \geq 0,$$

with

$$\begin{aligned} \rho_{n+1} &= \beta_n, \quad n \geq 0, \\ \varrho_{n+1} &= \gamma_n + \lambda, \quad n \geq 0, \end{aligned}$$

and with the convention $\gamma_0 = 0$.

P r o o f. Multiplying (3.15) by $x+1$, we obtain

$$(x+1)P_{n+2}(x) = (x-\beta_{n+1})(x+1)P_{n+1}(x) - \gamma_{n+1}(x+1)P_n(x), \quad n \geq 0.$$

Applying $\lambda\tau_{-1}$ to the (3.15) and taking the difference between the two resulting equations, we obtain

$$\begin{aligned}(x+1)P_{n+2}(x) - \lambda(\tau_{-1}P_{n+2})(x) &= (x-\beta_{n+1})\big((x+1)P_{n+1}(x) - \lambda(\tau_{-1}P_{n+1})(x)\big)\\ &-\gamma_{n+1}\big((x+1)P_n(x) - \lambda(\tau_{-1}P_n)(x)\big) - \lambda P_{n+1}(x+1).\end{aligned}$$

Substituting (3.13) in the last equation, we get

$$Q_{n+3}(x) = (x-\beta_{n+1})Q_{n+2}(x) - \gamma_{n+1}Q_{n+1}(x) - \lambda P_{n+1}(x+1), \quad n \geq 0.$$

Using the three-term recurrence relation (3.16), we get

$$\lambda P_{n+1}(x+1) = \big(\rho_{n+2} - \beta_{n+1}\big)Q_{n+2}(x) + \big(\varrho_{n+2} - \gamma_{n+1}\big)Q_{n+1}(x), \quad n \geq 0.$$

In fact, this result is valid for $n+1$ replaced by n. Then, by comparing the degrees in the previous equation, we get $\rho_{n+1} = \beta_n$ and $\varrho_{n+1} = \gamma_n + \lambda$, $n \geq 0$, and $Q_n(x) = \tau_{-1}P_n(x)$, $\quad n \geq 0$, with the convention $\gamma_0 = 0$. □

The following result is a straightforward consequence of Proposition 3.

Lemma 5. *The forms u_0 and v_0 satisfy the following relation*

$$\tau_1 v_0 = u_0.$$

According to Lemma 5, and based on some characterizations of Charlier polynomials [1], we can state the following theorem.

Theorem 2. *The Charlier polynomial sequence $\{C_n^{\lambda}(x)\}_{n\geq 0}$ where $\lambda > 0$, is the only $\mathcal{S}_\lambda$-classical orthogonal sequence. More precisely, we have for $n \geq 0$:*

$$P_n(x) = C_n^{\lambda}(x), \tag{3.17}$$

$$Q_n(x) = C_n^{\lambda}(x+1). \tag{3.18}$$

P r o o f. Assume that $\{P_n\}_{n\geq 0}$ is a monic $\mathcal{S}_\lambda$-classical orthogonal sequence. Then there exists a monic orthogonal sequence $\{Q_n\}_{n\geq 0}$ satisfying (3.13), which gives by transposition the following system

$$\langle v_0, (x+1)P_n(x) - \lambda P_n(x+1)\rangle = \langle v_0, Q_{n+1}(x)\rangle = 0, \quad n \geq 0.$$

But the left hand side reads as

$$\langle (x+1)v_0 - \lambda\tau_1 v_0, P_n(x)\rangle = 0, \quad n \geq 0.$$

In other words,

$$(x+1)v_0 - \lambda\tau_1 v_0 = 0.$$

Applying the operator τ_{-1}, we obtain

$$(x+2)\tau_{-1}v_0 - \lambda v_0 = 0.$$

Equivalently,

$$(x+1)\tau_{-1}v_0 + \tau_{-1}v_0 - (x+1)v_0 + (x+1)v_0 - \lambda v_0 = 0,$$

which also gives

$$(x+1)[\tau_{-1}v_0 - v_0] + \tau_{-1}v_0 + (x+1)v_0 - \lambda v_0 = 0,$$

or equivalently

$$(x+1)D_1 v_0 + \tau_{-1}v_0 + (x+1)v_0 - \lambda v_0 = 0.$$

By using Lemma 4, the last relation becomes

$$D_1\big(x(\tau_1 v_0)\big) + (x-\lambda)(\tau_1 v_0) = 0,$$

which means that $v_0 = \tau_{-1}C(\lambda)$, where $C(\lambda)$ is the Charlier form with $\lambda > 0$. In addition, using the Proposition 3, we obtain that $P_n(x) = C_n^{\lambda}(x)$ are the monic Charlier polynomials and then

$$Q_n(x) = C_n^{\lambda}(x+1), \quad n \geq 0.$$

□

4. Conclusion and prospects

We described Hahn's problem for some perturbed raising operators of the operator $X - c$ using the Pearson equation, which is satisfied by the corresponding linear functionals. Indeed, we have proved that the q-Hermite (resp. Charlier) polynomial is the only $H_{\alpha,q}$-classical (resp. $\mathcal{S}_\lambda$-classical) orthogonal polynomial, where $H_{\alpha,q} := X + \alpha H_q$ and $\mathcal{S}_\lambda := (X+1) - \lambda\tau_{-1}$.

Now, using (3.17), (3.18) and (3.12), we obtain

$$\mathcal{S}_\lambda C_n^{\lambda}(x) = C_{n+1}^{\lambda}(x+1), \quad n \geq 0,$$

which gives, by induction, the following formula

$$\mathcal{S}_\lambda^{(m)} C_n^{\lambda}(x) = C_{n+m}^{\lambda}(x+m), \quad n \geq 0, \tag{4.1}$$

where $\mathcal{S}_\lambda^{(m)} = \mathcal{S}_\lambda^{(m)} \circ \cdots \circ \mathcal{S}_\lambda^{(m)}$.

Making $n = 0$ in (4.1) we get

$$\mathcal{S}_\lambda^{(m)}(1) = C_m^{\lambda}(x+m), \quad m \geq 0.$$

For prospects, we can replace the operator H_q in Subsection 3.1 by the Dunkl operator ($T_\mu := D + 2\mu H_{-1}$, see [6]) and study the same problem. Indeed, we have [6]

$$\Big(X - \frac{1}{2}T_\mu\Big)H_n^{\mu}(x) = \frac{\gamma_\mu(n+1)}{2\gamma_\mu(n)(n+1)}H_{n+1}^{\mu}(x), \quad n \geq 0, \tag{4.2}$$

where $H_n^{\mu}(x)$ is the monic generalized Hermite polynomial and where $\gamma_\mu(n)$ is defined by

$$\gamma_\mu(2m) = \frac{2^{2m} m!\Gamma(m+\mu+1/2)}{\Gamma(\mu+1/2)}, \quad \text{and} \quad \gamma_\mu(2m+1) = \frac{2^{2m+1} m!\Gamma(m+\mu+1/2)}{\Gamma(\mu+3/2)}.$$

In view of (4.2), we can say that $\{H_n^{\mu}\}_{n\geq 0}$ is an $\mathcal{O}$-classical polynomial sequence, since it fulfills Hahn's property relatively to the raising operator

$$\mathcal{O} := X - \frac{1}{2}T_\mu,$$

i.e., it is an orthogonal polynomial sequence whose sequence of $\mathcal{O}$-derivatives is also orthogonal.

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