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ON THE POTENTIALITY OF A CLASS OF OPERATORS RELATIVE TO LOCAL BILINEAR FORMS[1]

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Abstract: The inverse problem of the calculus of variations (IPCV) is solved for a second-order ordinary differential equation with the use of a local bilinear form. We apply methods of analytical dynamics, nonlinear functional analysis, and modern methods for solving the IPCV. In the paper, we obtain necessary and sufficient conditions for a given operator to be potential relative to a local bilinear form, construct the corresponding functional, i.e., found a solution to the IPCV, and define the structure of the considered equation with the potential operator. As a consequence, similar results are obtained when using a nonlocal bilinear form. Theoretical results are illustrated with some examples.

Keywords: Inverse problem of the calculus of variations, Local bilinear form, Potential operator, Conditions of potentiality.

1. Introduction

In the modern calculus framework, the classical inverse problem of the calculus of variations (IPCV) is a problem of constructing an integral functional such that its equations of extremals coincide with given equations. The issues considered in the paper are closely related to the following statement of the IPCV generalizing its classical statement. For a given equation, one needs to construct a functional such that its set of stationary points coincides with the set of solutions to this equation. These problems are also related to the mechanics of finite- and infinite-dimensional systems [7, 8, 11–13]. There is a large number of works devoted to IPCVs for different types of equations and their systems: in particular, for ordinary differential equations and differential equations with partial derivatives [4, 6, 13, 18, 19, 21], operator equations [2, 3, 14, 15], differential-difference equations [5, 9, 10], and stochastic differential equations [16, 17]. In these works, nonlocal bilinear forms were mainly used to solve an IPCV. Methods of investigating operators for the potentiality relative to local bilinear forms were developed in [6, 13, 20].

The main aim of the paper is to find a solution to an IPCV for a second-order ordinary differential equation. Local bilinear forms will play a significant role in the investigation.

Below, we use the notation and terminology of [2, 3, 13, 15].

Assume that U and V are linear normed spaces over $\mathbb{R}$.

The following definition and theorem will be needed for the sequel.

Definition 1 [13]. *An operator $N : D(N) \subset U \to V$ is called potential on the set $D(N)$ relative to a local bilinear form $\Phi(u;\cdot,\cdot) : V \times V \to \mathbb{R}$ if there exists a Gâteaux differentiable functional $F_N : D(F_N) = D(N) \to \mathbb{R}$ such that*

$$\delta F_N[u,h] = \Phi(u;N(u),h) \quad \forall u \in D(N), \quad \forall h \in D(N'_u). \tag{1.1}$$

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Theorem 1 [13]. *Consider a Gâteaux differentiable operator $N : D(N) \subset U \to V$ and a local bilinear form $\Phi(u;\cdot,\cdot) : V \times V \to \mathbb{R}$ such that, for any fixed elements $u \in D(N)$ and $g,h \in D(N'_u)$, the function $\psi(\varepsilon) = \Phi(u+\varepsilon h; N(u+\varepsilon h), g)$ belongs to the class $C^1[0,1]$. For N to be potential on the convex set $D(N)$ relative to Φ, it is necessary and sufficient to have*

$$\begin{gathered}\Phi\left(u; N'_u h, g\right) + \Phi'_u\left(h; N(u), g\right) = \Phi\left(u; N'_u g, h\right) + \Phi'_u\left(g; N(u), h\right) \\ \forall u \in D\left(N\right), \quad \forall h, g \in D\left(N'_u\right).\end{gathered} \tag{1.2}$$

Under this condition, the potential F_N is given as

$$F_N[u] = \int_0^1 \Phi(u_0 + \lambda(u-u_0); N(u_0+\lambda(u-u_0)), u-u_0)\, d\lambda + F_N[u_0], \tag{1.3}$$

where u_0 is a fixed element of $D(N)$.

Note that N'_u and Φ'_u are the Gâteaux derivatives of N and Φ at the point u.

2. Conditions of potentiality

Consider an ordinary differential equation of the second order

$$N(u) \equiv a(t,u(t))u''(t) + b(t,u(t))u'(t) + c(t,u(t))(u'(t))^2 + d(t,u(t)) = 0, \quad t \in [t_0,t_1]. \tag{2.1}$$

Here, $u = u(t)$ is an unknown function, $a \in C^2([t_0,t_1]\times T)$ and $b,c,d \in C^1([t_0,t_1]\times T)$ are given functions, and $T \subseteq \mathbb{R}$.

We define the domain of the operator N (2.1) as follows:

$$D(N) = \left\{u \in C^2[t_0,t_1] : u(t_0) = u_1,\ u(t_1) = u_2\right\}. \tag{2.2}$$

The domain $D(N'_u)$ consists of elements $h \in C^2[t_0,t_1]$ such that $(u+\varepsilon h) \in D(N)$ for all ε sufficiently small, i.e.,

$$D(N'_u) = \left\{h \in C^2[t_0,t_1] : h(t_0) = 0,\ h(t_1) = 0\right\}.$$

Let us introduce a local bilinear form

$$\Phi(u; v, g) = \int_{t_0}^{t_1} M(t,u(t))v(t)g(t)\, dt, \tag{2.3}$$

where $M \in C^2([t_0,t_1]\times T)$, $M(t,u(t)) \neq 0$.

Theorem 2. *For the operator N (2.1) to be potential on $D(N)$ (2.2) relative to the local bilinear form (2.3), it is necessary and sufficient that the following conditions hold for all $u \in D(N)$ and all $t \in [t_0,t_1]$:*

$$a'_u(t,u(t))M(t,u(t)) + a(t,u(t))M'_u(t,u(t)) - 2c(t,u(t))M(t,u(t)) = 0, \tag{2.4}$$

$$a'_t(t,u(t))M(t,u(t)) + a(t,u(t))M'_t(t,u(t)) - b(t,u(t))M(t,u(t)) = 0. \tag{2.5}$$

P r o o f. We have

$$N'_u h = a'_u(t,u(t))u''(t)h(t) + a(t,u(t))h''(t) + b'_u(t,u(t))u'(t)h(t) + \\ + b(t,u(t))h'(t) + c'_u(t,u(t))(u'(t))^2 h(t) + 2c(t,u(t))u'(t)h'(t) + d'_u(t,u(t))h(t).$$

In this case, criterion (1.2) becomes

$$\int_{t_0}^{t_1} \Big(a'_u(t,u(t))M(t,u(t))u''(t)h(t)g(t) + a(t,u(t))M(t,u(t))h''(t)g(t) + \\ + b'_u(t,u(t))M(t,u(t))u'(t)h(t)g(t) + b(t,u(t))M(t,u(t))h'(t)g(t) + \\ + c'_u(t,u(t))M(t,u(t))(u'(t))^2 h(t)g(t) + 2c(t,u(t))M(t,u(t))u'(t)h'(t)g(t) + \\ + M(t,u(t))d'_u(t,u(t))h(t)g(t) + a(t,u(t))M'_u(t,u(t))u''(t)h(t)g(t) + \\ + b(t,u(t))M'_u(t,u(t))u'(t)h(t)g(t) + c(t,u(t))M'_u(t,u(t))(u'(t))^2 h(t)g(t) + \\ + M'_u(t,u(t))d(t,u(t))h(t)g(t) \Big) dt = \int_{t_0}^{t_1} \Big(a'_u(t,u(t))M(t,u(t))u''(t)h(t)g(t) + \\ + a(t,u(t))M(t,u(t))g''(t)h(t) + b'_u(t,u(t))M(t,u(t))u'(t)h(t)g(t) + \\ + b(t,u(t))M(t,u(t))g'(t)h(t) + c'_u(t,u(t))M(t,u(t))(u'(t))^2 h(t)g(t) + \\ + 2c(t,u(t))M(t,u(t))u'(t)g'(t)h(t) + M(t,u(t))d'_u(t,u(t))h(t)g(t) + \\ + a(t,u(t))M'_u(t,u(t))u''(t)h(t)g(t) + b(t,u(t))M'_u(t,u(t))u'(t)h(t)g(t) + \\ + c(t,u(t))M'_u(t,u(t))(u'(t))^2 h(t)g(t) + M'_u(t,u(t))d(t,u(t))h(t)g(t) \Big) dt \\ \forall u \in D(N), \quad \forall h, g \in D\left(N'_u\right),$$

or

$$\int_{t_0}^{t_1} \Big(a(t,u(t))M(t,u(t))h''(t)g(t) + b(t,u(t))M(t,u(t))h'(t)g(t) + \\ + 2c(t,u(t))M(t,u(t))u'(t)h'(t)g(t) \Big) dt = \int_{t_0}^{t_1} \Big(a(t,u(t))M(t,u(t))g''(t)h(t) + \\ + b(t,u(t))M(t,u(t))g'(t)h(t) + 2c(t,u(t))M(t,u(t))u'(t)g'(t)h(t) \Big) dt \\ \forall u \in D(N), \quad \forall h, g \in D\left(N'_u\right). \tag{2.6}$$

Integrating by parts and taking into consideration that $h, g \in D(N'_u)$, we obtain

$$\int_{t_0}^{t_1} \Big(a(t,u(t))M(t,u(t))h''(t)g(t) + b(t,u(t))M(t,u(t))h'(t)g(t) + \\ + 2c(t,u(t))M(t,u(t))u'(t)h'(t)g(t) \Big) dt = \int_{t_0}^{t_1} \Big(a''_{tt}(t,u(t))M(t,u(t))h(t)g(t) + \\ + 2a''_{tu}(t,u(t))u'(t)M(t,u(t))h(t)g(t) + 2a'_t(t,u(t))M'_t(t,u(t))h(t)g(t) + \\ + 2a'_t(t,u(t))M'_u(t,u(t))u'(t)h(t)g(t) + 2a'_t(t,u(t))M(t,u(t))h(t)g'(t) + \\ + a''_{uu}(t,u(t))(u'(t))^2 M(t,u(t))h(t)g(t) + a'_u(t,u(t))u''(t)M(t,u(t))h(t)g(t) + \\ + 2a'_u(t,u(t))u'(t)M'_t(t,u(t))h(t)g(t) + 2a'_u(t,u(t))M'_u(t,u(t))(u'(t))^2 h(t)g(t) +$$

$$
\begin{aligned}
&+2a'_u(t,u(t))u'(t)M(t,u(t))h(t)g'(t)+2a(t,u(t))M'_t(t,u(t))g'(t)h(t)+\\
&+2a(t,u(t))M'_u(t,u(t))u'(t)h(t)g'(t)+a(t,u(t))M(t,u(t))h(t)g''(t)+\\
&+a(t,u(t))M''_{tt}(t,u(t))h(t)g(t)+2a(t,u(t))M''_{tu}(t,u(t))u'(t)h(t)g(t)+\\
&+a(t,u(t))M''_{uu}(t,u(t))(u'(t))^2h(t)g(t)+a(t,u(t))M'_u(t,u(t))u''(t)h(t)g(t)-\\
&-b'_t(t,u(t))M(t,u(t))h(t)g(t)-b'_u(t,u(t))u'(t)M(t,u(t))h(t)g(t)-\\
&-b(t,u(t))M'_t(t,u(t))h(t)g(t)-b(t,u(t))M'_u(t,u(t))u'(t)h(t)g(t)-\\
&-b(t,u(t))M(t,u(t))h(t)g'(t)-2c'_t(t,u(t))M(t,u(t))u'(t)h(t)g(t)-\\
&-2c'_u(t,u(t))M(t,u(t))(u'(t))^2h(t)g(t)-2c(t,u(t))(u'(t))^2M'_u(t,u(t))h(t)g(t)-\\
&-2c(t,u(t))M'_t(t,u(t))u'(t)h(t)g(t)-2c(t,u(t))M(t,u(t))u''(t)h(t)g(t)-\\
&-2c(t,u(t))M(t,u(t))u'(t)h(t)g'(t)\Big)dt.
\end{aligned}
$$

Thus, equality (2.6) can be written in the form

$$
\begin{aligned}
&\int_{t_0}^{t_1}\Big(a''_{tt}(t,u(t))M(t,u(t))h(t)g(t)+2a''_{tu}(t,u(t))u'(t)M(t,u(t))h(t)g(t)+\\
&+2a'_t(t,u(t))M'_t(t,u(t))h(t)g(t)+2a'_t(t,u(t))M'_u(t,u(t))u'(t)h(t)g(t)+\\
&+2a'_t(t,u(t))M(t,u(t))h(t)g'(t)+a''_{uu}(t,u(t))(u'(t))^2M(t,u(t))h(t)g(t)+\\
&+a'_u(t,u(t))u''(t)M(t,u(t))h(t)g(t)+2a'_u(t,u(t))u'(t)M'_t(t,u(t))h(t)g(t)+\\
&+2a'_u(t,u(t))M'_u(t,u(t))(u'(t))^2h(t)g(t)+2a'_u(t,u(t))u'(t)M(t,u(t))h(t)g'(t)+\\
&+2a(t,u(t))M'_t(t,u(t))g'(t)h(t)+2a(t,u(t))M'_u(t,u(t))u'(t)h(t)g'(t)+\\
&+a(t,u(t))M''_{tt}(t,u(t))h(t)g(t)+2a(t,u(t))M''_{tu}(t,u(t))u'(t)h(t)g(t)+\\
&+a(t,u(t))M''_{uu}(t,u(t))(u'(t))^2h(t)g(t)+a(t,u(t))M'_u(t,u(t))u''(t)h(t)g(t)-\\
&-b'_t(t,u(t))M(t,u(t))h(t)g(t)-b'_u(t,u(t))u'(t)M(t,u(t))h(t)g(t)-\\
&-b(t,u(t))M'_t(t,u(t))h(t)g(t)-b(t,u(t))M'_u(t,u(t))u'(t)h(t)g(t)-\\
&-2b(t,u(t))M(t,u(t))h(t)g'(t)-2c'_t(t,u(t))M(t,u(t))u'(t)h(t)g(t)-\\
&-2c'_u(t,u(t))M(t,u(t))(u'(t))^2h(t)g(t)-2c(t,u(t))(u'(t))^2M'_u(t,u(t))h(t)g(t)-\\
&-2c(t,u(t))M'_t(t,u(t))u'(t)h(t)g(t)-2c(t,u(t))M(t,u(t))u''(t)h(t)g(t)-\\
&-4c(t,u(t))M(t,u(t))u'(t)h(t)g'(t)\Big)dt=0\\
&\forall u\in D(N),\quad \forall h,g\in D\left(N'_u\right).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&a''_{tt}(t,u(t))M(t,u(t))+2a'_t(t,u(t))M'_t(t,u(t))+a(t,u(t))M''_{tt}(t,u(t))-\\
&-b'_t(t,u(t))M(t,u(t))-b(t,u(t))M'_t(t,u(t))=0,
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&2a''_{tu}(t,u(t))M(t,u(t))+2a'_t(t,u(t))M'_u(t,u(t))+2a'_u(t,u(t))M'_t(t,u(t))+\\
&+2a(t,u(t))M''_{tu}(t,u(t))-2c'_t(t,u(t))M(t,u(t))-2c(t,u(t))M'_t(t,u(t))-\\
&-b'_u(t,u(t))M(t,u(t))-b(t,u(t))M'_u(t,u(t))=0,
\end{aligned}
\tag{2.8}
$$

$$
a'_t(t,u(t))M(t,u(t))+a(t,u(t))M'_t(t,u(t))-b(t,u(t))M(t,u(t))=0,
\tag{2.9}
$$

$$
\begin{aligned}
&a(t,u(t))M''_{uu}(t,u(t))+a''_{uu}(t,u(t))M(t,u(t))+2a'_u(t,u(t))M'_u(t,u(t))-\\
&-2c'_u(t,u(t))M(t,u(t))-2c(t,u(t))M'_u(t,u(t))=0,
\end{aligned}
\tag{2.10}
$$

$$
a'_u(t,u(t))M(t,u(t))+a(t,u(t))M'_u(t,u(t))-2c(t,u(t))M(t,u(t))=0.
\tag{2.11}
$$

Note that conditions (2.7)–(2.11) are reduced to (2.4) and (2.5). □

Remark 1. If $M = M(t)$, then

$$\Phi(v,g) = \int_{t_0}^{t_1} M(t)v(t)g(t)\,dt \tag{2.12}$$

is a nonlocal bilinear form and conditions (2.4) and (2.5) are represented in the form

$$a'_u(t,u(t)) - 2c(t,u(t)) = 0, \tag{2.13}$$

$$a'_t(t,u(t))M(t) + a(t,u(t))M'(t) - b(t,u(t))M(t) = 0. \tag{2.14}$$

Remark 2. If $M = M(t)$ and $a = a(t)$, $b = b(t)$, $c = c(t)$, then conditions (2.4) and (2.5) can be written in the form

$$c(t) = 0, \tag{2.15}$$

$$a'(t)M(t) + a(t)M'(t) - b(t)M(t) = 0. \tag{2.16}$$

Remark 3. If $M(t,u(t)) \equiv 1$, then

$$\Phi(v,g) = \int_{t_0}^{t_1} v(t)g(t)\,dt \tag{2.17}$$

and conditions (2.4) and (2.5) take the form

$$a'_u(t,u(t)) - 2c(t,u(t)) = 0, \tag{2.18}$$

$$a'_t(t,u(t)) - b(t,u(t)) = 0. \tag{2.19}$$

Remark 4. If $M(t,u(t)) \equiv 1$ and $a = a(t)$, $b = b(t)$, $c = c(t)$, then conditions (2.4) and (2.5) are reduced to

$$c(t) = 0, \tag{2.20}$$

$$a'(t) - b(t) = 0. \tag{2.21}$$

3. Finding a solution to the IPCV

Theorem 3. *If conditions (2.4) and (2.5) hold, then the corresponding functional is given as*

$$F_N[u] = \int_{t_0}^{t_1} \Big(-\frac{1}{2}M(t,u(t))a(t,u(t))(u'(t))^2 + B_M(t,u(t))\Big)\,dt, \tag{3.1}$$

where

$$B_M(t,u(t)) = \int_0^1 M(t,\tilde{u}(t,\lambda))d(t,\tilde{u}(t,\lambda))(u(t)-u_0(t))\,d\lambda + B_M(t,u_0(t)), \tag{3.2}$$

$\tilde{u}(t,\lambda) = u_0(t) + \lambda(u(t)-u_0(t))$, $u_0 = u_0(t)$ *is a fixed element of* $D(N)$, *and* $B_M \in C^2([t_0,t_1]\times T)$.

P r o o f. According to formula (1.3) and conditions (2.4) and (2.5) we have

$$F_N[u] - F_N[u_0] =$$

$$= \int_{t_0}^{t_1}\int_0^1 \Big[M(t,\tilde u(t,\lambda))a(t,\tilde u(t,\lambda))\tilde u''_{tt}(t,\lambda)(u(t)-u_0(t)) +$$
$$+ M(t,\tilde u(t,\lambda))b(t,\tilde u(t,\lambda))\tilde u'_t(t,\lambda)(u(t)-u_0(t)) +$$
$$+ M(t,\tilde u(t,\lambda))c(t,\tilde u(t,\lambda))(\tilde u'_t(t,\lambda))^2(u(t)-u_0(t)) +$$
$$+ M(t,\tilde u(t,\lambda))d(t,\tilde u(t,\lambda))(u(t)-u_0(t))\Big]\, d\lambda dt =$$
$$= \int_{t_0}^{t_1}\int_0^1 \Big[-M'_t(t,\tilde u(t,\lambda))a(t,\tilde u(t,\lambda))\tilde u'_t(t,\lambda)(u(t)-u_0(t)) -$$
$$- M'_{\tilde u(t,\lambda)}(t,\tilde u(t,\lambda))a(t,\tilde u(t,\lambda))(\tilde u'_t(t,\lambda))^2(u(t)-u_0(t)) -$$
$$- M(t,\tilde u(t,\lambda))a'_t(t,\tilde u(t,\lambda))\tilde u'_t(t,\lambda)(u(t)-u_0(t)) -$$
$$- M(t,\tilde u(t,\lambda))a'_{\tilde u(t,\lambda)}(t,\tilde u(t,\lambda))(\tilde u'_t(t,\lambda))^2(u(t)-u_0(t)) -$$
$$- M(t,\tilde u(t,\lambda))a(t,\tilde u(t,\lambda))\tilde u'_t(t,\lambda)(u(t)-u_0(t))' +$$
$$+ M(t,\tilde u(t,\lambda))b(t,\tilde u(t,\lambda))\tilde u'_t(t,\lambda)(u(t)-u_0(t)) +$$
$$+ M(t,\tilde u(t,\lambda))c(t,\tilde u(t,\lambda))(\tilde u'_t(t,\lambda))^2(u(t)-u_0(t)) + \tag{3.3}$$
$$+ M(t,\tilde u(t,\lambda))d(t,\tilde u(t,\lambda))(u(t)-u_0(t))\Big]\, d\lambda dt =$$
$$= \int_{t_0}^{t_1}\int_0^1 \Big[\tilde u'_t(t,\lambda)(u(t)-u_0(t))\Big(-M'_t(t,\tilde u(t,\lambda))a(t,\tilde u(t,\lambda)) -$$
$$- M(t,\tilde u(t,\lambda))a'_t(t,\tilde u(t,\lambda)) + M(t,\tilde u(t,\lambda))b(t,\tilde u(t,\lambda))\Big) +$$
$$+ (\tilde u'_t(t,\lambda))^2(u(t)-u_0(t))\big(-M'_{\tilde u(t,\lambda)}(t,\tilde u(t,\lambda))a(t,\tilde u(t,\lambda)) -$$
$$- M(t,\tilde u(t,\lambda))a'_{\tilde u(t,\lambda)}(t,\tilde u(t,\lambda)) + M(t,\tilde u(t,\lambda))c(t,\tilde u(t,\lambda))\big) -$$
$$- M(t,\tilde u(t,\lambda))a(t,\tilde u(t,\lambda))\tilde u'_t(t,\lambda)(u(t)-u_0(t))' +$$
$$+ M(t,\tilde u(t,\lambda))d(t,\tilde u(t,\lambda))(u(t)-u_0(t))\Big]\, d\lambda dt =$$
$$= \int_{t_0}^{t_1}\int_0^1 \Big[-c(t,\tilde u(t,\lambda))M(t,\tilde u(t,\lambda))(\tilde u'_t(t,\lambda))^2(u(t)-u_0(t)) -$$
$$- a(t,\tilde u(t,\lambda))M(t,\tilde u(t,\lambda))\tilde u'_t(t,\lambda)(u(t)-u_0(t))' +$$
$$+ M(t,\tilde u(t,\lambda))d(t,\tilde u(t,\lambda))(u(t)-u_0(t))\Big]\, d\lambda dt.$$

Note that, using (2.4), we get

$$\int_0^1 \Big[-c(t,\tilde u(t,\lambda))M(t,\tilde u(t,\lambda))(\tilde u'_t(t,\lambda))^2(u(t)-u_0(t)) -$$
$$- a(t,\tilde u(t,\lambda))M(t,\tilde u(t,\lambda))\tilde u'_t(t,\lambda)(u(t)-u_0(t))'\Big]\, d\lambda =$$

$$= \int_0^1 \Big[-c(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))(\tilde{u}'_t(t,\lambda))^2(u(t)-u_0(t)) -$$

$$- a(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))\tilde{u}'_t(t,\lambda)\frac{\partial \tilde{u}'_t(t,\lambda)}{\partial\lambda}\Big]\,d\lambda =$$

$$= \int_0^1 \Big[-c(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))(\tilde{u}'_t(t,\lambda))^2(u(t)-u_0(t)) -$$

$$- \frac{\partial}{\partial\lambda}\left(a(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))(\tilde{u}'_t(t,\lambda))^2\right) +$$

$$+ a'_{\tilde{u}(t,\lambda)}(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))(\tilde{u}'_t(t,\lambda))^2(u(t)-u_0(t)) +$$

$$+ a(t,\tilde{u}(t,\lambda))M'_{\tilde{u}(t,\lambda)}(t,\tilde{u}(t,\lambda))(\tilde{u}'_t(t,\lambda))^2(u(t)-u_0(t)) +$$

$$+ a(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))\tilde{u}'_t(t,\lambda)(u(t)-u_0(t))'\Big]\,d\lambda =$$

$$= \int_0^1 \Big[c(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))(\tilde{u}'_t(t,\lambda))^2(u(t)-u_0(t)) -$$

$$- \frac{\partial}{\partial\lambda}\left(a(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))(\tilde{u}'_t(t,\lambda))^2\right) +$$

$$+ a(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))\tilde{u}'_t(t,\lambda)(u(t)-u_0(t))'\Big]\,d\lambda =$$

$$= \int_0^1 \Big[c(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))(\tilde{u}'_t(t,\lambda))^2(u(t)-u_0(t)) +$$

$$+ a(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))\tilde{u}'_t(t,\lambda)(u(t)-u_0(t))'\Big]\,d\lambda -$$

$$- a(t,u(t))M(t,u(t))(u'(t))^2 + a(t,u_0(t))M(t,u_0(t))(u'_0(t))^2.$$

Hence,

$$\int_0^1 \Big[-c(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))(\tilde{u}'_t(t,\lambda))^2(u(t)-u_0(t)) -$$

$$- a(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))\tilde{u}'_t(t,\lambda)(u(t)-u_0(t))'\Big]\,d\lambda =$$

$$= \int_0^1 \Big[c(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))(\tilde{u}'_t(t,\lambda))^2(u(t)-u_0(t)) +$$

$$+ a(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))\tilde{u}'_t(t,\lambda)(u(t)-u_0(t))'\Big]\,d\lambda -$$

$$- a(t,u(t))M(t,u(t))(u'(t))^2 + a(t,u_0(t))M(t,u_0(t))(u'_0(t))^2$$

and

$$\int_0^1 \Big[-c(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))(\tilde{u}'_t(t,\lambda))^2(u(t)-u_0(t)) -$$

$$- a(t,\tilde{u}(t,\lambda))M(t,\tilde{u}(t,\lambda))\tilde{u}'_t(t,\lambda)(u(t)-u_0(t))'\Big]\,d\lambda =$$

$$= -\frac{1}{2}a(t,u(t))M(t,u(t))(u'(t))^2 + \frac{1}{2}a(t,u_0(t))M(t,u_0(t))(u'_0(t))^2.$$

Thus, (3.3) becomes

$$F_N[u]-F_N[u_0]=\int\limits_{t_0}^{t_1}\Big(-\frac{1}{2}a(t,u(t))M(t,u(t))(u'(t))^2+\frac{1}{2}a(t,u_0(t))M(t,u_0(t))(u_0'(t))^2+$$
$$+\int\limits_0^1 M(t,\tilde{u}(t,\lambda))d(t,\tilde{u}(t,\lambda))(u(t)-u_0(t))\,d\lambda\Big)dt.$$

The use of (3.2) yields functional (3.1). □

Remark 5. If $M=M(t)$ and $a=a(t)$, $b=b(t)$, $c=c(t)$, then

$$F_N[u]=\int\limits_{t_0}^{t_1}\Big(-\frac{1}{2}M(t)a(t)(u'(t))^2+B_M(t,u(t))\Big)dt, \tag{3.4}$$

where

$$B_M(t,u(t))=\int\limits_0^1 M(t)d(t,\tilde{u}(t,\lambda))(u(t)-u_0(t))\,d\lambda+B_M(t,u_0(t)). \tag{3.5}$$

Remark 6. If $M(t,u(t))\equiv 1$ and $a=a(t)$, $b=b(t)$, $c=c(t)$, then

$$F_N[u]=\int\limits_{t_0}^{t_1}\Big(-\frac{a(t)}{2}(u'(t))^2+B(t,u(t))\Big)dt,$$

where

$$B(t,u(t))=\int\limits_0^1 d(t,\tilde{u}(t,\lambda))(u(t)-u_0(t))\,d\lambda+B(t,u_0(t)).$$

4. The structure of variational equation (2.1)

Theorem 4. *Conditions* (2.4) *and* (2.5) *hold if and only if equation* (2.1) *takes the form*

$$\begin{aligned}N(u)\equiv a(t,u(t))u''(t)+\frac{1}{M(t,u(t))}\Big[M_t'(t,u(t))a(t,u(t))+M(t,u(t))a_t'(t,u(t))\Big]u'(t)+\\+\frac{1}{2M(t,u(t))}\Big[M_u'(t,u(t))a(t,u(t))+M(t,u(t))a_u'(t,u(t))\Big](u'(t))^2+\frac{(B_M)_u'(t,u(t))}{M(t,u(t))}=0.\end{aligned} \tag{4.1}$$

P r o o f. According to (1.1), for functional (3.1), we have

$$\delta F_N[u,h]=\int\limits_{t_0}^{t_1}\Big(-\frac{1}{2}M_u'(t,u(t))a(t,u(t))(u'(t))^2h(t)-$$
$$-\frac{1}{2}M(t,u(t))a_u'(t,u(t))(u'(t))^2h(t)-M(t,u(t))a(t,u(t))u'(t)h'(t)+(B_M)_u'(t,u(t))h(t)\Big)dt=$$
$$=\int\limits_{t_0}^{t_1}\Big(-\frac{1}{2}M_u'(t,u(t))a(t,u(t))(u'(t))^2h(t)-\frac{1}{2}M(t,u(t))a_u'(t,u(t))(u'(t))^2h(t)+$$

$$+ M'_t(t,u(t))a(t,u(t))u'(t)h(t) + M'_u(t,u(t))a(t,u(t))(u'(t))^2h(t) + M(t,u(t))a'_t(t,u(t))u'(t)h(t) +$$
$$+ M(t,u(t))a'_u(t,u(t))(u'(t))^2h(t) + M(t,u(t))a(t,u(t))u''(t)h(t) + (B_M)'_u(t,u(t))h(t)\Big)dt =$$
$$= \int_{t_0}^{t_1}\Big(M(t,u(t))a(t,u(t))u''(t) + \big(M'_t(t,u(t))a(t,u(t)) + M(t,u(t))a'_t(t,u(t))\big)u'(t) +$$
$$+ \frac{1}{2}\big(M'_u(t,u(t))a(t,u(t)) + M(t,u(t))a'_u(t,u(t))\big)(u'(t))^2 + (B_M)'_u(t,u(t))\Big)h(t)dt = \Phi(u;N(u),h)$$
$$\forall u \in D(N), \quad \forall h \in D(N'_u).$$

Hence, equation (2.1) is represented in form (4.1).

On the other hand, equation (4.1) is derived from the stationarity condition of functional (3.1). This means that conditions (2.4) and (2.5) must be satisfied. □

5. Examples

Example 1. Consider the Emden–Fowler equation [1]

$$N(u) \equiv u''(t) + \frac{k_1}{t}u'(t) + k_2 t^{m-1}u^n(t) = 0, \quad t \in [t_0, t_1], \quad t_0 > 0, \tag{5.1}$$

where k_1, k_2, m, and n are constants, $n \in \mathbb{N}$.

In this case,

$$a = 1, \quad b(t) = \frac{k_1}{t}, \quad c = 0, \quad d(t,u(t)) = k_2 t^{m-1}u^n(t).$$

The operator N (5.1) is not potential on $D(N)$ (2.2) relative to bilinear form (2.17) because condition (2.21) is not satisfied.

We find $M = M(t)$ such that the operator N (5.1) is potential on $D(N)$ (2.2) relative to a bilinear form of type (2.12).

From condition (2.16), we obtain

$$M(t) = t^{k_1}.$$

Thus, the operator N (5.1) is potential on $D(N)$ (2.2) relative to the following bilinear form:

$$\Phi(v,g) = \int_{t_0}^{t_1} t^{k_1}v(t)g(t)\,dt.$$

By formula (3.5), we get

$$B_M(t,u(t)) = \frac{k_2}{n+1}t^{m-1+k_1}u^{n+1}(t),$$

and functional (3.4) takes the form

$$F_N[u] = \int_{t_0}^{t_1}\Big(-\frac{t^{k_1}}{2}(u'(t))^2 + \frac{k_2}{n+1}t^{m-1+k_1}u^{n+1}(t)\Big)\,dt. \tag{5.2}$$

Remark 7. The operator N of the Emden equation [8]

$$N(u) \equiv u''(t) + \frac{2}{t}u'(t) + u^5(t) = 0, \quad t \in [t_0, t_1], \quad t_0 > 0,$$

is potential on $D(N)$ (2.2) relative to the following bilinear form:

$$\Phi(v, g) = \int_{t_0}^{t_1} t^2 v(t) g(t)\, dt$$

(see Example 1; $k_1 = 2$, $k_2 = 1$, $m = 1$, and $n = 5$).

In this case, functional (5.2) becomes

$$F_N[u] = \int_{t_0}^{t_1} \Big(-\frac{t^2}{2}(u'(t))^2 + \frac{t^2}{6}u^6(t) \Big)\, dt. \tag{5.3}$$

Note that functional (5.3) was obtained in another way in [8].

Example 2. Consider the following equation:

$$N(u) \equiv 2tu''(t) + 2u'(t) + t(u'(t))^2 - u(t) - 1 = 0, \quad t \in [t_0, t_1]. \tag{5.4}$$

In this case,

$$a(t) = 2t, \quad b = 2, \quad c(t) = t, \quad d(u(t)) = -u(t) - 1.$$

The operator N (5.4) is not potential on $D(N)$ (2.2) relative to bilinear forms (2.12) and (2.17) because $c(t) \neq 0$.

We find $M = M(u(t))$ such that the operator N (5.4) is potential on $D(N)$ (2.2) relative to a bilinear form of type (2.3).

From conditions (2.4) and (2.5), we obtain

$$M(u(t)) = e^{u(t)}.$$

Thus, the operator N (5.4) is potential on $D(N)$ (2.2) relative to the following bilinear form:

$$\Phi(u; v, g) = \int_{t_0}^{t_1} e^{u(t)} v(t) g(t)\, dt.$$

By formula (3.2), we get

$$B_M(u(t)) = -e^{u(t)}u(t),$$

and functional (3.1) takes the form

$$F_N[u] = \int_{t_0}^{t_1} \Big(-te^{u(t)}(u'(t))^2 - e^{u(t)}u(t) \Big)\, dt.$$

Example 3. Consider the following equation [8]:

$$N(u) \equiv u''(t) - \frac{(u'(t))^2}{u(t)} + \frac{1}{u^2(t)} = 0, \quad t \in [t_0, t_1]. \tag{5.5}$$

Here,

$$a = 1, \quad b = 0, \quad c(u(t)) = -\frac{1}{u(t)}, \quad d(u(t)) = \frac{1}{u^2(t)}.$$

The operator N (5.5) is not potential on $D(N)$ (2.2) relative to bilinear forms (2.12) and (2.17) because conditions (2.13) and (2.18) do not hold.

We find $M = M(u(t))$ such that the operator N (5.5) is potential on $D(N)$ (2.2) relative to a bilinear form of type (2.3).

From conditions (2.4) and (2.5), we obtain

$$M(u(t)) = \frac{1}{u^2(t)}.$$

Thus, the operator N (5.5) is potential on $D(N)$ (2.2) relative to the following bilinear form:

$$\Phi(u; v, g) = \int_{t_0}^{t_1} \frac{1}{u^2(t)} v(t) g(t)\, dt.$$

By formula (3.2), we get

$$B_M(u(t)) = -\frac{1}{3u^3(t)},$$

and functional (3.1) takes the form

$$F_N[u] = \int_{t_0}^{t_1} \left(-\frac{(u'(t))^2}{2u^2(t)} - \frac{1}{3u^3(t)} \right) dt.$$

6. Conclusion

In the paper, we obtained the following results: the potentiality of the operator of a second-order ordinary differential equation relative to a local bilinear form was investigated, a formula for constructing the functional was given, and the structure of the corresponding Euler–Lagrange equation was defined. In particular, applications and extensions of the work consist in the possibility to establish connections between the invariance of the functional, the given equation, and its first integrals and to spread the proposed scheme of investigation to higher-order equations.

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