LINEARIZATION OF POISSON–LIE STRUCTURES ON THE 2D EUCLIDEAN AND (1 + 1) POINCARÉ GROUPS

Bousselham Ganbouri

Mohammed First University,
Mohammed V Avenue, P.O. Box 524, 60000 Oujda, Morocco

b.ganbouri@ump.ac.ma

Mohamed Wadia Mansouri

Ibn Tofail University,
P.O. Box 242, 14000 Kénitra, Morocco

mansourimohammed.wadia@uit.ac.ma

Abstract: The paper deals with linearization problem of Poisson-Lie structures on the (1 + 1) Poincaré and 2D Euclidean groups. We construct the explicit form of linearizing coordinates of all these Poisson-Lie structures. For this, we calculate all Poisson-Lie structures on these two groups mentioned above, through the correspondence with Lie Bialgebra structures on their Lie algebras which we first determine.

Keywords: Poisson-Lie groups, Lie bialgebras, Linearization.

1. Introduction

Poisson-Lie structure on a Lie group G is a Poisson structure $\{.,.\}$ on $C^\infty(G)$, such that the multiplication $\mu : G \times G \longrightarrow G$ is a Poisson map, namely

$$\{f \circ \mu, \, g \circ \mu\}_{C^\infty(G \times G)}(x,y) = \{f,g\}_{C^\infty(G)}(\mu(x,y)), \quad x,y \in G, \quad f,g \in C^\infty(G).$$

By Drinfel'd [5, 6], this is equivalent to giving an antisymmetric contravariant $2-$tensor π on G such that the Schouten–Nijenhuis bracket $[\pi, \pi] = 0$ and satisfies the multiplicativity relation

$$\pi(xy) = l_{x_*}\pi(y) + r_{y_*}\pi(x), \quad \forall x, y \in G,$$

where l_{x_*} and r_{y_*} are the left and right translations in G by x and y, respectively.

The relation above shows that the Poisson-Lie structure π must vanishing at the identity $e \in G$, so that its derivative $d_e\pi : \mathcal{G} \to \bigwedge^2 \mathcal{G}$ at e is well defined, where $\mathcal{G}$ is the Lie algebra of G. This linear homomorphism turns out to be a 1-cocycle with respect to the adjoint action, and the dual homomorphism $\bigwedge^2 \mathcal{G}^* \to \mathcal{G}^*$ satisfies the Jacobi identity; i.e., the dual $\mathcal{G}^*$ of $\mathcal{G}$ becomes a Lie algebra. Satisfying these properties, the map $d_e\pi$ is said to be a Lie bialgebra structure associated to π.

Recall that the preceding construction is in some sense invertible [10]. Namely, if G is simply connected then any Lie bialgebra structure $\delta : \mathcal{G} \to \bigwedge^2 \mathcal{G}$ on the Lie algebra $\mathcal{G} = \text{Lie}\,(G)$ carries uniquely defined Poisson–Lie structure π on G such that

$$(d_e\pi)(S) = \delta(S), \quad \forall S \in \mathcal{G}. \tag{1.1}$$

If we choose a local coordinates $(x_1, x_2, ..., x_n)$ in a neighborhood U of the unity e, the Poisson–Lie structure π is given by

$$\pi(x) = \sum_{1 \leq i < j \leq n} \pi_{ij}(x) \partial x_i \wedge \partial x_j, \quad \forall\, x \in U,$$

where π_{ij} are smooth functions vanishing at e and

$$\{x_i, x_j\}(x) = \pi_{ij}(x), \quad \forall\, x \in U.$$

The Taylor series of the functions π_{ij} is given by

$$\pi_{ij}(x) = \sum_{1 \leq k \leq n} c_{ij}^k x_k + \theta_{ij}(x),$$

where $order\,(\theta_{ij}) \geq 2$ and $c_{ij}^k = \partial \pi_{ij} / \partial x_k(e)$.

In particular, the terms $c_{ij}^k x_k$ define a linear Poisson structure π_0, called the *linear part* of π, there Poisson bracket is written in terms of the local coordinates $(x_1, x_2, ..., x_n)$ as

$$\{x_i, x_j\}_0 = \sum_{1 \leq k \leq n} c_{ij}^k x_k. \tag{1.2}$$

Further, since π satisfies the Jacobi identity, the $\{c_{ij}^k\}_{\substack{1 \leq i < j \leq n \\ 1 \leq k \leq n}}$ form a set of structure constants for the Lie algebra $(\mathcal{G}^*, \delta^*)$ dual of Lie algebra $(\mathcal{G}, [,])$. In other words, $\mathcal{G}^*$ is called the linearizing Lie algebra of Poisson–Lie structure π.

In this paper we are interested in the following *linearization problem:*
Are there new coordinates where the terms θ_{ij} vanish identically, so that the Poisson-Lie structure coincides with its linear part?

For a Poisson structure P vanishing at a point x_0, Weinstein [11] proved that if the linearizing Lie algebra is semisimple, then P is formally linearizable at x_0. Furthermore, Conn [3] proved that if the linearizing Lie algebra is semisimple, then P is analytically linearizable. Duffour [7] showed that semisimplicity does not imply smooth linearizability by giving a counterexample of a three-dimensional solvable Lie algebra. In the case of smooth Poisson structures, Conn [4] proved that if the linearizing Lie algebra is semisimple and of compact type then the linearization is smooth.

For a Poisson–Lie structures, Chloup–Arnould [2] gave examples of linearizable and non linearizable Poisson–Lie structures. Recently, Enriquez–Etingof–Marshal [8] constructed a Poisson isomorphism between the formal Poisson manifolds g^* and G^*, where g is a finite dimensional quasitriangular Lie bialgebra and Alekseev–Meinrenken [1] showed that for any coboundary Poisson–Lie group G, the Poisson structure on G^* is linearisable at the group unit.

The aim of this paper is the explicit construction of smooth linearizing coordinates for the Poisson-Lie structures on the $2D$ Euclidean group generated by the Lie algebra $s_3(0)$ and the $(1 + 1)$ Poincaré group generated by the Lie algebra $\tau_3(-1)$. We note that the notations $s_3(0)$ and $\tau_3(-1)$ are the same as in [9], where all real three-dimensional Lie algebras are classified. We adopt the same notification throughout this paper.

In this work we present a Lie bialgebra structures on the Lie algebras $s_3(0)$ and $\tau_3(-1)$ and we adopt the classification given in [9]. Then, we give the corresponding Poisson–Lie structures on $2D$ Euclidean and $(1 + 1)$ Poincaré groups and present their Casimir functions, which describe a symplectic leaves for all Poisson–Lie structures. Finally, we show that all these Poisson–Lie structures are linearizable near the unity by constructing the explicit forme of linearizing coordinates.

The paper is organized as follows. In Section 2 we treat the $2D$ Euclidean group and explain the technical methods, in Section 3 we investigate the $(1 + 1)$ Poincaré group for which we list in a schematic way our results in the same order and with the same notations.

2. Poisson–Lie structures on $2D$ Euclidean group

2.1. $2D$ Euclidean Lie algebra and group

The $2D$ Euclidean Lie algebra $s_3(0)$ is defined by the Lie brackets:

$$[e_3, e_1] = e_2, \quad [e_3, e_2] = -e_1, \quad [e_1, e_2] = 0.$$

The relation above defines a solvable three-dimensional real Lie algebra where its adjoint representation ρ is as follows:

$$\rho(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(e_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The generic Lie group element M with a local coordinates (x, y, z) "near $\{e\}$" is as follows

$$M = \exp(x\rho(e_1)) \exp(y\rho(e_2)) \exp(z\rho(e_3)) = \begin{pmatrix} \cos(z) & -\sin(z) & y \\ \sin(z) & \cos(z) & -x \\ 0 & 0 & 1 \end{pmatrix}.$$

If M' is another generic Lie group element with "local coordinates" (x', y', z'), then the multiplication of two group elements would be

$$M.M' = \begin{pmatrix} \cos(z + z') & -\sin(z + z') & y + y'\cos(z) + x'\sin(z) \\ \sin(z + z') & \cos(z + z') & -x - x'\cos(z) + y'\sin(z) \\ 0 & 0 & 1 \end{pmatrix}.$$

Therewith, the $2D$ Euclidean group can be identified by $\mathbb{R}^3$ associated with the group multiplication law:

$$(x, y, z).(x', y', z') = (x + x'\cos(z) - y'\sin(z), y + y'\cos(z) + x'\sin(z), z + z')$$

with the unity $e = (0, 0, 0)$.

The left invariant fields (E_1, E_2, E_3) associated to the basis (e_1, e_2, e_3) have this local expression

$$E_1 = \cos(z)\partial_x + \sin(z)\partial_y, \quad E_2 = -\sin(z)\partial_x + \cos(z)\partial_y, \quad E_3 = \partial_z.$$

2.2. Bialgebra and Poisson-Lie structures on $2D$ Euclidean group

Let δ be a bialgebra structure on the Lie algebra $s_3(0)$. In the basis (e_1, e_2, e_3) of $s_3(0)$ we write

$$\delta(e_1) = a_1 e_2 \wedge e_3 + b_1 e_3 \wedge e_1 + c_1 e_1 \wedge e_2,$$
$$\delta(e_2) = a_2 e_2 \wedge e_3 + b_2 e_3 \wedge e_1 + c_2 e_1 \wedge e_2,$$
$$\delta(e_3) = a_3 e_2 \wedge e_3 + b_3 e_3 \wedge e_1 + c_3 e_1 \wedge e_2,$$

this is equivalent to

$$\begin{pmatrix} \delta(e_1) \\ \delta(e_2) \\ \delta(e_3) \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} e_2 \wedge e_3 \\ e_3 \wedge e_1 \\ e_1 \wedge e_2 \end{pmatrix} = U \begin{pmatrix} e_2 \wedge e_3 \\ e_3 \wedge e_1 \\ e_1 \wedge e_2 \end{pmatrix}.$$

If $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is the dual basis of (e_1, e_2, e_3), then the Lie bracket on $s_3^*(0)$ given by δ^* can be written:

$$\delta^*(\varepsilon_2 \wedge \varepsilon_3) = a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3,$$
$$\delta^*(\varepsilon_3 \wedge \varepsilon_1) = b_1\varepsilon_1 + b_2\varepsilon_2 + b_3\varepsilon_3,$$
$$\delta^*(\varepsilon_1 \wedge \varepsilon_2) = c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3.$$

By a straightforward computation, we show that in order to ensure that δ is a 1-cocycle, the system below must to be verified

$$\begin{pmatrix} a_2 + b_1 & b_2 - a_1 & b_3 + c_2 \\ a_1 - b_2 & a_2 + b_1 & a_3 + c_2 \\ 0 & 0 & a_1 + b_2 \end{pmatrix} \begin{pmatrix} e_2 \wedge e_3 \\ e_3 \wedge e_1 \\ e_1 \wedge e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, the matrix U has the form

$$U = \begin{pmatrix} 0 & b_1 & c_1 \\ -b_1 & 0 & c_2 \\ -c_1 & -c_2 & c_3 \end{pmatrix}, \tag{2.1}$$

where the Jacobi identity fulfilled by δ^* is $b_1 c_3 = 0$.
Therefore, we get

Proposition 1. *The Lie bialgebra structures δ on 2D Euclidean Lie algebra are written in terms of the basis (e_1, e_2, e_3) as follows:*

$$\delta(e_1) = b_1 e_3 \wedge e_1 + c_1 e_1 \wedge e_2,$$
$$\delta(e_2) = -b_1 e_2 \wedge e_3 + c_2 e_1 \wedge e_2,$$
$$\delta(e_3) = -c_1 e_2 \wedge e_3 - c_2 e_3 \wedge e_1 + c_3 e_1 \wedge e_2,$$

where b_1, c_1, c_2 and c_3 are reals such that $b_1 c_3 = 0$.

Now, let π be the Poisson-Lie structures corresponding to the bialgebra structures δ. We set:

$$\pi = \pi_{23} E_2 \wedge E_3 + \pi_{31} E_3 \wedge E_1 + \pi_{12} E_1 \wedge E_2,$$

where $(E_2 \wedge E_3, E_3 \wedge E_1, E_1 \wedge E_2)$ is the basis of the bivector fields on the 2D Euclidean group.

For any element E_k of the basis (E_1, E_2, E_3), the Lie derivative of π in the direction of E_k is written as

$$L_{E_k}\pi = \sum_{1 \leq i < j \leq 3} E_k(\pi_{ij}) E_i \wedge E_j + \pi_{ij} \left([E_k, E_i] \wedge E_j - [E_k, E_j] \wedge E_i\right), \quad k = 1, 2, 3.$$

By a technical and explicit computation using the above relation, we show that the equation (1.1) which describes the correspondence between π and δ can be transformed into the following system

$$\begin{cases} E_1(\pi_{23}) = 0, \\ E_2(\pi_{23}) = -b_1, \\ E_3(\pi_{23}) - \pi_2 = -c_1, \end{cases} \quad \begin{cases} E_1(\pi_{31}) = b_1, \\ E_2(\pi_{31}) = 0, \\ E_3(\pi_{31}) + \pi_{23} = -c_2, \end{cases} \quad \begin{cases} E_1(\pi_{12}) + \pi_{31} = c_1, \\ E_2(\pi_{12}) - \pi_{23} = c_2, \\ E_3(\pi_{12}) = c_3. \end{cases} \tag{2.2}$$

The system (2.2) has for solutions:

$$\pi_{23}(x, y, z) = (b_1 x - c_1)\sin(z) - (b_1 y - c_2)\cos(z) - c_2,$$
$$\pi_{31}(x, y, z) = (b_1 x - c_1)\cos(z) + (b_1 y - c_2)\sin(z) + c_1,$$
$$\pi_{12}(x, y, z) = -\frac{b_1}{2}x^2 - \frac{b_1}{2}y^2 + c_1 x + c_2 y + c_3 z.$$

Since

$$E_2 \wedge E_3 = \cos(z)\partial_y \wedge \partial_z + \sin(z)\partial_z \wedge \partial_x,$$
$$E_3 \wedge E_1 = -\sin(z)\partial_y \wedge \partial_z + \cos(z)\partial_z \wedge \partial_x,$$
$$E_1 \wedge E_2 = \partial_x \wedge \partial_y,$$

we have:

Proposition 2. *In the local coordinates (x, y, z), the Poisson–Lie bracket $\{.,.\}$ on 2D Euclidean group is*:

$$\{y, z\} = -b_1 y - c_1 \sin(z) - c_2(\cos(z) - 1),$$
$$\{z, x\} = b_1 x - c_2 \sin(z) + c_1(\cos(z) - 1),$$
$$\{x, y\} = -\frac{b_1}{2}x^2 - \frac{b_1}{2}y^2 + c_1 x + c_2 y + c_3 z.$$

We will call this four-parametric Poisson–Lie brackets as $\mathcal{PL}(b_1, c_1, c_2, c_3)$.

The linear part π_0 of π is straightforwardly obtained as

$$\{y, z\}_0 = -b_1 y - c_1 z,$$
$$\{z, x\}_0 = b_1 x - c_2 z,$$
$$\{x, y\}_0 = c_1 x + c_2 y + c_3 z.$$

2.3. Classification of Poisson–Lie structures on $2D$ Euclidean group

The Poisson–Lie structures on a Lie group G are in one-to-one correspondence with the bialgebra structures on its Lie algebra $\mathcal{G}$. Thus, we obtain the complete classes of the Poisson-Lie structures on 2D Euclidean group by using the classification of Lie bialgebra structures on $s_3(0)$, which was given by Gomez in [9].

In [9], we find four nonequivalents (under Lie algebra automorphisms) classes of Lie bialgebra structures on $s_3(0)$. By taking into account the change of basis:

$$e_1 = \mathfrak{e}_1, \quad e_2 = \mathfrak{e}_2, \quad e_3 = -\mathfrak{e}_0,$$

we get a correspondence between each one of those classes and our presented cocommutator δ given in Proposition 1. This correspondence is specified by a fixed values of the parameters (b_1, c_1, c_2, c_3) of the matrix (2.1), as presented in the table below

Table 1. Correspondence with the classification [9] of Lie bialgebra structures on $s_3(0)$.

Lie bialgebra in [6]	b_1	c_1	c_2	c_3
(9)	$-\lambda$	0	0	0
15'	0	0	0	$-\omega$
11'	0	1	0	0
(14')	0	α	0	$-\lambda$

In Table 1, the first column describe the number that identifies the type of Lie bialgebra (last column of table III in [9]). The remaining of columns describe the particular values of the

parameters (b_1, c_1, c_2, c_3) for which the cocommutator given in Proposition 1 coincides with the Lie bialgebra parameters from [9]. Note, the parameters λ and ω are nonzero reals.

Thus, we have four nonequivalents (under group automorphisms) classes of Poisson–Lie structures on the $2D$ Euclidean group, that would be explicitly obtained by substituting the values of the parameters (b_1, c_1, c_2, c_3) into the full Poisson–Lie bracket expressions $\mathcal{PL}(b_1, c_1, c_2, c_3)$ given in Proposition 2 as shown in table below

Table 2. Classification of Poisson-Lie structures on the $2D$ Euclidean group corresponding to the Lie bialgebra structures given in Table 1.

$\{,\}$	$\{y, z\}$	$\{z, x\}$	$\{x, y\}$
$\mathcal{PL}(-\lambda, 0, 0, 0)$	λy	$-\lambda x$	$\lambda/2 \cdot (x^2 + y^2)$
$\mathcal{PL}(0, 0, 0, -\omega)$	0	0	$-\omega z$
$\mathcal{PL}(0, 1, 0, 0)$	$-\sin(z)$	$\cos(z) - 1$	x
$\mathcal{PL}(0, \alpha, 0, -\lambda)$	$-\alpha \sin(z)$	$\alpha(\cos(z) - 1)$	$\alpha x - \lambda z$

Now, recall that a local Casimir function on a Poisson–Lie group G is a function C such that $\{C, f\} = 0$ for any function f on G. Note that the local Casimir functions on a Poisson–Lie group (G, π) are constant in symplectic leaves of G.

Let $\mathcal{C}_{\mathcal{PL}(b_1, c_1, c_2, c_3)}$ be the Casimir functions for the Poisson–Lie structures $\mathcal{PL}(b_1, c_1, c_2, c_3)$. For the classes of Poisson–Lie structures given in Table 2, we get

$$\mathcal{C}_{\mathcal{PL}(-\lambda, 0, 0, 0)} = 2 \arctan\left(\frac{x}{y}\right) + z,$$

$$\mathcal{C}_{\mathcal{PL}(0, 0, 0, -\omega)} = f(z),$$

$$\mathcal{C}_{\mathcal{PL}(0, 1, 0, 0)} = \frac{x \sin(z)}{\cos(z) - 1} - y,$$

$$\mathcal{C}_{\mathcal{PL}(0, \alpha, 0, -\lambda)} = -\alpha y + \frac{(\alpha x - \lambda z)\sin(z)}{\cos(z) - 1} - \lambda \ln(1 - \cos(z)),$$

where f is a C^∞−function that depends only on z.

2.4. Linearization of Poisson–Lie structures on $2D$ Euclidean group

Now, we consider the formula (1.2), than the linear part π_0 of π can be written as

$$\pi_0(x) = \sum_{1 \leq i < j \leq n} \left(\sum_{1 \leq k \leq n} c_{ij}^k x_k \right) \partial_{x_i} \wedge \partial_{x_j}.$$

Note, the Lie bialgebra structure δ associated to π defines a linear Poisson–Lie structure on the additive group $\mathcal{G}$ $(\mathcal{G} \simeq \mathbb{R}^n)$, that can be expressed as

$$\delta(a) = \sum_{1 \leq i < j \leq n} \left(\sum_{1 \leq k \leq n} c_{ij}^k a_k \right) \partial_i \wedge \partial_j, \quad a = (a_1, ..., a_n) \in \mathbb{R}^n, \tag{2.3}$$

where $(\partial_1, ..., \partial_n)$ is the canonical basis of $\mathbb{R}^n$.

The expression (2.3) coincides with the linear part π_0, hence the linearization problem becomes as follows:

Is there a local Poisson diffeomorphism $\varphi : G \longrightarrow \mathcal{G}$ of a neighborhood of e in G into a neighborhood of 0 in $\mathcal{G}$ such that $\varphi(e) = 0$?

A such diffeomorphism preserves necessarily the subgroup of singular points: $\{x \in G : \pi(x) = 0\}$ and the symplectics leaves.

If $(\varphi_1, ..., \varphi_n)$ are the components of φ, then φ is solution of the system of equations

$$\{\varphi_i, \varphi_j\} = \sum_{1 \leq k \leq n} c_{ij}^k \varphi_k, \qquad 1 \leq i < j \leq n. \tag{2.4}$$

Method. We calculate the equations which determine the symplectics leaves for the four classes of Poisson–Lie structures given in Table 2, using the Casimir functions (each symplectic leaf is the common level manifold of Casimir functions) and we determine their subgroup of singular points.

The identification of the subgroup of the singular points and the symplectics leaves of the $2D$ Euclidean group with those of its Lie algebra $s_3(0)$ allows us to solve the system of equations (2.4) for each class of Poisson–Lie structures given in Table 2. Consequently, our main result is the following

Theorem 1. *All Poisson–Lie structures on $2D$ Euclidean group which are given in Table 2 are linearizable near the unity. The linearizing coordinates of each class are given in Table 3:*

Table 3. Components of linearizing diffeomorphisms φ corresponding to the Poisson–Lie structures given in Table 2.

$\varphi_i(x,y,z)$	$\varphi_1(x,y,z)$	$\varphi_2(x,y,z)$	$\varphi_3(x,y,z)$
$\mathcal{PL}(-\lambda,0,0,0)$	$x\cos(\frac{z}{2}) + y\sin(\frac{z}{2})$	$-x\sin(\frac{z}{2}) + y\cos(\frac{z}{2})$	z
$\mathcal{PL}(0,0,0,-\omega)$	x	y	z
$\mathcal{PL}(0,1,0,0)$	$x + y\tan(\frac{z}{2})$	y	$\tan(\frac{z}{2})$
$\mathcal{PL}(0,\alpha,0,-\lambda)$	$x - \frac{\lambda}{\alpha}z + $ $+\left(y - \frac{\lambda}{\alpha}\ln(1+\tan^2(\frac{z}{2}))\right)\tan(\frac{z}{2})$	y	$2\tan(\frac{z}{2})$

Remark 1. The class $\mathcal{PL}(0,0,0,-\omega)$ is linear in the local coordinates (x,y,z) (trivial case).

3. Poisson–Lie structures on $(1+1)$ Poicaré group

3.1. $(1+1)$ Poincaré Lie algebra and group

The $(1+1)$ Poincaré Lie algebra $\tau_3(-1)$ (presented in null coordinates) is defined by the Lie brackets

$$[e_3, e_1] = -e_1, \quad [e_3, e_2] = e_2, \quad [e_1, e_2] = 0.$$

1. Adjoint representation

$$\rho(e_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(e_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. Matrix group element

$$M = \exp(x\rho(e_1))\exp(y\rho(e_2))\exp(z\rho(e_3)) = \begin{pmatrix} \exp(-z) & 0 & x \\ 0 & \exp(z) & -y \\ 0 & 0 & 1 \end{pmatrix}.$$

3. Group multiplication law

The $(1+1)$ Poincaré group can be identified by $\mathbb{R}^3$ associated with the group multiplication law:

$$(x,y,z).(x',y',z') = (x + x'\exp(-z), y + y'\exp(z), z + z'),$$

with the unity $e = (0,0,0)$.

4. Basis of left invariant fields

$$E_1 = \exp(-z)\partial_x, \quad E_2 = \exp(z)\partial_y, \quad E_3 = \partial_z.$$

3.2. Lie bialgebra and Poisson–Lie structures on $(1+1)$ Poincaré group

1. Lie bialgebra structures on $\tau_3(-1)$

Proposition 3. *The Lie bialgebra structures δ on $(1+1)$ Poincaré Lie algebra are written in terms of the basis (e_1, e_2, e_3) as follows*:

$$\delta(e_1) = b_1 e_3 \wedge e_1 + c_1 e_1 \wedge e_2,$$
$$\delta(e_2) = -b_1 e_2 \wedge e_3 + c_2 e_1 \wedge e_2,$$
$$\delta(e_3) = -c_1 e_2 \wedge e_3 - c_2 e_3 \wedge e_1 + c_3 e_1 \wedge e_2,$$

with a_1, b_1, c_1, c_2 are real such that $b_1 c_3 = 0$.

P r o o f. Similar to the proof of Proposition 1.   □

2. Poisson–Lie structures on (1+1) Poincaré group

Proposition 4. *In the local coordinates (x, y, z), the Poisson–Lie bracket $\{.,.\}$ on $(1+1)$ Poincaré group is written as*

$$\{y, z\} = -b_1 y + c_1(1 - \exp(z)),$$
$$\{z, x\} = b_1 x + c_2(\exp(-z) - 1),$$
$$\{x, y\} = c_1 x + c_2 y + c_3 z - b_1 xy.$$

We will call this six-parametric Poisson–Lie brackets as $\mathcal{PL}(b_1, c_1, c_2, c_3)$.

P r o o f. Similar to the proof of Proposition 1.   □

3. The linear part is as follows

$$\{y, z\}_0 = -b_1 y - c_1 z,$$
$$\{z, x\}_0 = b_1 x - c_2 z,$$
$$\{x, y\}_0 = c_1 x + c_2 y + c_3 z.$$

3.3. Classification of Lie bialgebra and Poisson–Lie structures on $(1+1)$ Poincaré group

1. Isomorphic to the Lie algebra $\tau_3(-1)$ in [9] through the change of variables

$$e_1 = \mathfrak{e}_1, \quad e_2 = \mathfrak{e}_2, \quad e_3 = -\mathfrak{e}_0$$

2. Correspondence with the classification of Lie bialgebras on $\tau_3(-1)$

Table 4. Correspondence with the classification [9] of Lie bialgebra structures on $\tau_3(-1)$.

Lie bialgebra in [9]	b_1	c_1	c_2	c_3
6 $(\rho = -1, \chi = e_0 \wedge e_1)$	0	0	1	0
7 $(\rho = -1)$	$-\lambda$	0	0	0
(11)	0	$\alpha\beta$	α	0
5'	0	0	0	-1
8	0	$-\alpha$	0	-1
(14)	0	$\alpha\lambda$	α	-1

In Table 4, the first column describes the number that identifies the type of Lie bialgebra (last column of table III in [9]. Note, the parameters λ, α and β are nonzero reals.

3. Classification of Poisson Lie structures on (1+1) Poincaré group

Table 5. Correspondence with the Lie bialgebra structures given in Table 4 of Poisson–Lie structures on the $(1+1)$ Poincaré group.

$\{,\}$	$\{y,z\}$	$\{z,x\}$	$\{x,y\}$
$\mathcal{PL}(0,0,1,0)$	0	$\exp(-z) - 1$	y
$\mathcal{PL}(-\lambda,0,0,0)$	λy	$-\lambda x$	λxy
$\mathcal{PL}(0,\alpha\beta,\alpha,0)$	$\alpha\beta(1 - \exp(z))$	$\alpha(\exp(-z) - 1)$	$\alpha\beta x + \alpha y$
$\mathcal{PL}(0,0,0,-1)$	0	0	$-z$
$\mathcal{PL}(0,-\alpha,0,-1)$	$\alpha(\exp(z) - 1)$	0	$-\alpha x - z$
$\mathcal{PL}(0,\alpha\lambda,\alpha,-1)$	$\alpha\lambda(1 - \exp(z))$	$\alpha(\exp(-z) - 1))$	$\alpha\lambda x + \alpha y - z$

4. Casimir functions

$$\mathcal{C}_{\mathcal{PL}(0,0,1,0)} = \frac{y}{\exp(z) - 1}, \quad \mathcal{C}_{\mathcal{PL}(-\lambda,0,0,0)} = -\frac{x\exp(z)}{y}, \quad \mathcal{C}_{\mathcal{PL}(0,\alpha\beta,\alpha,0)} = \frac{\beta x\exp(z) + y}{\exp(z) - 1},$$

$$\mathcal{C}_{\mathcal{PL}(0,0,0,-1)} = f(z), \quad \mathcal{C}_{\mathcal{PL}(0,-\alpha,0,-1)} = \frac{\alpha x\exp(z) + z}{\exp(z) - 1} - \ln(\exp(-z) - 1),$$

$$\mathcal{C}_{\mathcal{PL}(0,\alpha\lambda,\alpha,-1)} = \frac{\alpha\lambda x\exp(z) + \alpha y - z}{1 - \exp(z)} - \ln(\exp(-z) - 1),$$

where f is a $C^\infty-$function of the only variable z.

3.4. Linearization of Poisson-Lie structures on (1+1) Poincaré group

Theorem 2. *All Poisson-Lie structures on $(1+1)$ Poincaré group which are given in Table 5 are linearizable near the unity. The linearizing coordinates of each class are given below*:

Table 6. Components of linearizing diffeomorphisms φ corresponding to the Poisson–Lie structures given in Table 5.

$\varphi_i(x,y,z)$	$\varphi_1(x,y,z)$	$\varphi_2(x,y,z)$	$\varphi_3(x,y,z)$
$\mathcal{PL}(0,0,1,0)$	x	$-y$	$\exp(z)-1$
$\mathcal{PL}(-\lambda,0,0,0)$	$x\exp(z)$	$-y$	z
$\mathcal{PL}(0,\alpha\beta,\alpha,0)$	$x+(\frac{1}{\beta}y+1)(\exp(-z)-1)$	y	$\exp(-z)-1$
$\mathcal{PL}(0,0,0,-1)$	x	y	z
$\mathcal{PL}(0,-\alpha,0,1)$	$-x-\frac{1}{\alpha}z\exp(-z)$	y	$\exp(-z)-1$
$\mathcal{PL}(0,-\alpha,0,1)$	$x+\frac{1}{\lambda}y(\exp(-z)-1)-\frac{1}{\alpha\lambda}z\exp(-z)$	y	$1-\exp(-z)$

P r o o f. We use the same method as in Theorem 1. □

Remark 2. The class $\mathcal{PL}(0,0,0,-1)$ is linear in the local coordinates (x,y,z) (trivial case).

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