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ON ZYGMUND-TYPE INEQUALITIES CONCERNING POLAR DERIVATIVE OF POLYNOMIALS

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Abstract: Let $P(z)$ be a polynomial of degree n, then concerning the estimate for maximum of $|P'(z)|$ on the unit circle, it was proved by S. Bernstein that $\|P'\|_\infty \le n\|P\|_\infty$. Later, Zygmund obtained an L_p-norm extension of this inequality. The polar derivative $D_\alpha[P](z)$ of $P(z)$, with respect to a point $\alpha \in \mathbb{C}$, generalizes the ordinary derivative in the sense that $\lim_{\alpha\to\infty} D_\alpha[P](z)/\alpha = P'(z)$. Recently, for polynomials of the form $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$ and having no zero in $|z| < k$ where $k > 1$, the following Zygmund-type inequality for polar derivative of $P(z)$ was obtained:

$$\|D_\alpha[P]\|_p \le n\Big(\frac{|\alpha| + k^\mu}{\|k^\mu + z\|_p}\Big)\|P\|_p, \quad \text{where} \quad |\alpha| \ge 1, \quad p > 0.$$

In this paper, we obtained a refinement of this inequality by involving minimum modulus of $|P(z)|$ on $|z| = k$, which also includes improvements of some inequalities, for the derivative of a polynomial with restricted zeros as well.

Keywords: L^p-inequalities, Polar derivative, Polynomials.

1. Zygmund type inequalities for polynomials

Let $\mathcal{P}_n$ denote the space of all complex polynomials of degree at most n. Define

$$\|P\|_p := \left(\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right)^{1/p}, \quad 0 < p < \infty.$$

It is well known that the supremum norm satisfies

$$\|P\|_\infty := \max_{|z|=1} |P(z)| = \lim_{p\to\infty} \|P\|_p\,.$$

It is also known [11] that $\lim_{p\to 0}\|P\|_p = \|P\|_0$, where

$$\|P\|_0 := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(e^{i\theta})\right| d\theta\right).$$

Let $D_\alpha[P](z)$ denote the polar differentiation (see [12]) of a polynomial $P(z)$ of degree n with respect to a complex number α, then

$$D_\alpha[P](z) := nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha[P](z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha[P](z)}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leq R,\ R > 0$.

If $P \in \mathcal{P}_n$, then

$$\|P'\|_p \leq n\|P\|_p. \tag{1.1}$$

Inequality (1.1) is due to Zygmund [21] for the case $p \geq 1$. In its proof, he uses M. Riesz's interpolation formula by means of Minkowski's inequality and obtained this inequality as an L_p-norm analogue of Bernstein's inequality (for details see [13] or [20]). A natural question was raised here: whether the restriction on p was indeed necessary? The question remained open for quite a long time despite some partial answers. Finally, it was Arestov [1] came up with some remarkable results which among other things proved that the inequality (1.1) remains valid for $0 < p < 1$ as well. This result is sharp as shown by $P(z) = az^n, a \neq 0$. Arestov [2] also obtained some sharp Bernstein–Zygmund type inequalities for the Szegö composition operators on the set of algebraic polynomials with restrictions on the location of their zeros.

For the class of polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, inequality (1.1) can be sharpened. In fact, if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < 1$, then

$$\left\|P'\right\|_p \leq \frac{n}{\|1+z\|_p}\|P\|_p, \quad p \geq 1. \tag{1.2}$$

Inequality (1.2) is due to De Bruijn [7]. Later Rahman and Schmeisser [16] followed Arestov's technique and proved that this inequality remains true for $0 < p < 1$ as well. The estimates is sharp and equality in (1.2) holds for $P(z) = az^n + b, |a| = |b| \neq 0$.

Govil and Rahman [10] generalized inequality (1.2) and proved that if $P \in \mathcal{P}_n$ does not vanish in $|z| < k$ where $k \geq 1$, then

$$\left\|P'\right\|_p \leq \frac{n}{\|k+z\|_p}\|P\|_p, \quad p \geq 1. \tag{1.3}$$

Let $\mathcal{P}_{n,\mu} \subset \mathcal{P}_n$ be a class of lacunary type polynomials

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,$$

where $1 \leq \mu \leq n$.

As a generalization of inequality (1.3), it was shown by Gardner & Weems [8] that if $P \in \mathcal{P}_{n,\mu}$ and $P(z) \neq 0$ for $|z| < k$, $k \geq 1$, then

$$\left\|P'\right\|_p \leq \frac{n}{\|k^\mu+z\|_p}\|P\|_p, \quad p > 0. \tag{1.4}$$

Aziz and Rather [5] extended inequality (1.2) to the polar derivative of a polynomial and proved that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, and $p \geq 1$,

$$\|D_\alpha[P]\|_p \leq n\left(\frac{|\alpha|+1}{\|1+z\|_p}\right)\|P\|_p. \tag{1.5}$$

Concerning the concept and properties of the polar derivative refer to [14].

Aziz et. al [6] also obtained an analogue of inequality (1.3) to the polar derivative and proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $p \geq 1$,

$$\|D_\alpha[P]\|_p \leq n\left(\frac{|\alpha|+k}{\|k+z\|_p}\right)\|P\|_p. \tag{1.6}$$

Rather [17, 18] showed that inequalities (1.5) and (1.6) remain valid for $0 < p < 1$ as well.

Recently, as a generalization of inequality (1.6), Rather et. al [19] proved that if $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $0 \leq p < \infty$,

$$\|D_\alpha[P]\|_p \leq n\left(\frac{|\alpha|+k^\mu}{\|k^\mu+z\|_p}\right)\|P\|_p. \tag{1.7}$$

2. Main results

In this paper, we obtain a refinement of inequality (1.7) by involving the minimum modulus of a polynomial. We prove the following main result.

Theorem 1. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, $0 \leq p \leq \infty$ and $0 \leq t \leq 1$,*

$$\left\| |D_\alpha[P]| + nmt\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right\|_p \leq n\left(\frac{|\alpha|+k^\mu}{\|z+k^\mu\|_p}\right)\|P\|_p, \tag{2.1}$$

where $m = \min_{|z|=k}|P(z)|$.

Since

$$\frac{nmt(|\alpha|-1)}{1+k^\mu} \geq 0 \quad \text{for} \quad |\alpha| \geq 1,$$

then one can easily observe that

$$\|D_\alpha[P]\|_p \leq \left\| |D_\alpha[P]| + nmt\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right\|_p,$$

and this implies that the Theorem 1 is a refinement of inequality (1.7).

If we divide both sides of inequality (2.1) by $|\alpha|$ and let $|\alpha| \to \infty$, we obtain the following refinement of inequality (1.4).

Corollary 1. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $0 \leq p \leq \infty$,*

$$\left\| |P'| + \frac{nmt}{1+k^\mu}\right\|_p \leq \frac{n}{\|z+k^\mu\|_p}\|P\|_p, \tag{2.2}$$

where $m = \min_{|z|=k}|P(z)|$. The result is best possible as shown by the polynomial

$$P(z) = (z^\mu + k^\mu)^{n/\mu},$$

where μ divides n.

Inequality (2.2) also includes a refinement of (1.3). By taking $k = 1$ and $\mu = 1$ in (2.2), the following improvement of inequality (1.2) follows immediately.

Corollary 2. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$ then for $0 \leq p \leq \infty$,*

$$\left\| |P'| + \frac{nmt}{2} \right\|_p \leq \frac{n}{\|1+z\|_p} \|P\|_p, \tag{2.3}$$

where $m = \min_{|z|=1} |P(z)|$. The result is sharp and equality in (2.3) holds for $P(z) = z^n + 1$.

3. Lemmas

For the proof of above theorem, we need the following lemmas.

Lemma 1. *If*

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \quad 1 \leq \mu \leq n,$$

is a polynomial of degree n having no zeros in $|z| < k$, where $k \geq 1$, then

$$k^\mu |P'(z)| \leq |Q'(z)| \quad for \quad |z| = 1,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

The above Lemma 1 is implicit in Qazi [15] and the proof of next lemma is implicit in [9].

Lemma 2. *If $P(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$,*

$$|Q'(z)| \geq |\lambda| mn \quad for \quad |z| = 1,$$

where

$$m = \min_{|z|=k} |P(z)|, \quad Q(z) = z^n \overline{P(1/\overline{z})}.$$

Lemma 3. *If*

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \quad 1 \leq \mu \leq n,$$

is a polynomial of degree n having no zeros in $|z| < k$, where $k \geq 1$, then for $0 \leq t \leq 1$,

$$k^\mu |P'(z)| \leq |Q'(z)| - mnt \quad for \quad |z| = 1, \tag{3.1}$$

where

$$Q(z) = z^n \overline{P(1/\overline{z})}, \quad m = \min_{|z|=k} |P(z)|.$$

P r o o f. By hypothesis, the polynomial $P(z)$ has no zero in $|z| < k, k \geq 1$. We first show for a given $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, the polynomial $F(z) = P(z) - \lambda m$ does not vanish in $|z| < k$. This is clear if $m = 0$, that is if $P(z)$ has a zero on $|z| = k$. We now suppose that all the zeros of $P(z)$ lie in $|z| > k$, then clearly $m > 0$ so that $m/P(z)$ is analytic in $|z| \leq k$ and

$$\left| \frac{m}{P(z)} \right| \leq 1 \quad \text{for} \quad |z| = k.$$

Since $m/P(z)$ is not a constant, by the Minimum modulus principle, it follows that

$$m < |P(z)| \quad \text{for} \quad |z| < k. \tag{3.2}$$

Now, if $F(z) = P(z) - \lambda m$ has a zero in $|z| < k$, say at $z = z_0$ with $|z_0| < k$, then

$$P(z_0) - \lambda m = 0.$$

This gives

$$|P(z_0)| = |\lambda m| = |\lambda| m \leq m, \quad \text{where} \quad |z_0| < k,$$

which contradicts (3.2). Hence, we conclude that in any case, the polynomial

$$F(z) = P(z) - \lambda m$$

does not vanish in $|z| < k$, $k \geq 1$, for every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. Applying Lemma 1 to

$$F(z) = P(z) - \lambda m,$$

we get

$$|Q'(z) - \overline{\lambda} mnz^{n-1}| \geq k^{\mu} |P'(z)| \quad \text{for} \quad |z| = 1. \tag{3.3}$$

Now choosing the argument of λ so that on $|z| = 1$,

$$|Q'(z) - \overline{\lambda} mnz^{n-1}| = |Q'(z)| - |\lambda| mn \tag{3.4}$$

which is possible due to lemma 2. By combining (3.3) and (3.4), we obtain

$$|Q'(z)| \geq k^{\mu} |P'(z)| + tmn \quad \text{for} \quad |z| = 1, \tag{3.5}$$

where $t = |\lambda|$ and $0 \leq t < 1$. For the case $t = 1$, the inequality (3.1) follows immediately by letting $t \to 1$ in (3.5) and this completes the proof.

The following lemma is due to Aziz and Rather [3].

Lemma 4. *If A, B and C are non-negative real numbers such that $B + C \leq A$, then for every real number β,*

$$|(A - C) + e^{i\beta}(B + C)| \leq |A + e^{i\beta} B|.$$

Lemma 5 [19]. *If a, b are any two positive real numbers such that $a \geq bc$ where $c \geq 1$, then for any $x \geq 1$, $p > 0$ and $0 \leq \beta < 2\pi$,*

$$(a + bx)^p \int_0^{2\pi} \left|c + e^{i\beta}\right|^p d\beta \leq (c + x)^p \int_0^{2\pi} \left|a + be^{i\beta}\right|^p d\beta.$$

We also need the following lemma due to Aziz and Rather [4].

Lemma 6 [4]. *If $P \in \mathcal{P}_n$ and $Q(z) = z^n \overline{P(1/\overline{z})}$, then for every $p > 0$ and β real , $0 \leq \beta < 2\pi$,*

$$\int_0^{2\pi} \int_0^{2\pi} \left|P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})\right|^p d\theta d\beta \leq 2\pi n^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

4. Proof of Theorem 1

P r o o f. By hypothesis $P \in \mathcal{P}_{n,\mu}$ and does not vanish in $|z| < k$, where $k \geq 1$ further if

$$Q(z) = z^n \overline{P(1/\overline{z})},$$

then, by Lemma 3, we have for $|z| = 1$,

$$k^\mu \left|P'(z)\right| \leq \left|Q'(z)\right| - mnt = \left|Q'(z)\right| - mnt \left(\frac{1+k^\mu}{1+k^\mu}\right).$$

Equivalently,

$$k^\mu \left(\left|P'(z)\right| + \frac{mnt}{1+k^\mu}\right) \leq |Q'(z)| - \frac{mnt}{1+k^\mu} \quad \text{for} \quad |z| = 1. \tag{4.1}$$

Setting

$$A = \left|Q'(e^{i\theta})\right|, \quad B = |P'(e^{i\theta})|, \quad C = \frac{mnt}{1+k^\mu}$$

in Lemma 4 we note by (4.1) that

$$B + C \leq k^\mu(B+C) \leq A - C \leq A, \quad \text{since} \quad k \geq 1.$$

Therefore, by Lemma 4 for each real β, we get

$$\left|\left(|Q'(e^{i\theta})| - \frac{mnt}{1+k^\mu}\right) + e^{i\beta}\left(|P'(e^{i\theta})| + \frac{mnt}{1+k^\mu}\right)\right| \leq \left||Q'(e^{i\theta})| + e^{i\beta}|P'(e^{i\theta})|\right|.$$

This implies for each $p > 0$

$$\int_0^{2\pi} \left|F(\theta) + e^{i\beta}G(\theta)\right|^p d\theta \leq \int_0^{2\pi} \left||Q'(e^{i\theta})| + e^{i\beta}|P'(e^{i\theta})|\right|^p d\theta, \tag{4.2}$$

where

$$F(\theta) = |Q'(e^{i\theta})| - \frac{mnt}{1+k^\mu} \quad \text{and} \quad G(\theta) = |P'(e^{i\theta})| + \frac{mnt}{1+k^\mu}. \tag{4.3}$$

Let $P'(\theta) = |P'(\theta)|e^{i\psi}$ and $Q'(\theta) = |Q'(\theta)|e^{i\phi}$, then

$$\begin{aligned}\int_0^{2\pi} \left|Q'(e^{i\theta})e^{i\beta} + P'(e^{i\theta})\right|^p d\beta &= \int_0^{2\pi} \left||Q'(e^{i\theta})|e^{i(\beta+\phi)} + e^{i\psi}|P'(e^{i\theta})|\right|^p d\beta \\ &= \int_0^{2\pi} \left||Q'(e^{i\theta})|e^{i(\beta+\phi-\psi)} + |P'(e^{i\theta})|\right|^p d\beta.\end{aligned}$$

Putting $\beta + \phi - \psi = \Phi$, then we obtain,

$$\int_0^{2\pi} \left|Q'(e^{i\theta})e^{i\beta} + P'(e^{i\theta})\right|^p d\beta = \int_{\phi-\psi}^{2\pi+\phi-\psi} \left||Q'(e^{i\theta})|e^{i\Phi} + |P'(e^{i\theta})|\right|^p d\Phi.$$

Since the function

$$T(\Phi) = |Q'(e^{i\theta})|e^{i\Phi} + |P'(e^{i\theta})|$$

is periodic with period 2π, hence we have

$$\begin{aligned}\int_0^{2\pi}\left|Q'(e^{i\theta})e^{i\beta}+P'(e^{i\theta})\right|^p d\beta &= \int_0^{2\pi}\left||Q'(e^{i\theta})|e^{i\Phi}+|P'(e^{i\theta})|\right|^p d\Phi \\ &= \int_0^{2\pi}\left||Q'(e^{i\theta})|e^{i\beta}+|P'(e^{i\theta})|\right|^p d\beta.\end{aligned} \tag{4.4}$$

Integrating (4.2) both sides with respect to β from 0 to 2π and using (4.4), we get

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\beta}G(\theta)\right|^p d\theta d\beta &\leq \int_0^{2\pi}\int_0^{2\pi}\left||Q'(e^{i\theta})|+e^{i\beta}|P'(e^{i\theta})|\right|^p d\beta d\theta \\ &= \int_0^{2\pi}\int_0^{2\pi}\left|Q'(e^{i\theta})+e^{i\beta}P'(e^{i\theta})\right|^p d\beta d\theta \\ &= \int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^p d\theta d\beta.\end{aligned}$$

By using Lemma 6 this implies,

$$\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\beta}G(\theta)\right|^p d\theta d\beta \leq 2\pi n^p\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta. \tag{4.5}$$

Now for $|z|=1$, $0\leq t\leq 1$ and $\alpha\in\mathbb{C}$ with $|\alpha|\geq 1$ and using the fact that

$$|nP(z)-zP'(z)|=|Q'(z)|$$

for z with unit modulus, we have

$$\begin{aligned}|D_\alpha[P](e^{i\theta})|+nmt\left(\frac{|\alpha|-1}{1+k^\mu}\right) &= \left|nP(z)+(\alpha-z)P'(z)\right|+nmt\left(\frac{|\alpha|-1}{1+k^\mu}\right) \\ &\leq |\alpha||P'(z)|+|nP(z)-zP'(z)|+nmt\left(\frac{|\alpha|-1}{1+k^\mu}\right) \\ &= |\alpha||P'(e^{i\theta})|+|Q'(e^{i\theta})|+nmt\left(\frac{|\alpha|-1}{1+k^\mu}\right) \\ &= |\alpha|\left(|P'(e^{i\theta})|+\frac{mnt}{1+k^\mu}\right)+\left(|Q'(e^{i\theta})|-\frac{mnt}{1+k^\mu}\right).\end{aligned}$$

By integrating both sides with respect to θ from 0 to 2π, for each $p>0$, we get

$$\begin{aligned}&\int_0^{2\pi}\left||D_\alpha[P](e^{i\theta})|+nmt\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right|^p d\theta \\ &\leq \int_0^{2\pi}\left||\alpha|\left(|P'(e^{i\theta})|+\frac{mnt}{1+k^\mu}\right)+\left(|Q'(e^{i\theta})|-\frac{mnt}{1+k^\mu}\right)\right|^p d\theta.\end{aligned}$$

Multiply both sides by

$$\int_0^{2\pi}|k^\mu+e^{i\beta}|^p d\beta,$$

we obtain

$$\begin{aligned}&\int_0^{2\pi}|k^\mu+e^{i\beta}|^p d\beta\int_0^{2\pi}\left||D_\alpha[P](e^{i\theta})|+nmt\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right|^p d\theta \\ &\leq \int_0^{2\pi}\left||\alpha|\left(|P'(e^{i\theta})|+\frac{mnt}{1+k^\mu}\right)+\left(|Q'(e^{i\theta})|-\frac{mnt}{1+k^\mu}\right)\right|^p d\theta\int_0^{2\pi}|k^\mu+e^{i\beta}|^p d\beta.\end{aligned} \tag{4.6}$$

Further, since $k^\mu \geq 1$, $1 \leq \mu \leq n$, and if

$$a = \left|Q'(e^{i\theta})\right| - \frac{mnt}{1+k^\mu}, \quad b = \left|P'(e^{i\theta})\right| + \frac{mnt}{1+k^\mu}, \quad c = k^\mu, \quad x = |\alpha|,$$

then from (4.1) one can observe that $a \geq bc$. Using Lemma 5, we get for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,

$$\begin{aligned}&\left\{\left(|Q'(e^{i\theta})| - \frac{mnt}{1+k^\mu}\right) + |\alpha|\left(|P'(e^{i\theta})| + \frac{mnt}{1+k^\mu}\right)\right\}^p \int_0^{2\pi} |k^\mu + e^{i\beta}|^p d\beta \\ &\leq (|\alpha| + k^\mu)^p \int_0^{2\pi} \left|\left(|Q'(e^{i\theta})| - \frac{mnt}{1+k^\mu}\right) + e^{i\beta}\left(|P'(e^{i\theta})| + \frac{mnt}{1+k^\mu}\right)\right|^p d\beta.\end{aligned}$$

Again, integrating both sides with respect to θ from 0 to 2π, we obtain

$$\begin{aligned}&\int_0^{2\pi} \left|\left(|Q'(e^{i\theta})| - \frac{mnt}{1+k^\mu}\right) + |\alpha|\left(|P'(e^{i\theta})| + \frac{mnt}{1+k^\mu}\right)\right|^p d\theta \int_0^{2\pi} |k^\mu + e^{i\beta}|^p d\beta \\ &\qquad\leq (|\alpha| + k^\mu)^p \int_0^{2\pi}\int_0^{2\pi} \left|F(\theta) + e^{i\beta}G(\theta)\right|^p d\beta d\theta,\end{aligned}$$

where $F(\theta)$ and $G(\theta)$ are given by (4.3). Using this in inequality (4.6), we get

$$\begin{aligned}&\int_0^{2\pi} |k^\mu + e^{i\beta}| d\beta \int_0^{2\pi} \left||D_\alpha[P](e^{i\theta})| + nmt\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right|^p d\theta \\ &\qquad\leq (|\alpha| + k^\mu)^p \int_0^{2\pi}\int_0^{2\pi} |F(\theta) + e^{i\beta}G(\theta)|^p d\beta d\theta.\end{aligned} \tag{4.7}$$

By using (4.5) in (4.7), we obtain for each $p > 0$ and $|\alpha| \geq 1$

$$\int_0^{2\pi} |k^\mu + e^{i\beta}| d\beta \int_0^{2\pi} \left||D_\alpha[P](e^{i\theta})| + nmt\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right|^p d\theta \leq (|\alpha| + k^\mu)^p 2\pi n^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

Equivalently,

$$\begin{aligned}&\left(\frac{1}{2\pi}\int_0^{2\pi} \left||D_\alpha[P](e^{i\theta})| + nmt\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right|^p d\theta\right)^{1/p} \\ &\leq \frac{n(|\alpha| + k^\mu)}{\left(1/(2\pi)\int_0^{2\pi} |k^\mu + e^{i\beta}| d\beta\right)^{1/p}} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right)^{1/p},\end{aligned}$$

which immediately leads to (2.1) for $0 < p < \infty$ and the cases $p = 0$ and $p = \infty$ follow by respectively taking the limits $p \to 0^+$ and $p \to \infty$. This completes the proof of Theorem 1. $\square$

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