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DISTANCE-REGULAR GRAPH WITH INTERSECTION ARRAY $\{27, 20, 7; 1, 4, 21\}$ DOES NOT EXIST[1]

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Abstract: In the class of distance-regular graphs of diameter 3 there are 5 intersection arrays of graphs with at most 28 vertices and noninteger eigenvalue. These arrays are $\{18, 14, 5; 1, 2, 14\}$, $\{18, 15, 9; 1, 1, 10\}$, $\{21, 16, 10; 1, 2, 12\}$, $\{24, 21, 3; 1, 3, 18\}$, and $\{27, 20, 7; 1, 4, 21\}$. Automorphisms of graphs with intersection arrays $\{18, 15, 9; 1, 1, 10\}$ and $\{24, 21, 3; 1, 3, 18\}$ were found earlier by A.A. Makhnev and D.V. Paduchikh. In this paper, it is proved that a graph with the intersection array $\{27, 20, 7; 1, 4, 21\}$ does not exist.

Keywords: Distance-regular graph, Graph Γ with strongly regular graph Γ_3, Automorphism.

1. Introduction

We consider undirected graphs without loops and multiple edges. For given vertex a of a graph Γ, we denote by $\Gamma_i(a)$ the subgraph of Γ induced by the set of all vertices at distance i from a. The subgraph $[a] = \Gamma_1(a)$ is called the *neighbourhood of the vertex* a.

Let Γ be a graph of diameter d and $i \in \{1, 2, 3, \ldots, d\}$. The graph Γ_i have the same set of vertices, and vertices u and w are adjacent in Γ_i if $d_\Gamma(u, w) = i$.

If vertices u and w are at distance i in Γ, then denote by $b_i(u, w)$ (by $c_i(u, w)$) the number of vertices in the intersection of $\Gamma_{i+1}(u)$ $(\Gamma_{i-1}(u))$ with $[w]$. A graph Γ of diameter d is called a *distance-regular graph with intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if the values $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of the vertices u and w at distance i in Γ for any $i = 0, \ldots, d$. For such graph and for $0 \le i, j, h \le d$, the number $p^h_{ij} = \begin{bmatrix} uv \\ ij \end{bmatrix}$ is independent of u and v for all vertices $u, v \in \Gamma$ with $d(u, v) = h$. The constants p^h_{ij} are called the intersection numbers of Γ [1].

The incidence system with set of points P and set of lines $\mathcal{L}$ is called *α-partial geometry of order* (s, t) (and is denoted by $pG_\alpha(s, t)$) if every line contains exactly $s + 1$ points, every point lies

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exactly on $t+1$ lines, any two points lie on at most one line, and, for any antiflag $(a,l) \in (P,\mathcal{L})$, there is exactly α lines passing through a and intersecting l. If $\alpha = t+1$, then the geometry is called a dual 2-scheme; and if $\alpha = t$, then the geometry is called a net.

The *point graph* of a geometry of points and lines is a graph whose vertices are points of the geometry, and two different vertices are adjacent if they lie on a common line. It is easy to understand that the point graph of a partial geometry $pG_\alpha(s,t)$ is strongly regular with parameters

$$v = (s+1)(1+st/\alpha), \quad k = s(t+1), \quad \lambda = (s-1)+(\alpha-1)t, \quad \mu = \alpha(t+1).$$

A strongly regular graph having these parameters for some positive integers α, s, t is called a *pseudogeometric graph* for $pG_\alpha(s,t)$.

In the class of distance-regular graphs Γ of diameter 3, there are 5 hypothetical graphs with at most 28 vertices and non-integer eigenvalues. They have intersection arrays $\{18,14,5;1,2,14\}$, $\{18,15,9;1,1,10\}$, $\{21,16,10;1,2,12\}$, $\{24,21,3;1,3.18\}$, and $\{27,20,7;1,4,21\}$. Earlier, automorphisms of graphs with intersection arrays $\{18,15,9;1,1,10\}$ and $\{24,21,3;1,3,18\}$ were found by A.A. Makhnev and D.V. Paduchikh [4], [5].

In this paper, we study the properties of a hypothetical distance-regular graph with intersection array $\{27,20,7;1,4,21\}$ and prove the following theorem.

Theorem 1. *A distance-regular graph with intersection array* $\{27,20,7;1,4,21\}$ *does not exist.*

2. Preliminary results

In the proof of Theorem 1, we use triple intersection numbers [2].

Let Γ be a distance-regular graph of diameter d. If u_1, u_2, and u_3 are vertices of Γ and r_1, r_2, and r_3 are non-negative integers not greater than d, then $\left\{\begin{matrix} u_1u_2u_3 \\ r_1r_2r_3 \end{matrix}\right\}$ is the set of vertices $w \in \Gamma$ such that

$$d(w,u_i) = r_i, \qquad \begin{bmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{bmatrix} = \left| \left\{\begin{matrix} u_1u_2u_3 \\ r_1r_2r_3 \end{matrix}\right\} \right|.$$

The numbers $\begin{bmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{bmatrix}$ are called triple intersection numbers. For a fixed triple of vertices u_1, u_2, u_3, we will write $[r_1r_2r_3]$ instead of $\begin{bmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{bmatrix}$. Unfortunately, there are no general formulas for the numbers $[r_1r_2r_3]$. However, a method for calculating some numbers $[r_1r_2r_3]$ was suggested in [2].

Assume that u, v, and w are vertices of the graph Γ, $W = d(u,v)$, $U = d(v,w)$, and $V = d(u,w)$. Since there is exactly one vertex $x = u$ such that $d(x,u) = 0$, the number $[0jh]$ is either 0 or 1. Hence, $[0jh] = \delta_{jW}\delta_{hV}$. Similarly, $[i0h] = \delta_{iW}\delta_{hU}$ and $[ij0] = \delta_{iU}\delta_{jV}$.

Another set of equations can be obtained by fixing the distance between two vertices from $\{u,v,w\}$ and counting the number of vertices located at all possible distances from the third:

$$\sum_{l=1}^{d}[ljh] = p^U_{jh} - [0jh], \quad \sum_{l=1}^{d}[ilh] = p^V_{ih} - [i0h], \quad \sum_{l=1}^{d}[ijl] = p^W_{ij} - [ij0]. \tag{2.1}$$

At the same time, some triplets disappear. For $|i-j| > W$ or $i+j < W$, we have $p^W_{ij} = 0$; therefore, $[ijh] = 0$ for all $h \in \{0,\ldots,d\}$.

Let

$$S_{ijh}(u,v,w) = \sum_{r,s,t=0}^{d} Q_{ri}Q_{sj}Q_{th} \begin{bmatrix} uvw \\ rst \end{bmatrix}.$$

If Krein's parameter $q_{ij}^h = 0$, then $S_{ijh}(u, v, w) = 0$.

We fix vertices u, v, and w of a distance-regular graph Γ of diameter 3 and put

$$\{ijh\} = \left\{ \begin{matrix} uvw \\ ijh \end{matrix} \right\}, \quad [ijh] = \begin{bmatrix} uvw \\ ijh \end{bmatrix}, \quad [ijh]' = \begin{bmatrix} uwv \\ ihj \end{bmatrix}, \quad [ijh]^* = \begin{bmatrix} vuw \\ jih \end{bmatrix}, \quad [ijh]^\sim = \begin{bmatrix} wvu \\ hji \end{bmatrix}.$$

In the cases $d(u, v) = d(u, w) = d(v, w) = 2$ or $d(u, v) = d(u, w) = d(v, w) = 3$, the calculation of the numbers

$$[ijh]' = \begin{bmatrix} uwv \\ ihj \end{bmatrix}, \quad [ijh]^* = \begin{bmatrix} vuw \\ jih \end{bmatrix}, \quad [ijh]^\sim = \begin{bmatrix} wvu \\ hji \end{bmatrix}$$

(symmetrizing an array of triple intersection numbers) can give new relations for the prove of the nonexistence of the graph.

3. Proof of Theorem 1

In this section, we prove Theorem 1.

Let Γ be a distance-regular graph with intersection array $\{27, 20, 7; 1, 4, 21\}$. Then Γ has $1 + 27 + 135 + 45 = 208$ vertices, the spectrum $27^1, (2 + \sqrt{13})^{45}, -1^{117}, (5 - 2\sqrt{13})^{45}$, and the dual matrix Q of eigenvalues

$$\begin{pmatrix} 1 & 45 & 117 & 45 \\ 1 & \frac{10}{3}\sqrt{13} + \frac{5}{3} & -13/3 & -\frac{10}{3}\sqrt{13} + \frac{5}{3} \\ 1 & -\frac{2}{3}\sqrt{13} + \frac{5}{3} & -13/3 & \frac{2}{3}\sqrt{13} + \frac{5}{3} \\ 1 & -7 & 13 & -7 \end{pmatrix}.$$

By [3, Lemma 3], the complement of Γ_3 is a pseudo-geometric graph for $pG_{21}(27, 5)$.

Lemma 1. *The intersection numbers of the graph* Γ *are*:

(1) $p_{11}^1 = 6$, $p_{21}^1 = 20$, $p_{32}^1 = 35$, $p_{22}^1 = 80$, $p_{33}^1 = 10$;

(2) $p_{11}^2 = 4$, $p_{12}^2 = 16$, $p_{13}^2 = 7$, $p_{22}^2 = 90$, $p_{23}^2 = 28$, $p_{33}^2 = 10$;

(3) $p_{12}^3 = 21$, $p_{13}^3 = 6$, $p_{22}^3 = 84$, $p_{23}^3 = 30$, $p_{33}^3 = 8$.

P r o o f. The lemma is proved by direct calculations. □

We fix vertices u, v, and w of the graph Γ and put

$$\{ijh\} = \left\{ \begin{matrix} uvw \\ ijh \end{matrix} \right\}, \quad [ijh] = \begin{bmatrix} uvw \\ ijh \end{bmatrix}.$$

Let $\Delta = \Gamma_2(u)$ and $\Lambda = \Delta_2$. Then Λ is a regular graph of degree 90 on 135 vertices.

Lemma 2. *Let* $d(u, v) = d(u, w) = 2$ *and* $d(v, w) = 1$. *Then the triple intersection numbers are*:

(1) $[111] = r_4$, $[112] = [121] = -r_4 + 4$, $[122] = -r_1 + r_3 + r_4 + 5$; $[123] = [132] = r_1 - r_3 + 7$, $[133] = -r_1 + r_3$;

(2) $[211] = -r_2 - r_4 + 6$, $[212] = [221] = r_2 + r_4 + 9$, $[222] = r_1 - r_2 - r_4 + 53$, $[223] = [232] = -r_1 + 28$, $[233] = r_1$;

(3) $[311] = r_2$, $[312] = [321] = -r_2 + 7$, $[322] = r_2 - r_3 + 21$, $[323] = [332] = r_3$, $[333] = -r_3 + 10$,

where $r_1, r_3 \in \{0, 1, \ldots, 10\}$, $r_2 \in \{0, 1, \ldots, 6\}$, *and* $r_4 \in \{0, 1, \ldots, 4\}$.

P r o o f. Let $[111] = r_4$. Then $[113] = 0$ and $[111] + [112] = c_2 = 4$; thus, $[112] = -r_4 + 4$. Similarly, $[121] = -r_4 + 4$.

Let $[311] = r_2$. Then $[313] = 0$ and $[311] + [312] = p^2_{13} = 7$; thus, $[312] = -r_2 + 7$.

Using formulas (2.1), we obtain all the equalities. □

By Lemma 2, we have $43 \le [222] = r_1 - r_2 - r_4 + 53 \le 63$. Since $\{v, w\} \cup \Lambda(v) \cup \Lambda(w)$ contains $182 - [222]$ vertices, we have $182 - [222] \le 135$; hence, $47 \le [222] \le 63$ and $-r_1 + r_2 + r_4 \le 6$.

Lemma 3. *Let* $d(u, v) = d(u, w) = 2$ *and* $d(v, w) = 3$. *Then the triple intersection numbers are*:

(1) $[113] = r_5 + r_6 + r_7 + r_8 - r_9 - 26$, $[121] = -r_5 - r_6 - r_7 - r_8 + r_{10} + 30$, $[122] = r_5 + r_6 + r_7 + r_8 - r_9 - r_{10} - 14$, $[123] = r_9$, $[131] = r_5 + r_6 + r_7 + r_8 - r_{10} - 26$, $[132] = r_{10}$, $[133] = -r_5 - r_6 - r_7 - r_8 + 33$;

(2) $[212] = r_5 + r_7 + r_8 - r_9 - 9$, $[213] = -r_5 - r_7 - r_8 + r_9 + 25$, $[221] = r_5 + r_6 + r_8 - r_{10} - 9$, $[222] = -r_5 - r_6 - r_7 - 2r_8 + r_9 + r_{10} + 97$, $[223] = r_7 + r_8 - r_9 + 2$, $[231] = -r_5 - r_6 - r_8 + r_{10} + 25$, $[232] = r_6 + r_8 - r_{10} + 2$, $[233] = r_5$;

(3) $[312] = r_6$, $[313] = -r_6 + 7$, $[321] = r_7$, $[322] = r_8$, $[323] = -r_7 - r_8 + 28$, $[331] = -r_7 + 7$, $[332] = -r_6 - r_8 + 28$, $[333] = r_6 + r_7 + r_8 - 25$,

where $r_5 \in \{0, 1, \ldots, 8\}$, $r_6, r_7 \in \{1, 2, \ldots, 7\}$, $r_8 \in \{11, 12, \ldots, 27\}$, *and* $r_9, r_{10} \in \{0, 1, \ldots, 7\}$.

P r o o f. Using (2.1), we arrive at relations (1)–(3) of the Lemma 3. □

By Lemma 3, we have $47 \le [222] = -r_5 - r_6 - r_7 - 2r_8 + r_9 + r_{10} + 97 \le 90$.

Consider the appropriate symmetrization. Let $d(u, v) = d(u, w) = 2$ and $d(v, w) = 3$. Then the following equalities are true: $[123] = r_9 = [132]' = r'_{10}$, $[233] = r_5 = r'_5$, $r_6 = [312] = [321]' = r'_7$, $[322] = r_8 = r'_8$. Further, $r_7 + r_8 - r_9 + 2 = [223] = [232]' = r'_6 + r'_8 - r'_{10} + 2$.

Lemma 4. *Let* $d(u, v) = d(u, w) = d(v, w) = 2$. *Then the triple intersection numbers are*:

(1) $[111] = r_9 + r_{10} - r_{11} - 24$, $[112] = [121] = r_{15}$, $[113] = [131] = r_{11}$, $[122] = -r_{10} - r_{15} + 16$, $[123] = [132] = r_{10}$, $[133] = 7 - r_{11} - r_{10}$;

(2) $[211] = r_{15}$, $[212] = [221] = -r_{10} - r_{15} + 16$, $[213] = r_{10}$, $[222] = 2r_9 + 2r_{10} - 11$, $[223] = [232] = 28 - r_9 - r_{10}$, $[231] = r_{10}$, $[233] = r_9$;

(3) $[311] = r_{11}$, $[312] = [321] = r_{10}$, $[313] = [331] = 7 - r_{11} - r_{10}$, $[322] = -r_{10} - r_{15} + 16$, $[323] = [332] = r_9$, $[333] = r_{11} + r_{10} + 3$,

where $r_{11} + 24 \le r_9 + r_{10} \le 28$, $r_{11} + r_{10} \le 7$, $r_{10} + r_{15} \le 16$, *and* $r_{12} \le 22$.

P r o o f. Using formulas (2.1), we get the equalities:

$[111] = -r_{11} - r_{12} + 4$, $[112] = r_{15}$, $[113] = r_{11}$, $[121] = r_{10} + r_{12} + r_{15} + r_{16} - 28$, $[122] = -r_{10} - r_{15} + 16$, $[123] = -r_{12} - r_{16} + 28$, $[131] = -r_{10} + r_{11} - r_{12} - r_{16} + 28$, $[132] = r_{10}$, $[133] = -r_{11} + r_{12} + r_{16} - 21$;

$[211] = r_{12} + r_{13} + r_{14} + r_{15} - 28$, $[212] = -r_{13} - r_{15} + 16$, $[213] = -r_{12} - r_{14} + 28$, $[221] = -r_9 - r_{10} - r_{12} - r_{13} - r_{15} + 44$, $[222] = r_9 + r_{10} + r_{13} + r_{14} + 45$, $[223] = r_{12}$, $[231] = r_9 + r_{10} - r_{14}$, $[232] = -r_9 - r_{10} + 28$, $[233] = r_{14}$;

$[311] = r_{11} - r_{12} - r_{13} - r_{14} + 28$, $[312] = r_{13}$, $[313] = -r_{11} + r_{12} + r_{14} - 21$, $[321] = r_9 + r_{13} - r_{16}$, $[322] = -r_9 - r_{13} + 28$, $[323] = r_{16}$, $[331] = -r_9 - r_{11} + r_{12} + r_{14} + r_{16} - 21$, $[332] = r_9$, $[333] = r_{11} - r_{12} - r_{14} - r_{16} + 31$.

Now consider symmetrization. The following equalities are true:

$[112] = r_{15} = r^*_{15}$, $[113] = r_{11} = r^*_{11}$, $[223] = r_{12} = r^*_{12}$, $[233] = r_{14} = r'_{14}$, $[323] = r_{16} = r^{\sim}_{16}$, $[332] = r_9 = r^*_9$, $r_{10} = [132] = [312]^* = r^*_{13}$.

Further, $r_9 + r_{10} + r_{13} + r_{14} + 45 = [222] = [222]^* = r^*_9 + r^*_{10} + r^*_{13} + r^*_{14} + 45 = r_9 + r_{13} + r_{10} + r^*_{14} + 45$; therefore, $[233] = r_{14} = r^*_{14} = [323] = r_{16}$.

We have $[111] = -r_{11} - r_{12} + 4$; hence $r_{11} + r_{12} = r'_{11} + r'_{12} = r^{\sim}_{11} + r^{\sim}_{12}$. Similarly, $[122] = -r_{10} - r_{15} + 16$; therefore, $r_{10} + r_{15} = r'_{10} + r'_{15}$, $[123] = -r_{12} - r_{16} + 28$, and $r_{12} + r_{16} = r'_{12} + r'_{16}$.

Finally, $[133] = -r_{11} + r_{12} + r_{16} - 21 = -r'_{11} + r'_{12} + r'_{16} - 21$; thus, $r_{11} = r'_{11}$, $r_{12} = r'_{12}$, and $r_{16} = r'_{16}$. Hence $r_{11} = [113] = [131] = -r_{10} + r_{11} - r_{12} - r_{16} + 28$ and $r_{10} + r_{12} + r_{16} = 28$. Further, $r_{12} = [223] = [232] = -r_9 - r_{10} + 28$, $r_{12} + r_9 + r_{10} = 28$, and $r_9 = r_{16}$.

The equalities $[113] = [131] = r_{11}$, $r_{11} = r^*_{11}$, and $[311] = r_{11} + r_{10} - r_{13}$ imply that $r_{10} = r_{13}$. Hence, we obtain the equalities from the conclusion of the lemma. □

By Lemma 4, we have $r_{11} + 24 \le r_9 + r_{10} \le 28$; hence $45 \le [222] = 2r_9 + 2r_{10} - 11 \le 56 - 11 = 45$. Thus, Λ is an edge-regular graph with parameters (135,90,45).

In view of Lemmas 2 and 3, the following inequalities hold for the number of edges e between $\Lambda(w)$ and $\Lambda - (\{w\} \cup \Lambda(w))$:

$$2068 = 47 \cdot 16 + 47 \cdot 28 \le e = 63 \cdot 16 + 90 \cdot 28 \le 3528.$$

Contrariwise, we have $e = 90 \cdot 89 - \sum_i [222]^i$; therefore, $2068 \le e = 90 \cdot 89 - \sum_i [222]^i \le 3528$, $4482 \le \sum_i [222]^i \le 5942$, and $49.8 \le \sum_i [222]^i/90 \le 66.03$.

The resulting contradiction completes the proof of Theorem 1.

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