URAL MATHEMATICAL JOURNAL, Vol. 6, No. 2, 2020, pp. 68–75
DOI: 10.15826/umj.2020.2.007

POSITIONAL IMPULSE AND DISCONTINUOUS CONTROLS FOR DIFFERENTIAL INCLUSION[1]

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Abstract: Nonlinear control systems presented in the form of differential inclusions with impulse or discontinuous positional controls are investigated. The formalization of the impulse-sliding regime is carried out. In terms of the jump function of the impulse control, the differential inclusion is written for the ideal impulse-sliding regime. The method of equivalent control for differential inclusion with discontinuous positional controls is used to solve the question of the existence of a discontinuous system for which the ideal impulse-sliding regime is the usual sliding regime. The possibility of the combined use of the impulse-sliding and sliding regimes as control actions in those situations when there are not enough control resources for the latter is discussed.

Keywords: Impulse position control, Discontinuous position control, Differential inclusion, Impulse-sliding regime, Sliding regime.

Introduction

Impulse-sliding regimes for differential equations arise in problems of impulse optimal control when the system is affected by perturbations. The formalization of impulse-sliding regimes for differential equations was done in [8]. When describing the motions of systems subject to perturbations, the right-hand side can also be not uniquely defined. Therefore, under the action of perturbations on the system, it is natural to describe the motion of the system using differential inclusions and impulse control (see [3, 5]). In [5], the formalization of the impulse-sliding regime for systems of this type is given. In this paper, we investigate the properties of impulse-sliding regimes. In addition, an equivalent control method is applied to systems of this type [6, 7]. We also discuss the issue of the combined use of impulse-sliding and sliding regimes.

[1]This work was supported by Russian Foundation for Basic Research (project No. 19-01-00371).

1. Description of impulse-sliding regime

We study a dynamical system

$$\dot{x}(t) \in F(t, x(t)) + B(t, x(t))u, \quad t \in I = [t_0, \vartheta], \tag{1.1}$$

with the initial condition $x(t_0) = x_0$. Here, $F(\cdot, \cdot)$ is a multivalued function with convex compact values in $\mathbb{R}^n$, the matrix function $B(\cdot, \cdot)$ of dimension $n \times m$ is continuous in the set of variables in the considered domain, and $u = (u_1, \dots, u_m)^T$ is a function that describes some control action on the system.

For $F(\cdot, \cdot)$, we write the following basic assumptions.

(B1) For almost all $t \in \mathbb{R}$, the mapping $F(t, x)$ is upper semicontinuous in x. This means that, for arbitrary $\varepsilon > 0$, there exists $\delta = \delta(t, x, \varepsilon) > 0$ such that $F(t, x') \subset F^\varepsilon(t, x)$ for all $x' \in W_\delta(x)$, where $F^\varepsilon(t, x)$ is the ε-neighborhood of the set $F(t, x)$ and $W_\delta(x)$ is the δ-neighborhood of the point x.

(B2) For any x, the multivalued mapping $t \to F(t, x)$ has a measurable selector, i.e., there is a measurable function $f(t) \in F(t, x)$ for almost all $t \in I$.

(B3) The multivalued mapping $F(t, x)$ satisfies the condition of sublinear growth: the inequality $\|w\| \le l(1 + \|x\|)$ holds for any $(t, x) \in \mathbb{R}^{n+1}$ and $w \in F(t, x)$.

Under these assumptions, the differential inclusion

$$\dot{x} \in F(t, x) \tag{1.2}$$

has a solution $x(t)$, which can be extended to the entire number axis $\mathbb{R}^1$ (see [1]). It is assumed that the matrix $B(t, x)$ satisfies the Frobenius condition

$$\sum_{\nu=1}^{n} \frac{\partial b_{ij}(t, x)}{\partial x_\nu} b_{\nu l}(t, x) = \sum_{\nu=1}^{n} \frac{\partial b_{il}(t, x)}{\partial x_\nu} b_{\nu j}(t, x),$$

which will provide the unique reaction of system (1.1) on the control u in the case when u is an impulse action on this system (see [9]). By impulse positional control, we mean some abstract operator $(t, x) \longrightarrow U(t, x)$ that maps the space of variables t, x into the space m of vector distributions [8] according to the rule

$$U(t, x) = r(t, x(t))\, \delta_t,$$

where $r(t, x)$ is a vector function with values in $\mathbb{R}^m$ and δ_t is the Dirac impulse function concentrated at the point t. "Running impulse" $r(t, x(t))\, \delta_t$ as a generalized function does not make sense. An impulse control of this type is understood as a discrete implementation of a "running impulse" in the form of a sequence of correcting impulses concentrated at the points of some partition $h\colon t_0 < t_1 < \ldots < t_N = \theta$ of the segment I. The result of such a sequential correction is a discontinuous curve $x^h(\cdot)$, here called "Euler's broken line" or impulse-sliding regime. Let us describe more precisely the impulse-sliding regime. Let us define a network of "Euler's broken lines" $x^h(\cdot)$ corresponding to the set of partitions directed in magnitude

$$d(h) = \max(t_{k+1} - t_k), \quad h\colon t_0 < t_1 < \cdots < t_p = \vartheta$$

of the segment I. For this purpose, we first define the jump function by means of the equations

$$S(t, x, r(t, x)) = z(1) - z(0), \quad \dot{z}(\xi) = B(t, z(\xi))r(t, x), \quad z(0) = x.$$

Here, we take into account that, in fact, the dependence $z = z(\xi, t, x, r(t, x))$ takes place. Note also that the jump function is a vector function $S = (S^1, \dots, S^n)$.

The jumps of the "Euler's broken lines" at the points of the partitions h of the segment I are determined by the equations

$$S\big(t_i, x^h(t_i), r(t_i, x^h(t_i))\big) = z(1) - z(0), \quad \dot z(\xi) = B(t_{t_i}, z(\xi)) r(t_i, x^h(t_i))$$

with initial conditions $z(0) = x^h(t_i)$. The "Euler's broken line" $x^h(\cdot)$ is constructed as a function of bounded variation, which coincides with the solution of the differential inclusion (1.2) on each interval $(t_i, t_{i+1}]$ with initial conditions

$$x(t_i) = x^h(t_i) + S(t_i, x^h(t_i), r(t_i, x^h(t_i))), \quad x(t_0) = x_0, \quad i = 0, \dots, p-1.$$

We will assume that the following equality is valid for all admissible t and x:

$$r\big(t, x + S(t, x, r(t, x))\big) = 0, \tag{1.3}$$

which means that, after an impulsive action on the system at time t, the phase point $x(t)$ will be on the manifold (target set)

$$\Phi = \big\{(t, x) : r(t, x) = 0\big\}.$$

Note that the definition of the jump function and condition (1.3) imply the relation

$$S = 0 \Leftrightarrow r = 0,$$

which is further used without reservation. It is also assumed that the functions $S(t, x, r)$ and $r(t, x)$ are continuously differentiable.

Under some additional assumptions, the sequence of "Euler's broken lines" has a convergent subsequence, the limit of which will be on the surface Φ. It is called the ideal sliding mode. The purpose of the impulse control is to keep the phase point on the manifold Φ. In [5], the differential inclusion of an ideal pulse-sliding mode is obtained in the form

$$\begin{gathered} \dot x \in \frac{\partial S(t, x, r(t, x))}{\partial t} + \frac{\partial S(t, x, r(t, x))}{\partial r} \frac{\partial r(t, x)}{\partial t} \\ + \left(E + \frac{\partial S(t, x, r(t, x))}{\partial x} + \frac{\partial S(t, x, r(t, x))}{\partial r} \frac{\partial r(t, x)}{\partial x}\right) F(t, x), \\ x(t_0 + 0) = x(t_0) + S\big(t_0, x(t_0), r(t_0, x(t_0))\big). \end{gathered} \tag{1.4}$$

Controls of the positional-impulse type were used to solve various problems of game theory and control, in particular, when constructing positional impulse controls in degenerate linear-quadratic optimal control problems. Note also that "Euler's broken lines" for the same positional impulse control may differ in the way of constructing jumps. One of them is listed above. Another one can be found in [4]. Accordingly, the equations of ideal sliding-impulse modes will differ.

In literature, you can find other methods for constructing jumps of impulse control, where the term "impulse-sliding regime" is used in a broader sense. As for processes of "sliding" type, to a greater extent, they are an attribute of controlled systems with discontinuous positional controls (feedbacks) and the theory of discontinuous systems in general, where such movements are called sliding regime. In this paper, a differential inclusion with discontinuous positional controls with constraints on control resources is constructed for which the ideal "impulse-sliding regime" of inclusion (1.1) is the usual sliding regime in the sense of the theory of discontinuous systems. It is the main mode of functioning of a discontinuous controlled system and allows solving such problems as stabilization, complete controllability, and tracking (movement along a predetermined trajectory). A huge number of works are devoted to these questions.

2. Multivalued equivalent controls

We will consider a controlled differential inclusion

$$\dot{x} \in F(t,x) + \tilde{u}, \tag{2.1}$$

where $\tilde{u} = (\tilde{u}_i, \dots, \tilde{u}_n)$, $\tilde{u}_i(t,x) = -H_i(t,x)\operatorname{sgn} S^i$, $H_i(t,x) \geq 0$ are some continuous functions, and S^i is the ith component of the jump function, $i = 1, \dots, n$.

If $S^i = 0$, then denote by $\tilde{U}_i(t,x)$ the segment $[-H_i(t,x), H_i(t,x)]$ and if $S^i \neq 0$, then $\tilde{U}_i(t,x) = \tilde{u}_i$. Let $\tilde{U}(t,x) = \tilde{U}_1(t,x) \times \dots \times \tilde{U}_n(t,x)$. Under a solution to problem (2.1), we mean a solution to the differential inclusion

$$\dot{x} \in F(t,x) + \tilde{U}(t,x), \tag{2.2}$$

those, absolutely continuous function satisfying (2.2) almost everywhere on the considered segment I.

We will represent inclusion (2.2) in the form of a controlled system

$$\begin{cases} \dot{x} \in F(t,x) + \tilde{u}, \\ \tilde{u} \in \tilde{U}(t,x). \end{cases} \tag{2.3}$$

A solution to problem (2.3), defined on the segment I, is a pair $(x(t), \tilde{u}(t))$ consisting of an absolutely continuous function $x(t)$ (trajectory) and a measurable function $\tilde{u}(t)$ (control) satisfying inclusions (2.3) almost everywhere on I.

Lemma 1. *Let the multivalued mapping $F(t,x)$ satisfy conditions* (B1)–(B3)*, and let the functions $r(t,x)$ and $H_i(t,x)$ be continuous. Then, for any initial conditions $x(t_0) = x_0$, there exists a solution to inclusion* (2.2) *and, for any solution $x(t)$ to inclusion* (2.2)*, there exists a measurable function $\tilde{u}(t)$ such that the pair $(x(t), \tilde{u}(t))$ is a solution to problem* (2.3)*.*

P r o o f. It is easy to check that the multivalued mapping $\tilde{U}(t,x)$ is upper semicontinuous and locally bounded. Then the right-hand side of inclusion (2.2) is upper semicontinuous, as the algebraic sum of two upper semicontinuous multivalued mappings. In addition, it is easy to check that the right-hand side of inclusion (2.2) possesses property (B1) and is integrally bounded. Then there exist a solution to inclusion (2.2) (see [1]).

Let $x(t)$ be a solution to inclusion (2.2). Then

$$\dot{x}(t) \in F(t, x(t)) + \tilde{U}(t, x(t))$$

for almost all $t \in I$ and Filippov's implicit function lemma (see [1, Theorem 1.5.15]) implies the existence of a measurable function $\tilde{u}(t) \in \tilde{U}(t, x(t))$ such that $\dot{x}(t) \in F(t, x(t)) + \tilde{u}(t)$ for almost all $t \in I$. Then the pair $(x(t), \tilde{u}(t))$ is a solution to the controlled system (2.3) and the lemma is proved.

We consider sliding regimes to inclusion (2.1) in relation to the surface

$$\Gamma = \big\{(t,x) : S(t,x,r(t,x)) = 0\big\}$$

or, which is equivalent, to the surface Φ.

A solution $x(t)$ to inclusion (2.1) satisfying the condition $(t, x(t)) \in \Phi$ will be called the sliding regime. One of the main ways to obtain equations of sliding regimes of discontinuous control systems is the method of equivalent controls (see [2]). The controls should be chosen so that the velocity vector $\dot{x}(t)$ at the points $(t, x(t))$ of the discontinuity surface lies in the tangent plane to this surface. Such controls $\tilde{u}^{eq}$ are called equivalent if they satisfy the given constraints. In the problem under consideration, these constraints have the form $\tilde{u}^{eq} \in \tilde{U}(t, x)$.

We denote by S_t the partial derivative of the mapping $t \to S(t, x, r(t, x))$ with respect to the variable t and by S_x the Jacobi matrix of the mapping $x \to S(t, x, r(t, x))$ with respect to the variable x. Let

$$\tilde{U}^{eq}(t, x) = S_t + S_x F(t, x).$$

Define a multivalued analogue of equivalent control for differential inclusion (2.1) in the form

$$\tilde{U}^{*eq}(t, x) = \tilde{U}^{eq}(t, x) \cap \tilde{U}(t, x).$$

Theorem 1. *Let $x(t)$ be a sliding regime of inclusion (2.1) and*

$$S_x = -E_n \tag{2.4}$$

for any $(t, x) \in \Gamma$, where E_n is an $n \times n$ identity matrix. Then

$$\tilde{U}^{*eq}\big(t, x(t)\big) \neq \emptyset \tag{2.5}$$

for almost all t and the function $x(t)$ is the trajectory of the controlled system

$$\begin{cases} \dot{x} \in F(t, x) + \tilde{u}, \\ \tilde{u} \in \tilde{U}^{*eq}(t, x). \end{cases} \tag{2.6}$$

P r o o f. Since the function $x(t)$ is a solution to inclusion (2.2), according to Lemma 1, there is a measurable function $\tilde{u}(t) \in \tilde{U}(t, x(t))$ such that the inclusion $\dot{x}(t) \in F(t, x(t)) + \tilde{u}(t)$ holds for almost all t. Since $(t, x(t))$ is a sliding regime, we have $(t, x(t)) \in \Gamma$ and, from the condition $S_x(t, x(t), r(t, x(t)) = -E_n$, we get

$$0 \in S_t(t, x(t), r)t, x(t)) + S_x(t, x(t), r(t, x(t))F(t, x(t)) - \tilde{u}(t). \tag{2.7}$$

It follows from (2.7) that $\tilde{u}(t) \in \tilde{U}^{eq}(t, x(t))$ for almost all t. Hence, $\tilde{u}(t) \in \tilde{U}^{*eq}(t, x(t))$ for almost all $t \in I$, condition (2.5) holds, and the pair $(x(t), \tilde{u}(t))$ is a solution to the controlled system (2.6). The theorem is proved.

Theorem 1 gives a necessary condition for the existence of a sliding mode for a differential inclusion (2.1).

We investigate sufficient conditions for the existence of sliding regimes S using the function

$$V(t, x) = \frac{1}{2}\langle S, S\rangle,$$

where $\langle \cdot, \cdot \rangle$ stand for the scalar product.

For any $\delta > 0$, we use the notation

$$W_\delta(t, x) = \big\{(t', x')\colon \|x' - x\| < \delta,\ |t - t'| < \delta\big\}.$$

Theorem 2. *Let condition (2.4) hold and, for every point $(t, x) \in \Gamma$, there exist $\varepsilon > 0$ and a neighborhood $W_\delta(t, x)$ such that*

$$\max_{w \in F(t', x')} |S_t^i + w_i| < H_i(t, x) - \varepsilon \tag{2.8}$$

for all indices $i = 1, \ldots, n$ and all $(t', x') \in W_\delta(t, x)$.

Then the following statements are true.

(1) *For any solution to inclusion* (2.1) *with initial conditions* $(t_0, x_0) \in \Gamma$*, there holds* $(t, x(t)) \in \Gamma$ *for all points* $t \geq t_0$ *at which this solution exists.*

(2) *For any initial conditions* $(t_0, x_0) \in \Gamma$*, there is a sliding regime of inclusion* (2.1) *defined as a solution to inclusion* (2.2)*, and any solution* $x(t)$ *with initial condition* $(t_0, x_0) \in \Gamma$ *is a sliding regime if and only if it is a trajectory of the controlled system* (2.3) *with the same initial condition.*

Theorem 2 follows from statements 1 and 3 of Theorem 3 from [4] with the replacement of the function $\sigma(t, x)$ by the function $S(t, x, r(t, x))$ and the use of Lemma 1, condition (2.4), and inequality (2.8).

3. Impulse-sliding and sliding regimes of differential inclusions

It follows immediately from the definitions that the differential inclusion (1.4) of the ideal impulse-sliding regime is written as

$$\dot{x} \in F(t, x) + \tilde{U}^{eq}(t, x). \tag{3.1}$$

Then the results of the previous section can be applied to it.

Theorem 3. *Let conditions* (B1)–(B3) *be satisfied, and let* (2.4)*,* (1.3)*, and inequality*

$$\|S(\tau, y, r(\tau, y)) - S(t, x, r(t, x))\| \leq L(|\tau - t| + \|y - x\|), \tag{3.2}$$

also hold for all admissible t*,* τ*,* x*, and* y*. Then:*

(1) *For inclusion* (1.1)*, any sequence of "Euler's broken lines" has a subsequence uniformly converging to the ideal impulse-sliding mode, any ideal impulse-sliding regime* $\tilde{x}(t)$ *satisfies the condition* $S(t, \tilde{x}, r(t, \tilde{x})) = 0$ *and is a solution to the discontinuous system* (2.2) *and the trajectory of the controlled system*

$$\begin{cases} \dot{x} \in F(t, x) + \tilde{u}, \\ \tilde{u} \in \tilde{U}^{eq}(t, x) \end{cases} \tag{3.3}$$

with the initial condition $\tilde{x}(t_0 + 0) = x_0 + S(t_0, x_0, r(t_0, x_0))$*.*

(2) *If, in addition, inequalities* (2.8) *hold, then any ideal impulse-sliding regime* $\tilde{x}(t)$ (1.1) *is a sliding regime* (2.2) *with discontinuous positional control* $\tilde{u}$*.*

Note that the controlled system (3.3) and the differential inclusion (3.1) are equivalent in the sense that any trajectory from the pair $(x(t), \tilde{u}(t))$ is a solution to inclusion (3.1) and any solution to this inclusion is a trajectory of system (3.3).

Note also that the sliding mode in Theorem 3 is stable with respect to the target set Φ. If this is not the case (outside the scope of Theorem 2), then the usual sliding mode can be terminated and the solution can be kept on the target set using the impulse-sliding regime.

4. Example

Consider a controlled system

$$\begin{cases} \dot{x}_1(t) = -\mathrm{sgn}(x_2(t) - 1) + x_1(t)u_1, \\ \dot{x}_2(t) = -\mathrm{sgn}(x_1(t) - 1) + x_2(t)u_2. \end{cases} \tag{4.1}$$

It is required to organize a sliding mode on the set $x_1 \equiv 1$, $x_2 \equiv 1$. In this case, impulse control can be omitted. We put $u_1 \equiv u_1 \equiv 0$. Then the trajectory of the system (4.1) in the space x_1, x_2, t reaches the plane $x_1 = 1$ or $x_2 = 1$. After that, moving along this plane, it reaches the straight line $x_1 = x_2 = 1$ and will stay on this straight line in sliding regime.

If we consider the system

$$\begin{cases} \dot{x}_1(t) = \mathrm{sgn}(x_2(t) - 1) + x_1(t)u_1, \\ \dot{x}_2(t) = \mathrm{sgn}(x_1(t) - 1) + x_2(t)u_2, \end{cases} \tag{4.2}$$

then it is possible to provide sliding on the set $x_1 = x_2 = 1$ only with the help of the impulse-sliding regime.

The vector function $r = (r_1, r_2)^T$ is defined by the equalities

$$r_1(t, x) = -\ln x_1, \quad r_2(t, x) = -\ln x_2.$$

The control u has the form

$$U(t, x(t)) = r(t, x)\delta_t.$$

The problem for the impulse control u is to keep the phase point at the intersection of the straight lines $x_1 = 1$ and $x_2 = 1$, which are determined from the conditions $\ln x_1 = 0$ and $\ln x_2 = 0$. The jump function $S(t, x, r)$ has the form

$$S(t, x, r) = \begin{cases} x_1(e^{r_1} - 1), \\ x_2(e^{r_2} - 1). \end{cases}$$

The multivalued function $F(x)$ on the right-hand side of system (4.2) is defined as follows:

$$F_i = \begin{cases} 1, & x_i > 0, \\ -1, & x_i < 0, \qquad i = 1, 2. \\ [-1, 1], & x_i = 0, \end{cases}$$

This corresponds to the simplest convex extension of the right-hand side of the discontinuous equation (4.2) in Filippov's sense.

The fulfillment of conditions (1.3) and (2.4) for these functions $r(t, x)$ and $S(t, x, r(t, x))$ is verified directly.

The impulse-sliding regime is described by the equations $\dot{x}_1 = 0$ with the initial condition $x_1(0+) = 1$ and $\dot{x}_2 = 0$ with the initial condition $x_2(0+) = 1$. In order, in accordance with Theorem 3, to write a differential inclusion of the form (2.2), it is necessary to specify the coefficients H_1 and H_2 satisfying inequalities (2.8). It is easy to see that these can be any numbers exceeding one.

5. Conclusion

The impulse control that transfers the manipulator from a given position to its final position is constructed in the work. A computational experiment showing the efficiency of the proposed algorithm is presented. The proposed algorithm is simulated in the case when the ideal impulse is approximated by the usual bounded control.

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