ESTIMATES OF REACHABLE SETS OF CONTROL SYSTEMS WITH BILINEAR–QUADRATIC NONLINEARITIES

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Abstract: The problem of estimating reachable sets of nonlinear impulsive control systems with quadratic nonlinearity and with uncertainty in initial states and in the matrix of system is studied. The problem is studied under uncertainty conditions with set-membership description of uncertain variables, which are taken to be unknown but bounded with given bounds. We study the case when the system nonlinearity is generated by the combination of two types of functions in related differential equations, one of which is bilinear and the other one is quadratic. The problem may be reformulated as the problem of describing the motion of set-valued states in the state space under nonlinear dynamics with state velocities having bilinear-quadratic kind. Basing on the techniques of approximation of the generalized trajectory tubes by the solutions of control systems without measure terms and using the techniques of ellipsoidal calculus we present here a state estimation algorithms for the studied nonlinear impulsive control problem bilinear-quadratic type.

Key words: Nonlinear control systems, Impulsive control, Ellipsoidal calculus, Trajectory tubes, Estimation.

Introduction

The paper deals with the problem of parameter estimation for control problems and of the evaluation of related estimating sets describing uncertainty. We study the case when a probabilistic description of noise and errors is not available, but only a bound on them is known [1, 15, 19, 22, 23]. Such models may be found in many applied areas ranged from engineering problems in physics to economics as well as to biological and ecological modeling when it occurs that a stochastic nature of the errors is questionable because of limited data or because of nonlinearity of the model. Unlike the classical estimation approach, set-membership estimation is not concerned with minimizing any objective function and instead of finding a single optimal parameter vector, a set of feasible parameters vectors, consistent with the model structure, measurements and bounded uncertainty characterization, should usually be found.

The solution of many control and estimation problems under uncertainty involves constructing reachable sets and their analogs. For models with linear dynamics under such set-membership uncertainty there are several constructive approaches which allow finding effective estimates of reachable sets. We note here two of the most developed approaches to research in this area. The first one is based on ellipsoidal calculus [3, 15] and the second one uses the interval analysis [23]. Among other interesting approaches to solving the problems of estimation of the dynamics of the control systems we also note results [2, 6, 13].

Many applied problems are mostly nonlinear in their parameters and the set of feasible system states is usually non-convex or even non-connected. The key issue in nonlinear set-membership estimation is to find suitable techniques, which produce related bounds for the set of unknown system states without being too computationally demanding. Some approaches to the nonlinear set-membership estimation problems and discrete approximation techniques for differential inclusions through a set-valued analogy of well-known Euler's method were developed in [14, 16].

In this paper the modified state estimation approaches which use the special quadratic structure of nonlinearity of studied control system and use also the advantages of ellipsoidal calculus [3, 15] are presented. We study here more complicated case than in [12] and we assume now that the system nonlinearity is generated by the combination of two types of functions in related differential equations, one of which is bilinear and the other one is quadratic. The problem may be reformulated as the problem of describing the motion of set-valued states in the state space under nonlinear dynamics with state velocities having bilinear-quadratic kind. Using results of the theory of trajectory tubes of control systems and techniques of differential inclusions theory we find set-valued estimates of related reachable sets of such nonlinear uncertain control system. The algorithms of constructing the ellipsoidal estimates for studied nonlinear systems are given. Numerical simulation results related to the proposed techniques and to the presented algorithms are also included.

1. Problem formulation

Let us introduce the following basic notations. Let $\mathbb{R}^n$ be the n–dimensional Euclidean space, $\operatorname{comp}\mathbb{R}^n$ is the set of all compact subsets of $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ stands for the set of all $n\times n$–matrices and $x'y = (x,y) = \sum_{i=1}^n x_i y_i$ be the usual inner product of $x, y \in \mathbb{R}^n$ with prime as a transpose, $\|x\| = (x'x)^{1/2}$. We denote as $B(a,r)$ the ball in $\mathbb{R}^n$, $B(a,r) = \{x \in \mathbb{R}^n : \|x-a\| \le r\}$, I is the identity $n\times n$-matrix. Denote by $E(a,Q) = \{x \in \mathbb{R}^n :\ (Q^{-1}(x-a),(x-a)) \le 1\}$ the ellipsoid in $\mathbb{R}^n$ with a center $a \in \mathbb{R}^n$ and a symmetric positive definite $n\times n$–matrix Q, $\operatorname{Tr}(Y)$ denotes the trace of $n\times n$–matrix Y (the sum of its diagonal elements). For $x, y \in \mathbb{R}^n$ we will use the notation $x\cdot y' = Z$, where matrix $Z = \{z_{ij} = x_i y_j :\ 1 \le i,j \le n\} \in \mathbb{R}^{n\times n}$.

Consider the following system

$$\dot{x} = A(t)x + f(x)d + u(t), \quad x_0 \in X_0, \quad t \in [t_0, T], \tag{1.1}$$

where $x, d \in \mathbb{R}^n$, $\|x\| \le K$ $(K > 0)$, $f(x)$ is the nonlinear function, which is quadratic in x, $f(x) = x'Bx$, with a given symmetric and positive definite $n\times n$-matrix B. Control functions $u(t)$ in (1.1) are assumed Lebesgue measurable on $[t_0, T]$ and satisfying the constraint $u(t) \in U$, for a.e. $t \in [t_0, T]$, (here U is a given set, $U \in \operatorname{comp}\mathbb{R}^n$). The $n\times n$–matrix function $A(t)$ in (1.1) has the form

$$A(t) = A^0 + A^1(t), \tag{1.2}$$

where the $n\times n$–matrix A^0 is given and the measurable $n\times n$–matrix $A^1(t)$ with elements $\{a^{(1)}_{ij}(t)\}$ $(i,j = 1,\dots,n)$ is unknown but bounded, $A^1(t) \in \mathcal{A}^1$,

$$A(t) \in \mathcal{A} = A^0 + \mathcal{A}^1, \quad \mathcal{A}^1 = \big\{A = \{a_{ij}\} \in \mathbb{R}^{n\times n} : |a_{ij}| \le c_{ij}, \quad i,j = 1,\dots n\big\}, \quad t \in [t_0, T], \tag{1.3}$$

where $c_{ij} \ge 0$ $(i,j = 1,\dots n)$ are given.

We will assume that X_0 in (1.1) is an ellipsoid, $X_0 = E(a_0, Q_0)$, with a symmetric and positive definite matrix $Q_0 \in \mathbb{R}^{n\times n}$ and with a center a_0.

Let the absolutely continuous function $x(t) = x\big(t; u(\cdot), A(\cdot), x_0\big)$ be a solution to dynamical system (1.1) with initial state $x_0 \in X_0$, with admissible control $u(\cdot)$ and with a matrix $A(\cdot)$ satisfying (1.2)–(1.3). The reachable set $X(t)$ at time t $(t_0 < t \le T)$ of system (1.1)–(1.3) is defined as the following set

$$X(t) = \big\{x \in \mathbb{R}^n : \exists\, x_0 {\in} X_0,\ \exists\, u(\cdot){\in} U,\ \exists\, A(\cdot){\in}\mathcal{A},\ \ x = x(t) = x\big(t; u(\cdot), A(\cdot), x_0\big)\big\},\ t_0 < t \le T.$$

The main problem of the paper is to find the external ellipsoidal estimate $E(a^+(t), Q^+(t))$ (with respect to the inclusion of sets) of the reachable set $X(t)$ $(t_0 < t \le T)$ by using the analysis of a special type of nonlinear control systems with uncertain initial data.

2. Preliminaries

In this section we present some auxiliary results on the properties of reachable sets for different types of dynamical systems which we will need in the sequel.

2.1. Bilinear system

Bilinear dynamic systems are a special kind of nonlinear systems representing a variety of important physical processes. A great number of results related to control problems for such systems has been developed over past decades, among them we mention here [4,5,10,14,16,18,20]. Reachable sets of bilinear systems in general are not convex, but have special properties (for example, are star-shaped). We, however, consider here the guaranteed state estimation problem and use ellipsoidal calculus for the construction of external estimates of reachable sets of such systems.

Consider the bilinear system

$$\dot{x} = A(t)\,x, \quad t_0 \le t \le T, \tag{2.1}$$

$$x_0 \in X_0 = E(a_0, Q_0), \tag{2.2}$$

where $x, a_0 \in \mathbb{R}^n$, Q_0 is symmetric and positive definite matrix. The unknown matrix function $A(t) \in \mathbb{R}^{n\times n}$ is assumed to be of the form (1.2) with the assumption (1.3).

The external ellipsoidal estimate of reachable set $X(T)$ of the system (2.1)–(2.2) can be found by applying the following theorem.

Theorem 1 [4]. *Let $a^+(t)$ and $Q^+(t)$ be the solutions of the following system of nonlinear differential equations*

$$\dot{a}^+ = A^0 a^+, \quad a^+(t_0) = a_0, \tag{2.3}$$

$$\dot{Q}^+ = A^0 Q^+ + Q^+ A^{0\prime} + qQ^+ + q^{-1}G, \quad Q^+(t_0) = Q_0, \quad t_0 \le t \le T, \tag{2.4}$$

where

$$G = \operatorname{diag}\Big\{(n-v)\big[\sum_{i=1}^{n} c_{ji}|a_i^+| + \big(\max_{\sigma=\{\sigma_{ij}\}} \sum_{p,q=1}^{n} Q^+_{pq} c_{jp} c_{jq} \sigma_{jp}\sigma_{jq}\big)^{1/2}\big]^2\Big\}, \tag{2.5}$$

$$q = \big(n^{-1}\operatorname{Tr}\big((Q^+)^{-1}G\big)\big)^{1/2},$$

the maximum in (2.5) *is taken over all $\sigma_{ij} = \pm 1$, $i, j = 1, \dots, n$, such that $c_{ij} \neq 0$ and v is a number of such indices i for which we have: $c_{ij} = 0$ for all $j = 1, \dots, n$. Then the following external estimate for the reachable set $X(t)$ of the system* (2.1)–(2.2) *is true*

$$X(t) \subseteq E(a^+(t), Q^+(t)), \quad t_0 \le t \le T. \tag{2.6}$$

Corollary 1. *Under conditions of the Theorem 1 the following inclusion holds*

$$X(t_0+\sigma) \subseteq (I + \sigma\mathcal{A})\,X_0 + o_1(\sigma)B(0,1) \subseteq E(a^+(t_0+\sigma), Q^+(t_0+\sigma)) + o_2(\sigma)B(0,1), \tag{2.7}$$

where $\sigma^{-1}o_i(\sigma) \to 0$ for $\sigma \to +0$ $(i = 1, 2)$ and

$$(I + \sigma\mathcal{A})\,X_0 = \bigcup_{x\in X_0} \bigcup_{A\in\mathcal{A}} \{x + \sigma A x\}.$$

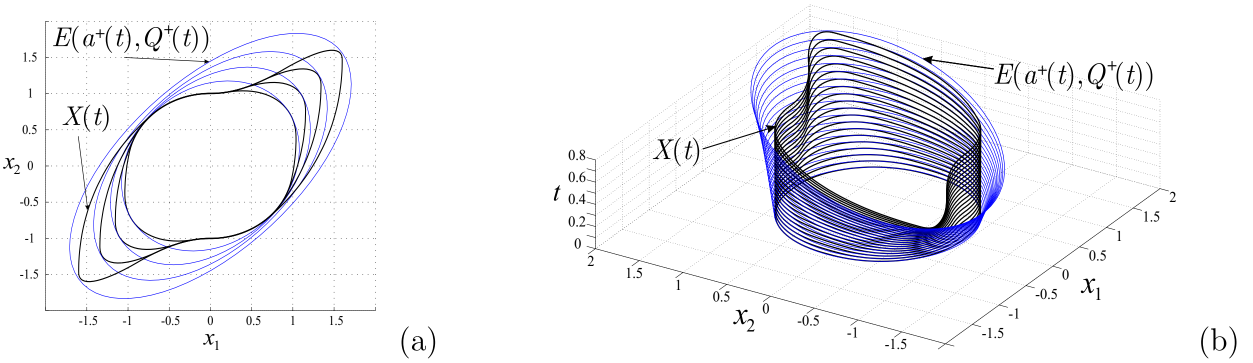

Figure 1. (a) Reachable sets $X(t)$ and their external estimates $E(a^+(t), Q^+(t))$ for $t = 0.2; 0.4; 0.6; 0.8$. (b) Trajectory tube $X(t)$ and its ellipsoidal estimating tube $E(a^+(t), Q^+(t))$ for the bilinear control system with uncertain initial states.

P r o o f. The inclusion (2.7) follows directly from (2.6) and presents a special case of the inclusion related to the discrete version of the integral funnel equation for the system (2.1)–(2.2) [14, 16].

The following example illustrates the result of Theorem 1.

Example 1. Consider the following system

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= c(t)\, x_1, \end{cases} \qquad 0 \le t \le 0.8, \tag{2.8}$$

where $x_0 \in X_0 = B(0,1)$, $c(t)$ is an unknown but bounded measurable function with $|c(t)| \le 1$ $(0 \le t \le 1)$. The reachable sets $X(t)$ and their external ellipsoidal estimates $E(a^+(t), Q^+(t))$ found by Theorem 1 are shown in Figure 1.

We see here that the trajectory tube $X(t)$ of bilinear system (2.8), issued from the convex set $X_0 = B(0,1)$, loses the convexity over time. External ellipsoidal tube $E(a^+(t), Q^+(t))$ contains the reachable set $X(t)$ and in some points is enough accurate (it touches the boundary of $X(t)$).

2.2. Systems with quadratic nonlinearity

Consider the control system of type (1.1) but with a known matrix $A = A^0$

$$\dot{x} = A^0 x + f(x)d + u(t), \tag{2.9}$$
$$x_0 \in X_0 = E(a_0, Q_0), \quad t_0 \le t \le T.$$

We assume here that $u(t) \in U = E(\hat{a}, \hat{Q})$, vectors d, a_0, $\hat{a}$ are given, a scalar function $f(x)$ has a form $f(x) = x'Bx$, matrices B, Q_0, $\hat{Q}$ are symmetric and positive definite.

Denote the maximal eigenvalue of the matrix $B^{1/2}Q_0B^{1/2}$ by k^2, it is easy to see this k^2 is the smallest number for which the inclusion $X_0 \subseteq E(a_0, k^2B^{-1})$ is true. The following result describes the external ellipsoidal estimate of the reachable set $X(t)$ of system (2.9) $(t_0 \le t \le T)$.

Theorem 2 [9]. *The following inclusion is true for any $t \in [t_0, T]$*

$$X(t) \subseteq E(a^+(t), r^+(t)B^{-1}),$$

where functions $a^+(t)$, $r^+(t)$ are the solutions of the following system of ordinary differential equations

$$\dot{a}^+(t) = A^0 a^+(t) + ((a^+(t))'Ba^+(t) + r^+(t))d + \hat{a}, \quad t_0 \le t \le T,$$

$$\dot{r}^+(t) = \max_{\|l\|=1} \Big\{ l' \big(2r^+(t)B^{1/2}(A^0 + 2d(a^+(t))'B)B^{-1/2} + q^{-1}(r^+(t))B^{1/2}\hat{Q}B^{1/2} \big) l \Big\} + q(r^+(t))r^+(t),$$

where $q(r) = \big((nr)^{-1}\mathrm{Tr}(B\hat{Q})\big)^{1/2}$, *with initial state* $a^+(t_0) = a_0, \quad r^+(t_0) = k^2$.

Corollary 2 [7]. *The following upper estimate for* $X(t_0+\sigma)$ $(\sigma > 0)$ *holds*

$$X(t_0+\sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)) + o(\sigma)B(0,1),$$

where $\sigma^{-1}o(\sigma) \to 0$ *when* $\sigma \to +0$ *and*

$$a^+(\sigma) = a(\sigma) + \sigma\hat{a}, \quad a(\sigma) = a_0 + \sigma(A^0 a_0 + a_0' B a_0 \cdot d + k^2 d),$$
$$Q^+(\sigma) = (p^{-1}+1)Q(\sigma) + (p+1)\sigma^2\hat{Q},$$
$$Q(\sigma) = k^2(I+\sigma R)B^{-1}(I+\sigma R)', \quad R = A^0 + 2d\cdot a_0' B$$

and p *is the unique positive root of the equation*

$$\sum_{i=1}^{n} \frac{1}{p+\alpha_i} = \frac{n}{p(p+1)}$$

with $\alpha_i \geq 0$ $(i = 1, ..., n)$ *being the roots of the following equation* $|Q(\sigma) - \alpha\sigma^2\hat{Q}| = 0$.

Numerical algorithms basing on Theorem 2 and producing the discrete-time external ellipsoidal tube estimating the reachable set of the system (2.9) (together with related examples) are given in [9, 12].

3. Main results

3.1. Bilinear-quadratic control system

Consider the general case

$$\dot{x} = A(t)x + f(x)d + u(t), \qquad t_0 \leq t \leq T, \tag{3.1}$$

where $f(x) = x'Bx$, initial state $x_0 \in X_0 = E(a_0, Q_0)$ and control constraints $u(t) \in U = E(\hat{a}, \hat{Q})$, and with the uncertain matrix

$$A(t) = A^0 + A^1(t), \quad A(t) \in \mathcal{A}, \tag{3.2}$$

where the set $\mathcal{A}$ is defined in (1.3). As before we assume that matrices B, $\hat{Q}$ and Q_0 are symmetric and positive definite.

The next theorem describes discrete external ellipsoidal estimates of reachable sets $X(t)$ of the uncertain control system (3.1)–(3.2), containing both bilinear and quadratic nonlinearities.

Theorem 3. *The following external ellipsoidal estimate holds*

$$X(t_0+\sigma) \subseteq E(a^*(t_0+\sigma), Q^*(t_0+\sigma)) + o(\sigma)B(0,1) \tag{3.3}$$

where $\sigma^{-1}o(\sigma) \to 0$ *for* $\sigma \to +0$ *and where*

$$a^*(t_0+\sigma) = \tilde{a}(t_0+\sigma) + \sigma(\hat{a} + a_0' B a_0\, d + k^2 d), \tag{3.4}$$
$$Q^*(t_0+\sigma) = (p^{-1}+1)\tilde{Q}(t_0+\sigma) + (p+1)\sigma^2\hat{Q}, \tag{3.5}$$

with functions $\tilde{a}(t)$, $\tilde{Q}(t)$ *calculated as* $a^+(t)$, $Q^+(t)$ *in Theorem 1 but when we replace matrices* Q_0 *and* A^0 *in* (2.3)–(2.5) *by*

$$\tilde{Q}_0 = k^2 B^{-1}, \quad \tilde{A}^0 = A^0 + 2d\cdot a_0' B \tag{3.6}$$

respectively, and p *is the unique positive root of the equation*

$$\sum_{i=1}^{n} \frac{1}{p+\alpha_i} = \frac{n}{p(p+1)} \tag{3.7}$$

with $\alpha_i \geq 0$ $(i = 1, ..., n)$ *being the roots of the following equation* $|Q(t_0+\sigma) - \alpha\sigma^2\hat{Q}| = 0$.

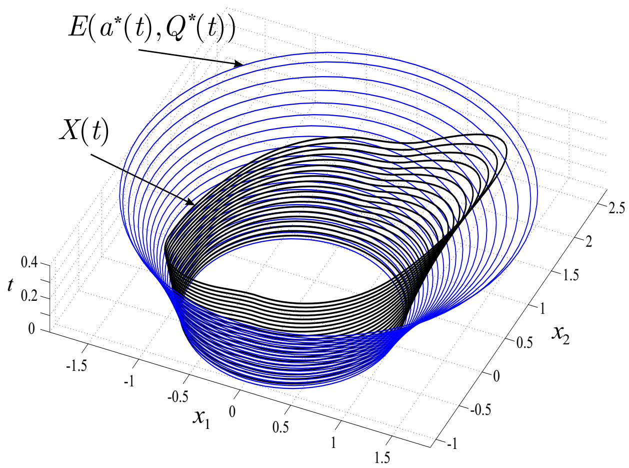

Figure 2. Trajectory tube $X(t)$ and its ellipsoidal estimating tube $E(a^*(t), Q^*(t))$ for the system with bilinear and quadratic nonlinearities.

P r o o f. Analyzing both results of Theorem 1 and Theorem 2 and of their corollaries and using the general scheme of the proof of Theorem 2 in [8] we obtain the formulas (3.3)–(3.7) of the Theorem 3.

The following iterative algorithm basing on Theorem 3 may be used to produce the external ellipsoidal tube estimating the reachable set $X(t)$ on the whole time interval $t \in [t_0, T]$.

Algorithm 1. Subdivide the time segment $[t_0, T]$ into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + ih$ $(i = 1, \ldots, m)$, $h = (T - t_0)/m$, $t_m = T$.

• Given $X_0 = E(a_0, Q_0)$, find the smallest $k = k_0 > 0$ such that

$$E(a_0, Q_0) \subseteq E(a_0, k^2 B^{-1})$$

(k^2 is the maximal eigenvalue of the matrix $B^{1/2} Q_0 B^{1/2}$).

• Take $\sigma = h$ and define by Theorem 3 the external ellipsoid $E(a_1, Q_1)$ such that

$$X(t_1) \subseteq E(a_1, Q_1) = E(a^*(t_0 + \sigma), Q^*(t_0 + \sigma)).$$

• Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, Q_1)$ as the initial ellipsoid at instant t_1.

• The following steps repeat the previous iteration.

At the end of the process we will get the external estimate $E(a(t), Q(t))$ of the tube $X(t)$ with accuracy tending to zero when $m \to \infty$.

Example 2. Consider the following control system $(t_0 \le t \le T)$

$$\begin{cases} \dot{x}_1 &= x_2 + u_1, \\ \dot{x}_2 &= c(t)x_1 + x_1^2 + x_2^2 + u_2. \end{cases}$$

Here we take $t_0 = 0$, $T = 0.4$, $x_0 \in X_0 = B(0, 1)$ and $U = B(0, 0.1)$, the uncertain but bounded measurable function $c(t)$ satisfies the inequality $|c(t)| \le 1$ $(t_0 \le t \le T)$. The trajectory tube $X(t)$ and its external ellipsoidal estimating tube $E(a^*(t), Q^*(t))$ calculated by the Algorithm 1 are given in Figure 2.

3.2. Impulsive bilinear-quadratic control system

Consider the following control system $(t_0 \le t \le T)$

$$dx(t) = \big(A(t)x(t) + f(x)d + u(t)\big)dt + C dv(t), \tag{3.8}$$
$$x \in \mathbb{R}^n, \quad A(t) = A^0 + A^1(t), \quad A^1(t) \in \mathcal{A},$$

where $f(x) = x'Bx$, B is positive definite and symmetric matrix, $A^0 \in \mathbb{R}^{n\times n}$, parameters d, C are n-vectors, $d, C \in \mathbb{R}^n$, the set $\mathcal{A}$ is defined in (1.3). Here the impulsive function $v : [t_0, T] \to R$ is of bounded variation on $[t_0, T]$, monotonically increasing and right-continuous. We assume that $\mu > 0$ and

$$\operatorname*{Var}_{t\in[t_0,T]} v(t) = \sup_{\{t_i\}} \sum_{i=1}^{k} |v(t_i) - v(t_{i-1})| \le \mu,$$

where $t_i : t_0 \le t_1 \le \ldots \le t_k = T$. We assume also $X_0 = E(a, k^2B^{-1})$ $(k \neq 0)$, $U = E(\hat{a}, \hat{Q})$.

Let us introduce a new time variable [21]:

$$\eta(t) = t + \int_{t_0}^{t} du(t),$$

and a new state coordinate

$$\tau(\eta) = \inf\{t \mid \eta(t) \ge \eta\}.$$

Consider the following inclusion

$$\frac{d}{d\eta}\begin{pmatrix} z \\ \tau \end{pmatrix} \in H(\tau, z), \tag{3.9}$$

$$z(t_0) = x_0 \in X_0 = E(a, k^2B^{-1}), \quad \tau(t_0) = t_0, \quad t_0 \le \eta \le T + \mu,$$

$$H(\tau, z) = \bigcup_{0\le\nu\le1} \left\{ \nu \begin{pmatrix} C \\ 0 \end{pmatrix} + (1-\nu) \begin{pmatrix} A(\tau)z + z'Bz\, d + E(\hat{a}, \hat{Q}) \\ 1 \end{pmatrix} \right\}.$$

Denote $w = \{z, \tau\}$ the extended state vector of the system (3.9) and the reachable set of the system (3.9) as $W(\eta) = W(\eta; t_0, w_0, \mathcal{A}, X_0 \times \{t_0\})$ $(t_0 \le \eta \le T + \mu)$).

Theorem 4. *The following inclusion holds true for $\sigma > 0$:*

$$W(t_0 + \sigma) \subseteq W(t_0, \sigma) + o(\sigma)B(0, 1), \quad \lim_{\sigma\to+0} \sigma^{-1}o(\sigma) = 0.$$

Here

$$W(t_0, \sigma) = \bigcup_{0\le\nu\le1} W(t_0, \sigma, \nu), \quad W(t_0, \sigma, \nu) = \begin{pmatrix} E\big(a^*(\sigma, \nu), Q^*(\sigma, \nu)\big) \\ t_0 + \sigma(1-\nu) \end{pmatrix},$$

$$a^*(\sigma, \nu) = \tilde{a}(\sigma, \nu) + \sigma(1-\nu)(a'Ba\, d + k^2d + \hat{a}) + \sigma\nu C,$$
$$Q^*(\sigma, \nu) = (p^{-1} + 1)\tilde{Q}(\sigma, \nu) + (p + 1)\sigma^2(1-\nu)^2\hat{Q},$$

with functions $\tilde{a}(\sigma, \nu)$, $\tilde{Q}(\sigma, \nu)$ calculated as $a^+(t)$, $Q^+(t)$ in Theorem 1 but when we replace matrices Q_0 and A^0 in (2.3)–(2.5) by

$$\tilde{Q}_0 = k^2B^{-1}, \quad \tilde{A}^0 = (1-\nu)(A^0 + 2d \cdot a_0'B)$$

respectively. Here $p = p(\sigma, \nu)$ is the unique positive root of the equation

$$\sum_{i=1}^{n} \frac{1}{p + \lambda_i} = \frac{n}{p(p+1)},$$

and $\lambda_i = \lambda_i(\sigma, \nu) \ge 0$ satisfy the equation $|\tilde{Q}(\sigma, \nu) - \lambda\sigma^2(1-\nu)^2\hat{Q}| = 0$.

P r o o f. The above generalization is based on a combination of the techniques described above and the results of [11, 12].

Remark 1. [11] To determinate simpler estimate of the reachable set $W(t_0+\sigma)$ we introduce small parameter $\varepsilon > 0$ and embed the degenerate ellipsoid $W(t_0,\sigma,\nu)$ in nondegenerate ellipsoid $E_\varepsilon\big(w(t_0,\sigma,\nu), O_\varepsilon(t_0,\sigma,\nu)\big)$:

$$W(t_0,\sigma,\nu) \subseteq E_\varepsilon\big(w(t_0,\sigma,\nu), O_\varepsilon(t_0,\sigma,\nu)\big),$$

$$w(t_0,\sigma,\nu) = \begin{pmatrix} a^*(\sigma,\nu) \\ t_0+\sigma(1-\nu) \end{pmatrix}, \quad O_\varepsilon(t_0,\sigma,\nu) = \begin{pmatrix} Q^*(\sigma,\nu) & 0 \\ 0 & \varepsilon^2 \end{pmatrix}.$$

Thus, for all small $\varepsilon > 0$ we get

$$W(t_0,\sigma) \subset W_\varepsilon(t_0,\sigma) = \bigcup_{0\le\nu\le1} E_\varepsilon\big(w(t_0,\sigma,\nu), O_\varepsilon(t_0,\sigma,\nu)\big) \subset E_\varepsilon(w^+(\sigma), O^+(\sigma))$$

and $\lim\limits_{\varepsilon\to+0} h(W(t_0,\sigma), W_\varepsilon(t_0,\sigma)) = 0$. The passage to the family of nondegenerate ellipsoids enables one to use the algorithms of [11, 17] and construct an external estimate $E_\varepsilon(w^+(\sigma), O^+(\sigma))$ of the union of ellipsoids $W_\varepsilon(t_0,\sigma)$. Therefore we get ellipsoidal estimates of the reachable set $W(t_0+\sigma)$

$$W(t_0+\sigma) \subset E_\varepsilon(w^+(\sigma), O^+(\sigma)) + o(\sigma)B(0,1).$$

The following lemma explains the construction of the differential inclusion (3.9).

Lemma 1 [11]. *The reachable set $X(T)$ is the projection of $W(T+\mu)$ at the subspace of variables z: $X(T) = \pi_z W(T+\mu)$.*

The following iterative algorithm basing on Theorem 4 may be used to produce the external ellipsoidal estimates for the reachable sets of the system (3.8) on the whole time interval $t \in [t_0, T]$.

Algorithm 2. Subdivide the time segment $[t_0, T+\mu]$ into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0+ih$ $(i = 1,\ldots,m)$, $h = (T+\mu-t_0)/m$, $t_m = T+\mu$. Subdivide the segment $[0,1]$ into subsegments $[\nu_j, \nu_{j+1}]$ where $\nu_i = ih_*$, $h_* = 1/m$, $\nu_0 = 0$, $\nu_m = 1$.

• Take $\sigma = h$ and for given $X_0 = E(a_0, k^2B^{-1})$ define by Theorem 4 the sets $W(t_0,\sigma,\nu_i)$ $(i = 0,\ldots,m)$.

• Find ellipsoid $E_\varepsilon(w_1^+(\sigma), O_1^+(\sigma))$ in $\mathbb{R}^{n+1}$ such that $W(t_0,\sigma,\nu_i) \subseteq E_\varepsilon(w_1^+(\sigma), O_1^+(\sigma))$ $(i = 0,\ldots,m)$. At this step we find the ellipsoidal estimate for the union of a finite family of ellipsoids [11, 17].

• Find the projection of $E(a_1, Q_1) = \pi_z E_\varepsilon(w_1^+(\sigma), O_1^+(\sigma))$ by Lemma 1.

• Find the smallest $k_1 > 0$ such that $E(a_1, Q_1) \subseteq E(a_1, k_1^2B^{-1})$ (k_1^2 is the maximal eigenvalue of the matrix $B^{1/2}Q_1B^{1/2}$).

• Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_0, k_1^2B^{-1})$ as the initial ellipsoid at instant t_1.

• The following steps repeat the previous iteration.

At the end of the process we will get the external estimate $E(a^+(T), Q^+(T))$ of the reachable sets of the system (3.8).

4. Conclusions

The paper deals with the problems of state estimation for nonlinear uncertain control systems for which we assume that the initial state is unknown but bounded with given constraints and the matrix in the linear part of state velocities is also unknown but bounded.

Basing on results of ellipsoidal calculus developed earlier for some classes of uncertain systems we present the modified state estimation approach which uses the special structure of nonlinearity and uncertainty in the control system and allows constructing the external ellipsoidal estimates of reachable sets.

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