

ON CHROMATIC UNIQUENESS OF SOME COMPLETE TRIPARTITE GRAPHS

Pavel A. Gein

Ural Federal University,
51, Lenin ave., Ekaterinburg, 620000, Russia
pavel.gein@gmail.com

Abstract: Let $P(G, x)$ be a chromatic polynomial of a graph G . Two graphs G and H are called chromatically equivalent iff $P(G, x) = P(H, x)$. A graph G is called chromatically unique if $G \simeq H$ for every H chromatically equivalent to G . In this paper, the chromatic uniqueness of complete tripartite graphs $K(n_1, n_2, n_3)$ is proved for $n_1 \geq n_2 \geq n_3 \geq 2$ and $n_1 - n_3 \leq 5$.

Keywords: Chromatic uniqueness, Chromatic equivalence, Complete multipartite graphs, Chromatic polynomial.

1. Introduction

All graphs in this paper are finite and simple, i.e., they do not contain loops and multiple edges. Basic terminology is used according to [1].

Let $G = (V, E)$ be a graph with a vertex set V and an edge set E . A *coloring* of the graph G with x colors is a map $\varphi: V \rightarrow \{1, 2, \dots, x\}$ such that $\varphi(u) \neq \varphi(v)$ for any two adjacent vertices u and v of the graph G . We will call the numbers $1, 2, \dots, x$ the colors. A graph is called *x -colorable* if there exists its coloring with x colors. The smallest integer x for which G is x -colorable is called the *chromatic number* of G and is denoted by $\chi(G)$. The number of colorings of a graph G with x colors is denoted by $P(G, x)$. It is well known (see, for example, [1]) that the function $P(G, x)$ is a polynomial in variable x . Two graphs G and H are called *chromatically equivalent* if $P(G, x) = P(H, x)$. A graph G is called *chromatically unique* if, for any graph H , the graphs G and H are chromatically equivalent iff they are isomorphic. Much attention of researches was drawn to the problem of chromatic uniqueness of complete multipartite graphs $K(n_1, n_2, \dots, n_t)$. Here are some results, a more complete list can be found in the survey [16] and the monograph [6].

- (1) A graph $K(n_1, n_2)$, where $n_1 \geq n_2 \geq 2$, is chromatically unique (see [11]).
- (2) A graph $K(n_1, n_2, \dots, n_{t-1}, 1)$ is chromatically unique iff $n_i \leq 2$ for all $i = 1, 2, \dots, t-1$ (see [14]).
- (3) A graph $K(n_1, n_2, \dots, n_t)$, where $n_1 \geq n_2 \geq \dots \geq n_t \geq 2$, is chromatically unique if $n_1 - n_t \leq 4$ (see [2, 4, 12, 13, 15]).

The main result of this paper is the following theorem.

Theorem 1. *A complete tripartite graph $K(n_1, n_2, n_3)$ is chromatically unique if $n_1 \geq n_2 \geq n_3 \geq 2$ and $n_1 - n_3 \leq 5$.*

The chromatic uniqueness of a graph $K(n_1, n_2, n_3)$ with $n_1 \geq n_2 \geq n_3 \geq 2$ and $n_1 - n_3 \leq 4$ was proved in [2, 12, 13]. The chromatic uniqueness of a graph $K(n_1, n_2, n_3)$ with $n_1 \geq n_2 \geq n_3 \geq 2$,

$n_1 - n_3 = 5$, and $n_1 + n_2 + n_3 \not\equiv 2 \pmod{3}$ was proved in [9]. The main aim of this paper is to prove the theorem in the case $n_1 - n_3 = 5$ and $n_1 + n_2 + n_3 \equiv 2 \pmod{3}$.

One of the most important tools for studying chromatic equivalence and chromatic uniqueness is chromatic invariants. Assume that a number is assigned to every graph. This number is called a *chromatic invariant* if it is the same for all chromatically equivalent graphs. For a chromatic invariant $\alpha(G)$ and two arbitrary graphs G_1 and G_2 , let $\Delta\alpha(G_2, G_1) = \alpha(G_2) - \alpha(G_1)$. It is well known (see, for example, [1] or [7]) that the number of vertices, the number of edges, the number of connected components, and the number of triangles are chromatic invariants.

According to Zykov's theorem (see, for example, [1]), the chromatic polynomial can be written as

$$P(G, x) = \sum_{i=\chi}^n pt(G, i)x^{(i)},$$

where $pt(G, i)$ is the number of partitions of the vertex set of the graph G into t independent sets, and $x^{(i)}$ is the falling factorial of x , i.e., $x^{(i)} = x(x-1)\dots(x-i+1)$. It follows from Zykov's theorem that the numbers $pt(G, i)$, $i = \chi, \dots, n$, are chromatic invariants. We are most interested in $pt(G, \chi+1)$, which we will write simply as $pt(G)$.

The rest of the paper is structured as follows. Section 2 describes a connection between integer partitions and chromatic uniqueness of complete multipartite graphs and presents a schema of the proof. Also, some properties of chromatic invariants are discussed in this section. In Sections 3, 4, and 5, upper bounds of the invariant pt are proved. Section 6 contains the proof of the main theorem.

2. Integer partition lattice and chromatic invariants

Let n be a positive integer. A *partition* of n is a sequence of nonnegative integers $u = (u_1, u_2, \dots)$ such that $u_1 \geq u_2 \geq \dots$ and $n = \sum_{i=1}^{\infty} u_i$. The *length* of a partition u is a number l such that $u_l > 0$ and $u_{l+1} = u_{l+2} = \dots = 0$. Writing a partition, we will often omit its zero elements.

Let $u = (u_1, u_2, \dots)$ and $v = (v_1, v_2, \dots)$ be two partitions of a number n . Define $v \leq u$ if

$$\begin{aligned} v_1 &\leq u_1, \\ v_1 + v_2 &\leq u_1 + u_2, \\ &\dots \\ v_1 + v_2 + \dots + v_{t-1} &\leq u_1 + u_2 + \dots + u_{t-1}, \end{aligned}$$

where t is the largest of the lengths of u and v . The relation \leq is called the *dominance order*. It was shown in [5] that all partitions of a number n form a lattice with respect to \leq .

As was proved in [3], all partitions of a number n with fixed length form a lattice with respect to \leq . Also, Baranskii and Sen'chonok introduced [3] a notion of *elementary transformation*. A partition $v = (v_1, v_2, \dots, v_t)$ is a result of an elementary transformation of a partition $u = (u_1, u_2, \dots, u_t)$ if there exist indices i and j such that

- (1) $1 \leq i < j \leq t$;
- (2) $u_i - 1 \geq u_{i+1}$ and $u_{j-1} \geq u_j + 1$;
- (3) $u_i - u_j = \delta \geq 2$;
- (4) $v_i = u_i - 1$, $v_j = u_j + 1$, $u_k = v_k$ for all $k = 1, 2, \dots, t$, $k \neq i, j$.

It was proved in [3] that $v \leq u$ holds if and only if the partition v can be obtained from the partition u by a with finite number of elementary transformations.

Every complete t -partite graph with n vertices can be identified with a partition of length t of the number n . Let $u = (u_1, u_2, \dots, u_t)$ be a partition of length t of the number n . We will write $K(u)$ instead of $K(u_1, u_2, \dots, u_t)$ and denote parts of the graph $K(u)$ by V_i , where $|V_i| = u_i$ for all $i = 1, 2, \dots, t$.

Let u be a partition of length t of a number n . We present the following schema for proving the chromatic uniqueness of the graph $K(u)$. By contradiction, we assume that the graph $K(u)$ is not chromatically unique. This means that there exists a graph H nonisomorphic to the graph $K(u)$, and the graphs H and $K(u)$ are chromatically equivalent. It is clear that the chromatic number of the graph H is t ; thus, the graph H can be obtained from a complete t -partite graph $K(v)$ by removing a set of edges E . It was shown in [17] that different complete multipartite graphs are not chromatically equivalent; hence, E must be non-empty. Denote by E_{ij} the set of all edges $e \in E$ such that one end of e is in V_i and the other is in V_j and define $e_{ij} = |E_{ij}|$. For an arbitrary set of edges E from $K(v)$, denote by $\langle E \rangle$ the subgraph edge-induced by E .

The following lemma was proved in [8].

Lemma 1 [8, Lemma 1]. *Let $u = (u_1, \dots, u_i, \dots, u_j, \dots, u_t) \rightarrow v = (\dots, u_i - 1, \dots, u_j + 1, \dots)$ be an elementary transformation of a partition u , where $u_t \geq 2$. Then the graphs $K(u)$ and H are not chromatically equivalent.*

It is clear that every complete t -partite graph is t -colorable, but is not $(t-1)$ -colorable; in other words, the chromatic number of a complete t -partite graph is equal to t . Let us compute $pt(K(u))$ for a complete multipartite graph $K(u_1, u_2, \dots, u_t)$. It is easy to show that any partition of the vertex set of the graph $K(u)$ into $t+1$ parts can be obtained by splitting exactly one part into two nonempty subsets; so $pt(K(u)) = \sum_{i=1}^n 2^{u_i-1} - t$.

It was investigated in [4] how the invariant pt changes from the graph $K(v)$ to the graph H . Let us introduce all necessary definitions and auxiliary statements.

A complete multipartite subgraph G_1 of a graph $K(v)$ is called an E -subgraph if each part of the graph G_1 is contained in some part of the graph $K(v)$, and the edge set of the graph G_1 is contained in the set E . An arbitrary disjoint set of E -subgraphs is called a *garland*. We will say that a garland G' *destroys* a part V_i if every vertex of V_i is contained in some E -subgraph of the garland G' . A garland of cardinality p which destroys exactly $p-1$ parts is called *interesting*. The set of all edges of all E -subgraphs of a garland is called the *edge aggregate* and is denoted by $E(G)$. The set of all vertices of all E -subgraphs of a garland is called the *vertex aggregate* and is denoted by $V(G)$. A garland is called k -edge if its edge aggregate contains exactly k edges. The following properties were proved in [4].

- (1) If the chromatic number of a graph H is equal to t and $pt(H) = 1$, then every garland of cardinality p destroys at most $p-1$ parts.
- (2) Every garland is uniquely defined by its edge aggregate.
- (3) $\Delta pt(H, K(v))$ is equal to the number of all interesting garlands.

The next lemma follows from these properties.

Lemma 2 [4, Corollary 2]. *If a graph H is obtained from a graph $K(v)$ by removing some set of edges E and the graphs $K(u)$ and H are chromatically equivalent, then $|E| \leq \Delta pt(H, K(v)) \leq 2^{|E|-1}$.*

An improvement of this estimate was obtained in [10]. A subgraph H' of a graph $\langle E \rangle$ is called a *coordinated subgraph of type $K(s, 1)$* if $H' \simeq K(s, 1)$ and all s vertices of degree 1 are in the same part of $K(v)$. A part V_j of a graph $K(v)$ is called *active* if there exist a vertex $x \in V_j$ and an edge $e \in E$ such that x and e are incident.

Lemma 3 [10, Theorem]. *Let every active part of a graph $K(v)$ contain at least three vertices. If E induces a coordinated subgraph of type $K(|E|, 1)$, then $\Delta pt(G, K(v)) = 2^{|E|} - 1$; otherwise, $\Delta pt(G, K(v)) \leq 2^{|E|-1} + 1$.*

Let $G' = \{G'_1, G'_2, \dots, G'_p\}$ be a garland. We will say that the garland G' is of type $H_1 \dot{\cup} H_2 \dot{\cup} \dots \dot{\cup} H_p$, where $\{H_1, H_2, \dots, H_p\}$ is a set of graphs, if $G'_i \simeq H_i$ for all $i = 1, 2, \dots, p$. We will say that an edge set $F \subseteq E$ induces a garland (an interesting garland) if there exists a garland (an interesting garland, respectively) G such that $F = E(G)$. A set $F \subseteq E$ is called *continuable* if there exists a garland G such that $F \subseteq E(G)$. A set $F \subseteq E' \subseteq E$ is called *continuable outside E'* if there exists a garland G such that $E(G) \cap E' = F$. We will say that an edge e is in the garland G if $e \in E(G)$.

Let e be an arbitrary edge from E . Denote by $\xi_1(e)$ the number of triangles of the graph $K(v)$ containing the edge e . Let $\xi_1 = \sum_{e \in E} \xi_1(e)$.

Consider a triangle in a graph G that contains exactly two edges from E . Denote them by e_1 and e_2 . We will call a subgraph induced by $\{e_1, e_2\}$ a Ξ_2 -subgraph. Denote by ξ_2 the number of such subgraphs and by ξ_3 the number of triangles in $\langle E \rangle$.

Denote by $I_3(G)$ the number of triangles in the graph G . In [2], the following equality was established:

$$\Delta I_3(K(v), H) = \xi_1 - \xi_2 - 2\xi_3.$$

Note that, the removal of an edge cannot produce a new triangle; so $\Delta I_3(K(v), H)$ is equal to the number of triangles in $K(v)$ destroyed by removing an edge set E from $K(v)$.

3. Upper bound for the invariant pt in the case where $\langle E \rangle$ contains a triangle

The main goal of this section is, in the case where $\langle E \rangle$ contains a triangle, to obtain an upper bound for the number of interesting garlands better than the bound in Lemma 3. Denote by Δ the edge set of this triangle. One the way to achieve this is to count number of garlands whose edge aggregates contain an edge from Δ and an edge that is not in the triangle. The following lemma was proved in [10].

Lemma 4 [10, Lemma 3]. *If a set of edges $\{e_1, e_2, e_3\} = E_1 \subset E$ induces a triangle, then there is no nonempty set $\hat{E} \subseteq E \setminus E_1$ such that the sets $\{e_i\} \dot{\cup} E_1$ induce interesting garlands for all $i = 1, 2, 3$.*

We can deduce the following statement from this lemma.

Corollary 1. *If Δ induces a triangle, $\Delta \subseteq E$, and F is an arbitrary nonempty subset of $E \setminus \Delta$, then there exist at most three nonempty subsets $F' \subseteq \Delta$ such that $F \cup F'$ induces an interesting garland.*

P r o o f. Let us fix some $F \subseteq E \setminus \Delta$ and consider all possible nonempty subsets $F' \subseteq \Delta$. Define $\Delta = \{e_1, e_2, e_3\}$. Assume that $F \cup F'$ induces an interesting garland G' .

Assume that $|F'| = 2$ and, without loss of generality, let $F' = \{e_1, e_2\}$. Define $e_1 = xy$ and $e_2 = yz$. Note that the vertices x and z are in different parts of $K(v)$. Since $F \cup F'$ induces a garland and the edges e_1 and e_2 are adjacent, they are in the same E -subgraph of G' . Denote this E -subgraph by H_1 . Since H_1 is a complete multipartite graph and the vertices x and z are in different parts, the edge xz is in the edge aggregate of G' . However, $xz \notin F'$ and $xz \in \Delta$; therefore, $xz \notin F$. Consequently, xz is not in the edge aggregate of G' , a contradiction. Hence, $|F'| \neq 2$.

Thus, there are four choices for F' : three one-element subsets and Δ . Using Lemma 4, one can deduce that at most two of the sets $F' \cup \{e_1\}$, $F' \cup \{e_2\}$, and $F' \cup \{e_3\}$ induce an interesting garland. Thus, there are at most three choices for F' and the corollary is proved. \square

The following lemma describes all sets $F \subseteq E \setminus \Delta$ such that there exist exactly three subsets $F' \subseteq \{e_1, e_2, e_3\}$ such that $F \cup F'$ induces an interesting garland.

Lemma 5. *Assume that each part of $K(v)$ contains at least three vertices and edges $e_1, e_2, e_3 \in E$ induce a triangle. Let $F \subseteq E \setminus \{e_1, e_2, e_3\}$. If there are exactly three nonempty subsets $F' \subseteq \{e_1, e_2, e_3\}$ such that $F \cup F'$ induces an interesting garland, then F induces an interesting garland, $|F| > 1$, and there is no edge $f \in F$ such that f is adjacent to any of e_1, e_2 , and e_3 .*

P r o o f. Define $\Delta = \{e_1, e_2, e_3\}$. Fix some $F \subseteq E \setminus \Delta$ and assume that there exist three different subsets $F' \subseteq \Delta$ such that $F \cup F'$ induces an interesting garland. Denote this subsets by E_1, E_2 , and E_3 . Using Corollary 1, one can deduce that two of E_1, E_2 , and E_3 are one-element sets and the third one is equal to Δ . Without loss of generality, let $E_1 = \{e_1\}$, $E_2 = \{e_2\}$, and $E_3 = \Delta$.

Let the set $E_1 \cup F$ induce an interesting garland G_1 , the set $E_2 \cup F$ induce an interesting garland G_2 , and the set $E_3 \cup F$ induce an interesting garland G_Δ . Denote the vertices of the triangle by x, y , and z .

Let us prove that an arbitrary edge $f \in F$ is not adjacent to any of the edges e_1, e_2 , and e_3 . By contradiction, assume that f is incident to some vertex from the triangle, without loss of generality, assume that f is incident to x . Denote the second end of f by w . Note that w and y or w and z are in different parts of $K(v)$. Without loss of generality, assume that w and y are in different parts of $K(v)$. Since the edges xy and $f = xw$ are adjacent and are in the same E -subgraph of the garland G_Δ , and w and y are in different parts of $K(v)$, the edge wy is in F . Therefore, since wx and wy are adjacent edges, they are in the same E -subgraph of the garland G_1 . So, the edge xy is in the same E -subgraph. Thus, xy is in the edge aggregate of the garland G_1 . By analogy, the edge xy is in the aggregate of the garland G_2 . But this is impossible since $xy \in \Delta$ and the edge aggregates of the garlands G_1 and G_2 cannot contain a common edge from Δ .

Since any edge from F is not adjacent to any edge from Δ , the set $\{e_1\}$ induces a E -subgraph of the garland G_1 . Therefore, the graph $\langle F \rangle$ consists of E -subgraphs, so F induces a garland. Denote this garland by G_F .

Note that all the garlands G_1, G_2 and G_Δ have the same cardinality. Denote this cardinality by p . Since all this garlands are interesting, each of them destroys exactly $p - 1$ parts of $K(v)$. Note also that $|G_F| = p - 1$.

Let $x \in V_i, y \in V_j, z \in V_k, e_1 = xy$, and $e_2 = yz$. Note that $V(G_\Delta) = V(G_1) \cup V(G_2)$. This means that if G_1 or G_2 destroy some part, then G_Δ also destroys this part. Note also that if one of the garlands G_1, G_2 , and G_Δ destroys some part V other than V_i, V_j , and V_k , then G_F also destroys V . Since no edge from $E(G_1)$ is incident to z , the garland G_1 cannot destroy V_k . If G_2 destroys V_k , then G_Δ also destroys V_k and thus G_Δ destroys more parts than G_1 does, a contradiction. Therefore, G_2 does not destroy V_k . If G_2 does not destroy V_j , then G_F destroys $p - 1$ parts and $|G_F| = p - 1$, a contradiction. So G_2 destroys V_j . Since no edge from F is incident to y , the garland G_F cannot destroy V_j , thus G_F destroys one part less than G_2 does. Therefore, G_F destroys $p - 1$ parts, and G_F is an interesting garland.

It remains to prove that $|F| > 1$. If $|F| = 1$, then G_Δ is a garland of type $K(1, 1, 1) \dot{\cup} K(1, 1)$. Note that such a garland cannot destroy any part, since each active part contains at least three vertices. \square

Now we are ready to prove a better upper bound in the case where $\langle E \rangle$ contains a triangle.

Lemma 6. *Assume that each part of $K(v)$ contains at least three vertices. If $\langle E \rangle$ contains a triangle, then the number of interesting garland does not exceed $2^{|E|-2} + 2^{|E|-3} - 3|E| + 13$.*

P r o o f. Denote edges of the triangle by e_1, e_2 , and e_3 , and let $\Delta = \{e_1, e_2, e_3\}$. Define $E' = E \setminus \Delta$.

Case 1. Assume that E' induces a coordinated subgraph of type $K(|E| - 3, 1)$. Then all edges of E are located in one of the ways shown in Fig. 1.

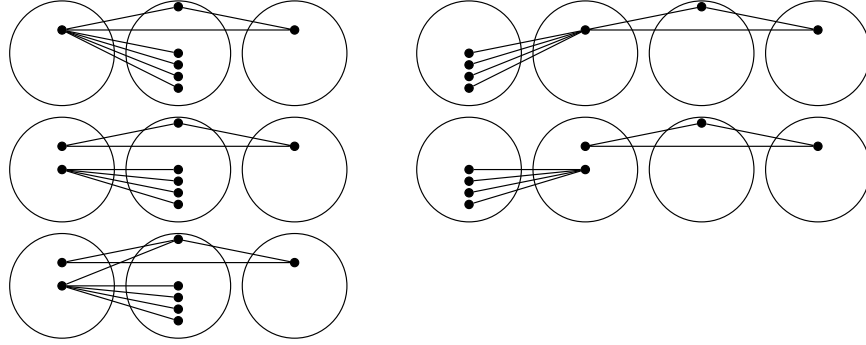


Figure 1. Triangle and a coordinated subgraph of type $K(|E| - 3, 1)$.

In the first case, there are $2 + 1 + 2^{|E|-2} - 1 = 2^{|E|-2} + 2$ interesting garlands.

In the second case, there are $4 + 2^{|E|-3} - 1 = 2^{|E|-3} + 3$ interesting garlands.

In the third case, there are at most $4 + 2^{|E|-3} - 1 + 3 = 2^{|E|-3} + 6$ interesting garlands.

In the fourth case, there are $4 + 2^{|E|-3} - 1 = 2^{|E|-3} + 3$ interesting garlands.

In the fifth case, there are at most $4 + 2^{|E|-3} - 1 + 1 = 2^{|E|-3} + 4$ interesting garlands.

In the sixth case, there are at most $2^{|E|-3} + 6 < 2^{|E|-2} + 2^{|E|-3} - 3|E| + 13$ interesting garlands.

In all the cases, the number of interesting garlands does not exceed $2^{|E|-3} + 6 < 2^{|E|-2} + 2^{|E|-3} - 3|E| + 13$.

Case 2. Now consider the case where E' does not induce a coordinated subgraph of type $K(|E| - 3, 1)$.

There are four interesting garlands G such that $E(G) \subseteq \Delta$. By Lemma 3, there exist at most $2^{|E|-4} + 1$ interesting garlands G such that $E(G) \subseteq E'$.

Estimate the number of interesting garlands whose edge aggregates contain some edge from E' and some edge from Δ . Define $F = E(G)$, $F_\Delta = F \cap \Delta$, and $F' = F \cap E'$. By Lemma 5, if for F there exist three F' such that $F \cup F'$ induces an interesting garland, then F' induce an interesting garland, $|F'| > 1$, and any edge from F' is not adjacent to any edge from Δ . Denote the number of such F by X .

Case 2.1. Assume that any edge from E' is not adjacent to any edge from Δ . Using Lemma 3 and the fact that $|F'| > 1$, one can deduce that

$$X \leq 2^{|E|-4} + 1 - (|E| - 3) = 2^{|E|-4} - |E| + 4.$$

For any other $F' \subseteq E'$, $|F'| > 1$, there are at most two subsets $F_\Delta \subseteq \Delta$ such that $F' \cup F_\Delta$ induces an interesting garland. If $F' \subseteq E'$ and $|F'| = 1$, there exist no $F_\Delta \subseteq \Delta$ such that $F' \cup F_\Delta$ induces

an interesting garland. Therefore, the number of interesting garlands does not exceed

$$\begin{aligned}
& 3X + 2(2^{|E|-3} - 1 - X - (|E| - 3)) + 2^{|E|-4} + 1 + 4 = \\
& = 3X + 2^{|E|-2} - 2 - 2X - 2|E| + 6 + 2^{|E|-4} + 1 + 4 = \\
& = 2^{|E|-2} + X - 2|E| + 2^{|E|-4} + 9 \leq \\
& \leq 2^{|E|-2} + 2^{|E|-4} - |E| + 4 - 2|E| + 2^{|E|-4} + 9 = 2^{|E|-2} + 2^{|E|-3} - 3|E| + 13.
\end{aligned}$$

Case 2.2. Now consider the case when there are exactly $k \geq 1$ edges from E' such that each of them is adjacent to some edge from Δ . Denote the set of this edges by H .

If for a set $F' \subseteq E'$ there exists exactly three sets $F_\Delta \subseteq \Delta$ such that $F' \cup F_\Delta$ induces an interesting garland, then $F' \subseteq E' \setminus H$ and $|F'| > 1$, so

$$X \leq 2^{|E|-3-k} - (|E| - 3 - k) - 1.$$

For any other $F' \subseteq E'$, $|F'| > 1$, there are at most two such sets F_Δ . If $|F'| = 1$, then if $F' \subseteq H$, then there is at most one such F_Δ and if $F' \not\subseteq H$, then such F_Δ does not exist. Therefore, the number of interesting garland does not exceed

$$\begin{aligned}
& 3X + 2(2^{|E|-3} - 1 - X - (|E| - 3)) + k + 2^{|E|-4} + 1 + 4 = \\
& 3X + 2^{|E|-2} - 2 - 2X - 2|E| + 6 + 2^{|E|-4} + 1 + 4 + k = \\
& = 2^{|E|-2} + X - 2|E| + 2^{|E|-4} + 9 + k \leq \\
& \leq 2^{|E|-2} - 2|E| + 2^{|E|-4} + 9 + 2^{|E|-3-k} - (|E| - 3 - k) - 1 + k = \\
& = 2^{|E|-2} + 2^{|E|-4} - 3|E| + 11 + 2^{|E|-3-k} + 2k.
\end{aligned}$$

It remains to prove that

$$2^{|E|-2} + 2^{|E|-4} - 3|E| + 11 + 2^{|E|-3-k} + 2k \leq 2^{|E|-2} + 2^{|E|-3} - 3|E| + 13.$$

It is sufficient to prove that

$$2^{|E|-3-k} + 2k \leq 2^{|E|-3} + 2.$$

Let $m = |E| - 3$. Consider the function $f(x) = 2^{m-x} + 2x$ defined on the closed interval from 1 to m . Calculate the first and second derivatives:

$$\begin{aligned}
f'(x) &= -2^{m-x} \ln 2 + 2; \\
f''(x) &= 2^{m-x} \ln^2 2 > 0.
\end{aligned}$$

Therefore, f is a convex function, so it takes its maximum value at the end point of the interval. So it suffices to verify the inequality for $k = 1$ and $k = m = |E| - 3$. For $k = 1$, the inequality takes the form

$$2^{|E|-3} + 2 \leq 2^{|E|-3} + 2$$

and is valid. Consider the case $k = m = |E| - 3$. In this case, we need to prove that $2m \leq 2^m + 2$. Dividing both sides by 2, one can obtain $m - 1 \leq 2^{m-1}$. By Bernulli's inequality,

$$2^{m-1} = (1 + 1)^{m-1} \geq 1 + m - 1 = m > m - 1,$$

and the lemma is proved. \square

In some cases, we can use a better upper bound than Lemma 3 gives. In such cases, the following lemma is useful.

Lemma 7. *Assume that each part of $K(v)$ contains at least three vertices and edges $e_1, e_2, e_3 \in E$ induce a triangle. Let $E' = E \setminus \{e_1, e_2, e_3\}$. If there are X interesting garlands G such that $E(G) \subseteq E'$, Y nonempty subsets of E' continuable outside E' , k edges $h \in E'$ for each of which there exists an edge $h' \in \Delta$ such that h and h' induce a coordinated subgraph of type $K(2, 1)$, then the number of interesting garlands does not exceed $2X + 2Y - 5|E| + 19 + k$.*

P r o o f. Consider an arbitrary subset $F \subseteq E'$ and estimate the number of interesting garlands G such that $F \subseteq E(G)$ and $E(F) \cap \{e_1, e_2, e_3\} \neq \emptyset$. Let $H = E(F) \cap \{e_1, e_2, e_3\}$. Note that F is a nonempty subset of E' continuable outside E' .

Case 1. $|F| = 1$. Let $F = \{f\}$. If f does not induce a coordinated subgraph of type $K(2, 1)$ with any of edges from $\{e_1, e_2, e_3\}$, then H must be an E -subgraph. This means that either H is a one-element set or $H = \{e_1, e_2, e_3\}$. In both cases, $F \cup H$ does not induce an interesting garland. Therefore, f must induce a coordinated subgraph of type $K(2, 1)$ with some edge from $\{e_1, e_2, e_3\}$. By the lemma hypothesis, there are k such edges. Note that, in this case, H must be a one-element set and it is uniquely defined by the edge f . So, in this case, there are at most k interesting garlands.

Case 2. $|F| > 1$ and F does not induce interesting garlands. By Lemma 5, there are at most two subsets H . Note that F can be chosen in

$$Y - X - (|E| - 3) = Y - X - |E| + 3$$

ways; so, in this case, the number of interesting garlands does not exceed

$$2(Y - X - |E| + 3) = 2Y - 2X - 2|E| + 6.$$

Case 3. $|F| > 1$ and F induces interesting garlands. In this case, by Lemma 7, there are at most three sets H , so the number of interesting garlands does not exceed

$$3(X - (|E| - 3)) = 3X - 3|E| + 9.$$

There are X interesting garlands whose edge aggregates are in the set E' and four interesting garlands whose edge aggregates are in the set $\{e_1, e_2, e_3\}$. Therefore, the number of interesting garlands does not exceed

$$2Y - 2X - 2|E| + 6 + 3X - 3|E| + 9 + k + X + 4 = 2Y + 2X - 5|E| + 19,$$

and the lemma is proved. □

Lemma 8. *Assume that each active part of $K(v)$ contains at least three vertices. If $\langle E \rangle$ contains two triangles that do not have a common edge, then the number of interesting garlands does not exceed $2^{|E|-2} - 12|E| + 58$.*

P r o o f. Denote by Δ the edge set of one triangle, and let $E' = E \setminus \Delta$. Let X be the number of interesting garlands G such that $E(G) \subseteq E'$, and let Y be the number of nonempty subsets of E' continuable outside E' .

By Lemma 6,

$$X \leq 2^{|E|-5} + 2^{|E|-6} - 3(|E| - 3) + 13 = 3 \cdot 2^{|E|-6} - 3|E| + 22.$$

Denote by Δ' the edge set of the other triangle. To estimate the number of nonempty subsets of E' continuable outside E' , consider an arbitrary $F \subseteq E' \setminus \Delta'$ and estimate the number of sets

$F' \subseteq \Delta'$ such that $F \cup F'$ is a continuable set. Note that if $F \cup F'$ is a continuable set, then F' is either empty, or a one-element set, or Δ' . Since F can be chosen in $2^{|E|-6}$ ways, there are at most $5 \cdot 2^{|E|-6}$ continuable sets; thus, there are at most $5 \cdot 2^{|E|-6}$ nonempty continuable sets.

Using Lemma 7 and assuming $k \leq |E| - 3$, one can conclude that the number of interesting garlands does not exceed

$$\begin{aligned} & 2 \cdot (3 \cdot 2^{|E|-6} - 3|E| + 22) + 2 \cdot (5 \cdot 2^{|E|-6} - 1) - 5|E| + 19 + |E| - 3 = \\ & = 16 \cdot 2^{|E|-6} - 12|E| + 55 = 2^{|E|-2} - 12|E| + 58. \end{aligned}$$

□

4. Upper bound for the invariant pt in the case where $\langle E \rangle$ contains a Ξ_2 -subgraph

The main goal of this section is, in the case where $\langle E \rangle$ contains a Ξ_2 -subgraph, to obtain a better upper bound for the number of interesting garlands. Recall that a pair of edges $e, f \in E$ induces a Ξ_2 -subgraph if it induces a triangle in $K(v)$ whose third edge is not in E .

Lemma 9. *If edges f and g induce a Ξ_2 -subgraph, then there is no a garland G such that $f, g \in E(G)$.*

P r o o f. Define $f = xy$ and $g = yz$ and note that the vertices x and z are in different parts of $K(v)$.

By contradiction, assume that there is a garland G such that $f, g \in E(G)$. The edges f and g are adjacent, so they are in the same E -subgraph H' of the garland G . Since the vertices y and z are in different parts of $K(v)$, the edge yz must be in the subgraph H' , so $yz \in E$, a contradiction. □

Now we are ready to prove a better upper bound.

Lemma 10. *Let $|E| \geq 4$ and each active part of $K(v)$ contains at least three vertices. If $\langle E \rangle$ contains a Ξ_2 -subgraph, then the number of interesting garlands does not exceed $2^{|E|-1}$.*

P r o o f. Let edges f and g induce a Ξ_2 -subgraph. Consider the cases.

Case 1. Assume that $\langle E \rangle$ contains a coordinated subgraph of type $K(|E| - 1, 1)$. Without loss of generality, assume that f is not in the coordinated subgraph of type $K(|E| - 1, 1)$. Then there are two possible configurations for E , they are shown in Fig. 2.

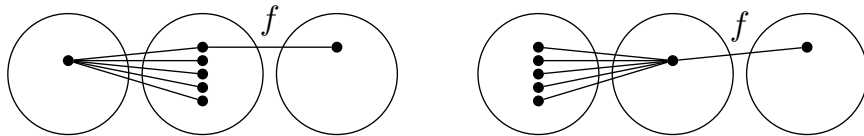


Figure 2. Ξ_2 -subgraph and coordinated subgraph of type $K(|E| - 1, 1)$.

In both cases, the edge f is in exactly one interesting garland. There are exactly $2^{|E|-1} - 1$ interesting garlands that do not contain the edge f . Thus, there are exactly

$$2^{|E|-1} - 1 + 1 = 2^{|E|-1}$$

interesting garlands.

Case 2. Now consider the case where $\langle E \rangle$ does not contain a coordinated subgraph of type $K(|E| - 1, 1)$. By Lemma 9, the edge aggregate of any garland is in the set $E \setminus \{f\}$ or in the set $E \setminus \{e\}$. By Lemma 3, there are at most $2^{|E|-2} + 1$ interesting garlands whose edge aggregates are in the set $E \setminus \{f\}$ and at most $2^{|E|-2} + 1$ interesting garlands whose edge aggregates are in the set $E \setminus \{e\}$. Note that $|E| - 2$ interesting one-edge garlands were counted twice, so the number of interesting garlands does not exceed

$$2 \cdot (2^{|E|-2+1}) - (|E| - 2) = 2^{|E|-1} - |E| + 4 \leq 2^{|E|-1},$$

and the lemma is proved. \square

In the case where $\langle E \rangle$ does not contain a coordinated subgraph of type $K(|E| - 1, 1)$, we can obtain a better upper bound.

Lemma 11. *Let $|E| \geq 5$, and let each part of $K(v)$ contain at least three vertices. If $\langle E \rangle$ does not contain a coordinated subgraph of type $K(|E| - 1, 1)$ and contains a Ξ_2 -subgraph, then the number of interesting garlands does not exceed $2^{|E|-1} - |E| + 1$.*

P r o o f. By Lemma 6, it is sufficient to consider the case where $\langle E \rangle$ does not contain a triangle.

Let edges e and g induce a Ξ_2 -subgraph. Note that, by Lemma 9, the edge aggregate of any garland cannot contain both the edges e and g .

Case 1. Assume that the edge g is in two different Ξ_2 -subgraphs. Define $E_e = E \setminus \{g\}$ and note that if $e \in E(G)$, then $E(G) \subseteq E_e$. By the lemma hypothesis, E_e does not induce a coordinated subgraph of type $K(|E| - 1, 1)$, so, by Lemma 3, there are at most $2^{|E|-2} + 1$ interesting garlands whose edge aggregates are subsets of E_e . In this case, it remains to estimate the number of interesting garlands whose edge aggregates contain the edge g .

Let $f \neq e$ be an edge such that the edges f and g induce a Ξ_2 -subgraph. Define $E_g = E \setminus \{f, e\}$ and note that if the edge g is in G , then $E(G) \subseteq E_g$.

Case 1.1. Assume that E_g induces a coordinated subgraph of type $K(|E_g|, 1)$. Then there are exactly $2^{|E|-2} - 1$ interesting garlands whose edge aggregates are subsets of E_g . Recall that there are at most $2^{|E|-2} + 1$ interesting garlands whose edge aggregates are subsets of E_e . Note also that there are $2^{|E|-3}$ interesting garlands whose edge aggregates are subsets of $E \setminus \{e, g\}$ (which are $2^{|E|-3} - 1$ interesting garlands of type $K(s, 1)$ whose edge aggregates are subsets of $E \setminus \{e, g, f\}$ and a one-edge garland induced by f). All these garlands were counted twice, so the number of interesting garlands does not exceed

$$2^{|E|-2} - 1 + 2^{|E|-2} + 1 - 2^{|E|-3} = 2^{|E|-1} - 2^{|E|-3}.$$

It remains to prove that

$$2^{|E|-1} - 2^{|E|-3} \leq 2^{|E|-1} - |E| + 1.$$

To do this, it suffices to prove that the inequality $n - 1 \leq 2^{n-3}$ holds for all integer $n \geq 5$. Let us prove this by induction on n .

The base case. If $n = 5$, then $5 - 1 = 4 = 2^2$ and the inequality is verified.

The induction step. Assume that the inequality is proved for n and prove it for $n + 1$. We need to prove $2^{n-2} \geq n$. By the induction hypothesis,

$$2^{n-2} = 2 \cdot 2^{n-3} \geq 2 \cdot (n - 1) = n + n - 2 \geq n + 3 > n,$$

and the inequality is proved.

Case 1.2. Now consider the case where E_g does not induce a coordinated subgraph of type $K(|E_g|, 1)$. In this case, by Lemma 3, there are at most $2^{|E|-3} + 1$ interesting garlands whose edge aggregates are subsets of E_g . Note also that there are $|E| - 3$ interesting one-edge garlands induced by edges from $E \setminus \{e, g\}$. So, the number of interesting garlands does not exceed

$$2^{|E|-3} + 1 + 2^{|E|-2} + 1 - (|E| - 3) = 3 \cdot 2^{|E|-3} + 5.$$

It remains to prove that

$$3 \cdot 2^{|E|-3} - |E| + 5 \leq 2^{|E|-1} - |E| + 1.$$

It suffices to check that

$$3 \cdot 2^{|E|-3} + 4 \leq 2^{|E|-1} = 4 \cdot 2^{|E|-3},$$

which is equivalent to $4 \leq 2^{|E|-3}$, which is true because $|E| \geq 5$.

Case 2. Now consider the case where the edges f and g are in only one Ξ_2 -subgraph. Define $E_g = E \setminus \{e\}$ and $E_e = E \setminus \{e\}$. Note that, by Lemma 3, there are at most $2^{|E|-1} + 1$ interesting garlands whose edge aggregates are subsets of E_g and at most $2^{|E|-1} + 1$ interesting garlands whose edge aggregates are subsets of E_e .

Assume that there exist at least $|E| + 1$ interesting garlands whose edge aggregates are subsets of $E \setminus \{e, g\}$. In this case, there are at most

$$2(2^{|E|-1} + 1) - (|E| + 1) = 2^{|E|} + 2 - |E| - 1 = 2^{|E|} - |E| + 1$$

interesting garlands. Therefore, it suffices to prove the lemma in the case when there are at most $|E|$ garlands whose edge aggregates are subsets of $E \setminus \{e, g\}$.

Note that, in any case, there are $|E| - 2$ interesting one-edge garlands induced by edges from $E \setminus \{e, g\}$. Consequently, there are at most two interesting garlands whose edge aggregates are subsets of $E \setminus \{e, g\}$ and contain at least two edges.

Note that it suffices to prove that the edge f is in at most $2^{|E|-2} - |E|$ interesting garlands. Indeed, in this case, since there are at most $2^{|E|-1} + 1$ interesting garland whose edge aggregates are subsets of E_g , the total number of interesting garlands does not exceed

$$2^{|E|-2} - |E| + 2^{|E|-2} + 1 = 2^{|E|-1} - |E| + 1.$$

By analogy, it suffices to prove that the edge g is in at most $2^{|E|-2} - |E|$ interesting garlands.

To prove this, we build either $|E|$ subsets $F \subseteq E \setminus \{e, f\}$ such that $F \cup \{f\}$ does not induce an interesting garland or $|E|$ subsets $F \subseteq E \setminus \{e, f\}$ such that $F \cup \{g\}$ does not induce an interesting garland.

Case 2.1. Assume that the edge e is in three different coordinated subgraphs of type $K(2, 1)$. In this case, either $E \setminus \{e, g\}$ contains a coordinated subgraph of type $K(3, 1)$ and so contains at least three interesting garlands with more than one edge or g is in two different Ξ_2 -subgraphs. Both cases are impossible.

Case 2.2. Assume that the edge e is in two different coordinated subgraphs of type $K(2, 1)$. Let edges h_1 and h_2 be such that the pairs of edges $\{e, h_1\}$ and $\{e, h_2\}$ induce Ξ_2 -subgraphs. Note that, in this case, the edge e is in the coordinated subgraph of type $K(3, 1)$ because, otherwise, the edge g is in two Ξ_2 -subgraphs.

Consider an arbitrary edge $\hat{h} \in E \setminus \{e, g, h_1, h_2\}$. Note that the set $\{e, \hat{h}\}$ does not induce a coordinated subgraph of type $K(2, 1)$; the set $\{e, h_1, \hat{h}\}$ does not induce an interesting garland, since an interesting three-edge garland must be of type $K(3, 1)$, a triangle, or $K(2, 1) \dot{\cup} K(1, 1)$, and

all cases are impossible. By analogy, the set $\{e, h_2, \hat{h}\}$ does not induce an interesting garland. So, we have built

$$3 \cdot (|E| - 4) = 3|E| - 12 = |E| + 2(|E| - 6)$$

necessary subsets. If $|E| \geq 6$, then the lemma is proved.

Consider the case $|E| = 5$. Denote by g' a single edge from $E \setminus \{e, g, h_1, h_2\}$. Note, that the sets $\{g, h_1\}$, $\{g, h_2\}$, $\{g, h_1, h_2\}$, $\{g, g', h_1\}$, and $\{g, g', h_2\}$ do not induce interesting garlands (see Fig. 3) and, in this case, the lemma is proved.

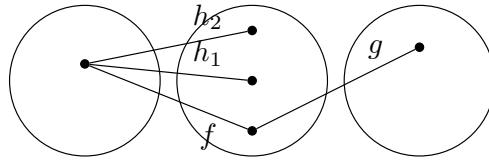


Figure 3. Coordinated subgraph of type $K(3, 1)$ and Ξ_2 -subgraph.

Case 2.3. Assume that the edge e is in exactly one coordinated subgraph of type $K(2, 1)$. Denote the second edge of this subgraph by h .

Case 2.3.1. Assume that the edge g is not in any coordinated subgraph of type $K(2, 1)$. Consider an arbitrary edge $f \in E \setminus \{e, g\}$. Note that the pair of edges $\{g, f\}$ does not induce a coordinated subgraph of type $K(2, 1)$. Note also that, for an arbitrary edge $f \in E \setminus \{e, g, h\}$, the triple of edges $\{g, h, f\}$ does not induce an interesting garland. Therefore, we have built

$$|E| - 2 + |E| - 3 = 2|E| - 5 \geq |E|$$

necessary subsets for the edge g .

Case 2.3.2. Assume that the edge g is in the coordinated subgraph of type $K(2, 1)$. By the previous cases, it suffices to consider the case where g is in exactly one coordinated subgraph of type $K(2, 1)$. Denote the second edge of this subgraph by h' . Note that the edges e, g, h , and h' form one of the two configurations shown in Fig. 4.

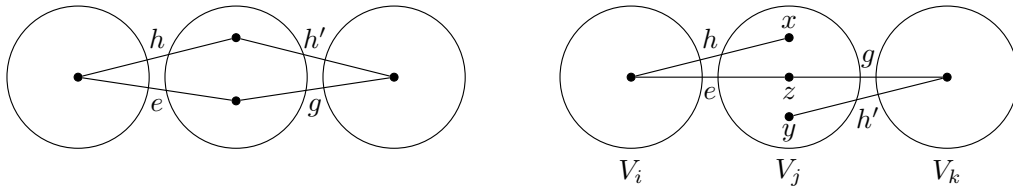


Figure 4. Ξ_2 -subgraph and two coordinated subgraphs of type $K(2, 1)$.

Consider an arbitrary edge $f \in E \setminus \{e, g, h'\}$. Note that the pair of edges $\{g, f\}$ does not induce a coordinated subgraph of type $K(2, 1)$. Note also that, for an arbitrary edge $f \in E \setminus \{e, g, h, h'\}$, the triple of edges $\{g, h, f\}$ does not induce an interesting garland. Thus, we have built

$$|E| - 3 + |E| - 4 = 2|E| - 7$$

necessary subsets for the edge g . By analogy, we can build $2|E| - 7$ necessary subsets for the edge f .

Let \hat{h} be an arbitrary edge from $E \setminus \{e, g, h, h'\}$. Consider the set $\{h, h', \hat{h}\}$. If the set $\{e, h, h', \hat{h}\}$ induces an interesting garland, this garland must be of type $K(2, 1) \dot{\cup} K(2, 1)$; the same is true for

the set $\{g, h, h', \hat{h}\}$. Since both the sets $\{\hat{h}, h\}$ and $\{\hat{h}, h'\}$ cannot induce a coordinated subgraph of type $K(2, 1)$, at least one of the sets $\{e, h, h', \hat{h}\}$ and $\{g, h, h', \hat{h}\}$ does not induce an interesting garland. Therefore, we have built $2|E| - 6$ necessary subsets for the edge e or for the edge g , and the lemma is proved if $|E| > 6$.

In the case where the edges e, g, h , and h' form the first configuration shown in Fig. 4, the sets $\{e, h, h'\}$ and $\{g, h, h'\}$ do not induce interesting garlands; consequently, we have built $2|E| - 5$ necessary subsets for the edge e or for the edge g .

It remains to prove the lemma in the case where $|E| = 5$ and the edges e, g, h , and h' form the second configuration shown in Fig. 4. Note that there are at most eight interesting garlands whose edge aggregates are subsets of $\{g, f, h', h\}$: four one-edge garlands, two garlands of type $K(2, 1)$, and possibly two garlands of type $K(2, 1) \dot{\cup} K(1, 1)$. So it suffices to prove that there are at most five interesting garlands that contain the edge \hat{h} .

Case 2.3.2.1. Assume that the edge \hat{h} is incident neither to the vertex x nor the vertex y . In this case, the edge \hat{h} is not in any coordinated subgraph of type $K(2, 1)$ because, otherwise, either the edge e is in two coordinated subgraphs of type $K(2, 1)$ or the edge e is in two coordinated subgraphs of type $K(2, 1)$, both the cases lead to a contradiction. Therefore, the edge \hat{h} is not in any two-edge garland. Moreover, since $\langle E \rangle$ does not contain a triangle, the edge \hat{h} is not in any one-element garland except for the one-edge garland.

Assume that the edge \hat{h} is in some garland G of cardinality more than one. In this case, the garland G must destroy some part. The vertex aggregate $V(G)$ has at most two common vertices with any part $V \neq V_j$. Thus, V_j is the only part that can be destroyed by G . Note also that the edge \hat{h} is not incident to the vertex z because, otherwise, the edge e or g is in two Ξ_2 -subgraphs. Therefore, either $e \in E(G)$ or $g \in E(G)$. Note also that this two edges cannot be in $E(G)$ simultaneously; thus, the edge \hat{h} is in at most two garlands with cardinality more than one. Therefore, in this case, the edge \hat{h} is in at most three garlands, and the lemma is proved in this case.

Case 2.3.2.2. Now assume that the edge \hat{h} is incident either to the vertex x or the vertex y . Without loss of generality, assume that \hat{h} is incident to x . Note that, in this case, the edge aggregate that contains the edge \hat{h} can be one of the following four sets: $\{\hat{h}\}$, $\{\hat{h}, h\}$, $\{\hat{h}, g, h'\}$, and $\{\hat{h}, h, g, h'\}$, and the lemma is proved in this case.

Case 2.4. It remains to consider the case where neither the edge e nor the edge g are in any coordinated subgraph of type $K(2, 1)$. In this case, neither the set $\{e, f\}$ nor the set $\{g, f\}$ induce an interesting garland for any edge $f \in E \setminus \{e, g\}$, so we have built $|E| - 2$ necessary subsets for the edges e and g .

Since $|E \setminus \{e, g\}| \geq 3$, consider three arbitrary edges from this set and denote them by h_1, h_2 and h_3 . Define $H = \{h_1, h_2, h_3\}$. Note that there exist exactly three two-element subsets $H' \subseteq H$. If, for any two-element subset $H' \subset H$, the set $H' \cup \{e\}$ does not induce an interesting garland, then we have built $|E| + 1$ necessary subsets for the edge e . So it suffices to consider the case where, for some two-element subset $H' \subset H$, the set $H' \cup \{e\}$ induces an interesting garland. Without loss of generality, $H' = \{h_1, h_2\}$. Since $\langle E \rangle$ does not contain a triangle and the edge e is not in any coordinated subgraph of type $K(2, 1)S$, an interesting three-edge garland must be of type $K(2, 1) \dot{\cup} K(1, 1)$ or $K(1, 1, 1)$.

Case 2.4.1. The set $H' \cup \{e\}$ induces an interesting garland of type $K(1, 1, 1)$. Note that, in this case, the set $\{g, h_1, h_2\}$ does not induce an interesting garland. Note also that at least one of the sets $\{g, h_3, h_1\}$ and $\{g, h_3, h_2\}$ does not induce an interesting garland. Therefore, we have built $|E|$ necessary subsets for the edge g , and, in this case, the lemma is proved.

Case 2.4.2. The set $H' \cup \{e\}$ induces an interesting garland of type $K(2, 1) \dot{\cup} K(1, 1)$. Note

that, in this case, sets $\{g, h_3, h_1\}$ and $\{g, h_3, h_2\}$ does not induce interesting garlands. Therefore, we have built $|E|$ necessary subsets for the edge g and the lemma is proved. \square

Lemma 12. *Assume that each active part of $K(v)$ contains at least three vertices. If an edge $e \in E$ is in $k \leq |E| - 2$ different Ξ_2 -subgraphs, then*

- (1) *the edge e is in at most $2^{|E|-k-1}$ interesting garlands;*
- (2) *if $\langle E \rangle$ does not contain a coordinated subgraph of type $K(|E| - k, 1)$, then the edge e is in at most $2^{|E|-k-1} - |E| + k + 2$ interesting garlands.*

P r o o f. Let edges h_1, h_2, \dots, h_k be such that the pair of edges $\{h_i, e\}$ induces a Ξ_2 -subgraph for all $i = 1, 2, \dots, k$. Define $E' = E \setminus \{h_1, h_2, \dots, h_k\}$.

Let F be an edge aggregate of some interesting garland, and let $e \in F$. Note that $h_i \notin F$ for all $i = 1, 2, \dots, k$. Therefore, $F \subseteq E'$.

If E' does not induce a coordinated subgraph of type $K(|E| - k, 1)$, then, by Lemma 3, there exist at most $2^{|E|-k-1}$ interesting garlands whose edge aggregates are subsets of E' . Note that among them there are $|E| - k - 1$ one-edge garlands that do not contain the edge e . Consequently, the edge e is in at most

$$2^{|E|-k-1} + 1 - (|E| - k - 1) = 2^{|E|-k-1} - |E| + k + 2$$

interesting garland, and the lemma is proved. \square

Lemma 13. *Assume that $|E| \geq 6$ and the subgraph $\langle E \rangle$ contains at least two Ξ_2 -subgraphs and does not contain a coordinated subgraph of type $K(|E| - 2)$.*

- (1) *If there exists an edge, which is in two Ξ_2 -subgraphs, then the number of interesting garlands does not exceed $2^{|E|-2} + 2^{|E|-3} - |E| + 5$.*
- (2) *If there exist two Ξ_2 -subgraphs without common edges, then the number of interesting garlands does not exceed $2^{|E|-1} - 3|E| + 6$.*

P r o o f. Prove the first statement of the lemma. Assume that an edge e is in two Ξ_2 -subgraphs. Since $\langle E \rangle$ does not contain a coordinated subgraph of type $K(|E| - 2)$, by Lemma 12, the edge e is in at most $2^{|E|-3} - |E| + 4$ interesting garlands. By Lemma 3, there are at most $2^{|E|-2} + 1$ interesting garlands whose edge aggregates are subsets of $E \setminus \{e\}$. Consequently, there are at most

$$2^{|E|-3} - |E| + 4 + 2^{|E|-2} + 1 = 2^{|E|-2} + 2^{|E|-3} - |E| + 5$$

interesting garlands.

Now prove the second statement of the lemma. Assume that there are two Ξ_2 -subgraphs without common edges. Let edges e_1 and e_2 induce a Ξ_2 -subgraph. By Lemma 9, the edge aggregate of any garland is a subset of $E \setminus \{e_1\}$ or $E \setminus \{e_2\}$. By Lemma 11, there are at most

$$2^{|E|-2} - (|E| - 1) + 1 = 2^{|E|-2} - |E| + 2$$

interesting garlands whose edge aggregates are subsets of $E \setminus \{e_1\}$ and there are at most $2^{|E|-2} - |E| + 2$ interesting garlands whose edge aggregates are subsets of $E \setminus \{e_2\}$. Note that $|E| - 2$ one-edge garlands were counted twice, so the number of interesting garlands does not exceed

$$2(2^{|E|-2} - |E| + 2) - (|E| - 2) = 2^{|E|-1} - 3|E| + 6,$$

and the lemma is proved. \square

Lemma 14. *Assume that each active part of $K(v)$ contains at least three vertices and $\langle E \rangle$ contains a triangle. If an edge e is in the triangle from $\langle E \rangle$ and is in k different Ξ_2 -subgraphs, then e is in at most $2^{|E|-k-2} - |E| + k + 3$ interesting garlands.*

P r o o f. Let Δ be the edge set of a triangle from $\langle E \rangle$ that contains the edge e , and let edges h_1, h_2, \dots, h_k be such that the pairs of edges $\{e, h_i\}$ induce Ξ_2 -subgraphs for all $i = 1, 2, \dots, k$. Define $E' = E \setminus (\Delta \cup \{h_1, h_2, \dots, h_k\})$.

Let G be an arbitrary garland such that $e \in E(G)$, and let $F = E(G)$. Note that $h_i \notin F$ for all $i = 1, 2, \dots, k$. Define $F' = F \cap E'$. Note also that either $F \cap \Delta = \{e\}$ or $F \cap \Delta = \Delta$.

If $F \cap \Delta = \{e\}$, then F' can be chosen in $2^{|E|-k-3}$ ways. If $F \cap \Delta = \Delta$, then $|F'| > 1$ and, consequently, F' can be chosen in

$$2^{|E|-k-3} - (|E| - k - 3)$$

ways.

Thus, the edge e is in at most

$$2^{|E|-k-3} + 2^{|E|-k-3} - (|E| - k - 3) = 2^{|E|-k-2} - |E| + k + 3$$

interesting garlands. □

Lemma 15. *Assume that each part of $K(v)$ contains at least three vertices and $\langle E \rangle$ contains two triangles with a common edge e . If the edge e is in k different Ξ_2 -subgraphs, then the edge e is in at most $2^{|E|-k-3} - 3|E| + 3k + 15$ interesting garlands.*

P r o o f. Let Δ_1 and Δ_2 be the edge sets of triangles from $\langle E \rangle$ that contain the edge e . Let edges h_1, h_2, \dots, h_k be such that the pair of edges $\{e, h_i\}$ induces a Ξ_2 -subgraph for all $i = 1, 2, \dots, k$. Define $E' = E \setminus (\Delta_1 \cup \Delta_2 \cup \{h_1, h_2, \dots, h_k\})$ and $E_\Delta = \Delta_1 = \Delta_2$.

Let G be an arbitrary garland such that $e \in E(G)$. Define $F = E(G)$ and note that $h_i \notin F$ for all $i = 1, 2, \dots, k$. Define $F' = F \cap E'$ and $F_\Delta = F \cap E_\Delta$. Note also that either $F_\Delta = \{e\}$, or $F_\Delta = \Delta_1$, or $F_\Delta = \Delta_2$, or $F_\Delta = \Delta$.

Consider an arbitrary $F' \subseteq E'$ and count the number of interesting garlands G such that $e \in E(G)$ and $F' \subseteq E(G)$. To do this, we estimate the number of ways to choose a subset $F_\Delta \subseteq \Delta$ such that $F' \cup F_\Delta$ induces an interesting garland. If $|F'| \neq 1$, then F_Δ can be chosen in four ways; if $|F'| = 1$, then $F_\Delta \neq \Delta_1$ and $F_\Delta \neq \Delta_2$. Let $f \in F'$. Note that if the set $\{f, e\}$ induces an interesting garland G' , then G' is of type $K(2, 1)$ and $\{f\} \cup \Delta$ does not induce an interesting garland. Therefore, if $|F'| = 1$, then F_Δ can be chosen in at most one way.

Thus, the edge e is in at most

$$4 \cdot (2^{|E|-k-5} - (|E| - k - 5)) + (|E| - k - 5) = 2^{|E|-k-3} - 3|E| + 3k + 15$$

interesting garlands. □

5. Upper bound for the invariant pt when $|E|$ is small

The main goal of this section is to prove some upper bounds for the number of interesting garlands in the case where E contains relatively small number of elements.

Lemma 16. *Assume that each active part of $K(v)$ contains at least three vertices, $\langle E \rangle$ contains a triangle, does not contain a coordinated subgraph of type $K(4, 1)$, and $|E| = 8$. Denote the edge set of the triangle by Δ . If $\langle E \setminus \Delta \rangle$ contains at least two Ξ_2 -subgraphs, then the number of interesting garlands does not exceed 46.*

P r o o f. Let X be the number of interesting garlands whose edge aggregates are subsets of E' , and let Y be the number of nonempty subsets of E' continuable outside E' . Let k be the number of edges $h \in E \setminus \Delta$ for which there exists an edge h' such that h and h' induce a coordinated subgraph of type $K(2, 1)$.

Case 1. Assume that there exist three edges $e, f_1, f_2 \in E \setminus \Delta$ such that the pairs of edges $\{e, f_1\}$ and $\{e, f_2\}$ induce Ξ_2 -subgraphs.

In this case, any continuable set can contain neither the set $\{e, f_1\}$ nor $\{e, f_2\}$. Consequently,

$$Y \leq 2^5 - 1 - 2 \cdot 2^3 + 2^2 = 32 - 1 - 16 + 4 = 1.$$

By Lemma 3, there are at most $2^3 + 1 = 9$ interesting garlands whose edge aggregates are subsets of $E' \setminus \{e\}$.

Case 1.1. Assume that $E' \setminus \{f_1, f_2\}$ induces a coordinated subgraph of type $K(3, 1)$. Denote this subgraph by H . In this case, the edge e is in exactly four interesting garlands whose edge aggregates are subsets of E' . Consequently, $X \leq 9 + 4 = 13$.

If there exists an edge from the triangle adjacent to all edges from H , then either $\langle E \rangle$ contains a coordinated subgraph of type $K(4, 1)$, which contradicts the lemma hypothesis, or any edge from H does not induce a coordinated subgraph of type $K(2, 1)$ with any edge from the triangle. Thus, $k \leq 2$. By Lemma 7, the number of interesting garlands does not exceed

$$2 \cdot 13 + 2 \cdot 19 - 5 \cdot 8 + 19 + 2 \leq 26 + 38 - 40 + 19 + 2 = 45.$$

Case 1.2. Define $E'' = E' \setminus \{f_1, f_2\}$ and assume that E'' does not induce a coordinated subgraph of type $K(3, 1)$. In this case, there exist at most 5 interesting garlands whose edge aggregates are subsets of E'' , and the edge e is not in two (one-edge) of them. Consequently, the edge e is in at most three interesting garlands whose edge aggregates are subsets of E' . Therefore, $X \leq 9 + 3 = 12$, $k \leq 5$, and, using Lemma 7, one can deduce that the number of interesting garlands does not exceed

$$2 \cdot 12 + 2 \cdot 19 - 5 \cdot 8 + 19 + 5 = 24 + 38 - 40 + 24 = 46.$$

Case 2. Now consider the case where $\langle E \rangle$ contains two Ξ_2 -subgraph without common edges. Let the pairs of edges $\{e_1, e_2\}$ and $\{g_1, g_2\}$ induce Ξ_2 -subgraphs.

Estimate the number of nonempty continuable sets. Let $F \subseteq E'$ be a nonempty continuable set, $F_e = F \cap \{e_1, e_2\}$, $F_g = F \cap \{g_1, g_2\}$, and let $F' = F \setminus \{e_1, e_2, g_1, g_2\}$. Note that $F = F_e \cup F_g \cup F'$. To obtain an upper bound, it suffices to count the number of ways to choose the sets F_e , F_g , and F' . Note that $F' \subseteq E' \setminus \{e_1, e_2, g_1, g_2\}$ and then it can be chosen in

$$2^{|E' \setminus \{e_1, e_2, g_1, g_2\}|} = 2^{5-4} = 2$$

ways. The set F_e can be either empty or one-element because, by Lemma 9, the edges e_1 and e_2 cannot be in the edge aggregate of any garland; so F_e can be chosen in three ways. By analogy, the set F_g can also be chosen in three ways. Excluding the empty set, one can conclude that $Y \leq 2 \cdot 3 \cdot 3 - 1 = 17$.

Note that the set $E' \setminus \{e_1\}$ contains a Ξ_2 -subgraph and by Lemma 10, there are at most

$$2^{|E' \setminus \{e_1\}| - 1} = 2^{4-1} = 8$$

interesting garlands whose edge aggregates are subsets of $E' \setminus \{e_1\}$. Note that three one-edge of them do not contain the edge e_2 ; consequently, e_2 is in at most 5 interesting garlands whose edge aggregates are subsets of E' . By analogy, there are at most 8 interesting garlands whose edge aggregates are subsets of $E' \setminus \{e_2\}$. Therefore, $X \leq 8 + 5 = 13$.

Thus, by Lemma 7, the number of interesting garlands does not exceed

$$2 \cdot 13 + 2 \cdot 17 - 5 \cdot 8 + 19 + 5 = 26 + 34 - 40 + 19 + 5 = 44,$$

and the lemma is proved. \square

Lemma 17. *Assume that each part of $K(v)$ contains at least three vertices and $\langle E \rangle$ does not contain a coordinated subgraph of type $K(5, 1)$. If $|E| = 7$ and $\langle E \rangle$ contains at least three Ξ_2 -subgraphs, then the number of interesting garlands does not exceed 41.*

P r o o f. Assume that there exists an edge $e \in E$ that is in three Ξ_2 -subgraphs. By Lemma 3, there exist at most $2^5 + 1 = 33$ interesting garlands whose edge aggregates are subsets of $E \setminus \{e\}$. By Lemma 12, one can deduce that e is in at most $2^{7-4} = 8$ interesting garlands. Therefore, there exist at most $33 + 8 = 41$ interesting garlands.

Now assume that there exists an edge $e \in E$ that is in exactly two Ξ_2 -subgraphs. Since $\langle E \rangle$ does not contain a coordinated subgraph of type $K(5, 1)$, by Lemma 12, one can deduce that e is in at most $2^{7-3} - 7 + 2 + 2 = 13$ interesting garlands. The graph $\langle E \setminus \{e\} \rangle$ contains a Ξ_2 -subgraph and does not contain a coordinated subgraph of type $K(5, 1)$. Thus, by Lemma 13, there exist at most $2^5 - 6 + 1 = 28$ interesting garlands whose edge aggregates are subsets of $E \setminus \{e\}$. Therefore, there exist at most $13 + 28 = 41$ interesting garlands.

It remains to consider the case when every edge is in at most one Ξ_2 -subgraph. Let pairs of edges $\{e_1, e_2\}, \{f_1, f_2\}$, and $\{g_1, g_2\}$ induce three different Ξ_2 -subgraphs. Denote the single left edge by h . By Lemma 13, there exist at most $2^5 - 3 \cdot 6 + 6 = 20$ interesting garlands whose edge aggregates are subsets of $E \setminus \{e_1\}$.

Estimate the number of interesting garlands that contain the edge e_1 . Let H be the edge aggregate of such a garland. Define $H' = H \cap \{h\}$, $H_f = H \cap \{f_1, f_2\}$, and $H_g = H \cap \{g_1, g_2\}$ and note that $H = \{e\} \cup H' \cup H_f \cup H_g$. Note also that H' is empty or equal to $\{h\}$. The set H_f can be chosen in three ways (because the edges f_1 and f_2 cannot be in the same garland). By analogy, the set H_g can be chosen in three ways. Therefore, the set H can be chosen in at most $2 \cdot 3 \cdot 3 = 18$ ways. Therefore, e is in at most 18 interesting garlands, the total number of interesting garlands does not exceed $20 + 16 = 36$, and the lemma is proved. \square

Lemma 18. *Assume that each part of $K(v)$ contains at least three vertices, $|E| = 7$, and $\langle E \rangle$ does not contain a coordinated subgraph of type $K(4, 1)$. If every edge is in at most three Ξ_2 -subgraphs and $\langle E \rangle$ contains at least four Ξ_2 -subgraphs, then the number of interesting garlands does not exceed 34.*

P r o o f. Assume that any edge is not in two Ξ_2 -subgraphs. Then

$$\xi_2 \leq \frac{|E|}{2} = \frac{7}{2} < 4,$$

a contradiction. Consequently, there exists an edge that is in at least two Ξ_2 -subgraphs.

Case 1. Assume that there exist an edge e that is in three Ξ_2 -subgraphs. By Lemma 12, there exist at most $2^{6-3-1} + 1 - 3 = 6$ interesting garlands such that e is in them. Also, by Lemma 11, there are at most 28 interesting garlands whose edge aggregates are subsets of $E \setminus \{e\}$. Therefore, there exist at most $28 + 6 = 34$ interesting garlands.

Case 2. Assume that every edge is in at most two Ξ_2 -subgraphs. Consider an edge e that is in exactly two Ξ_2 -subgraphs. Let edges f_1 and f_2 be such that the pairs of edges $\{e, f_1\}$ and $\{e, f_2\}$ induce Ξ_2 -subgraphs.

Case 2.1. Assume that $E \setminus \{f_1, f_2\}$ contains a Ξ_2 -subgraph. By Lemma 11, there exists at most $2^4 - 5 + 1 = 12$ interesting garlands whose edge aggregates are subsets of $E \setminus \{f_1, f_2\}$. Note that four one-edge garlands from them do not contain the edge e . Therefore, e is at most in $12 - 4 = 8$ interesting garlands. The set $E \setminus \{e\}$ contains at least two Ξ_2 -subgraphs and, by Lemma 13, there exist at most

$$\max(2^{6-2} + 2^{6-3} - 6 + 5, 2^{6-1} - 3 \cdot 6 + 6) = \max(2^4 + 2^3 - 1, 2^5 - 18 + 6) = \max(23, 20) = 23$$

interesting garlands whose edge aggregates are subsets of $E \setminus \{e\}$.

Case 2.2. Assume that $E \setminus \{f_1, f_2\}$ does not contain a Ξ_2 -subgraph. By Lemma 12, the edge e is in at most 13 interesting garlands.

Case 2.2.1. Assume that $E \setminus \{e\}$ contains two Ξ_2 -subgraphs without common edges. In this case, by Lemma 13, there exist at most $2^{6-1} - 3 \cdot 6 + 6 = 20$ interesting garlands whose edge aggregates are subsets of $E \setminus \{e\}$. Therefore, there exists at most $20 + 13 = 33$ interesting garlands.

Case 2.2.2. It remains to consider the case when there exists an edge $h \in E \setminus \{e\}$ that is in two Ξ_2 -subgraphs. Since every Ξ_2 -subgraph contains f_1 or f_2 (otherwise, $E \setminus \{f_1, f_2\}$ contains a Ξ_2 -subgraph), the pairs of edges $\{e, f_1\}$, $\{e, f_2\}$, $\{h, f_1\}$, and $\{h, f_2\}$ induce Ξ_2 -subgraphs. This means that the edge aggregate of any garland is a subset of $E \setminus \{f_1, f_2\}$ or $E \setminus \{e, h\}$. Applying Lemma 3 to each of them, one can conclude that there exist at most 17 interesting garlands whose edge aggregates are subsets of each of them. Since three one-edge garlands were counted twice, the total number of interesting garlands does not exceed $17+17-3 = 31$, and the lemma is proved. \square

6. Proof of Theorem 1

The main goal of this section is to prove that a graph $K(n_1, n_2, n_3)$ is chromatically unique if

$$n_1 \geq n_2 \geq n_3 \geq 2, \quad n_1 - n_3 = 5 \quad \text{and} \quad n_1 + n_2 + n_3 \equiv 2 \pmod{3}.$$

The bottom levels of the lattice $NPL(n, 3)$ in the case $n \equiv 2 \pmod{3}$ are shown in Fig. 5. The label above the covering relation shows how the number of edges changes, and the label under the covering relation shows how the invariant pt changes. To prove the theorem, it suffices to verify that the graph $K(q+3, q+1, q-2)$ is chromatically unique for $q \geq 4$ and the graph $K(q+4, q-1, q-1)$ is chromatically unique for $q \geq 3$.

Proposition 1. *The graph $K(q+3, q+1, q-2)$ is chromatically unique for $q \geq 4$.*

P r o o f. By contradiction, assume that the graph $K(u) = K(q+3, q+1, q-2)$ is not chromatically unique. This means that there exists a graph H such that the graphs $K(u)$ and H are chromatically equivalent. Let H be a graph obtained from the graph $K(v)$ by removing some

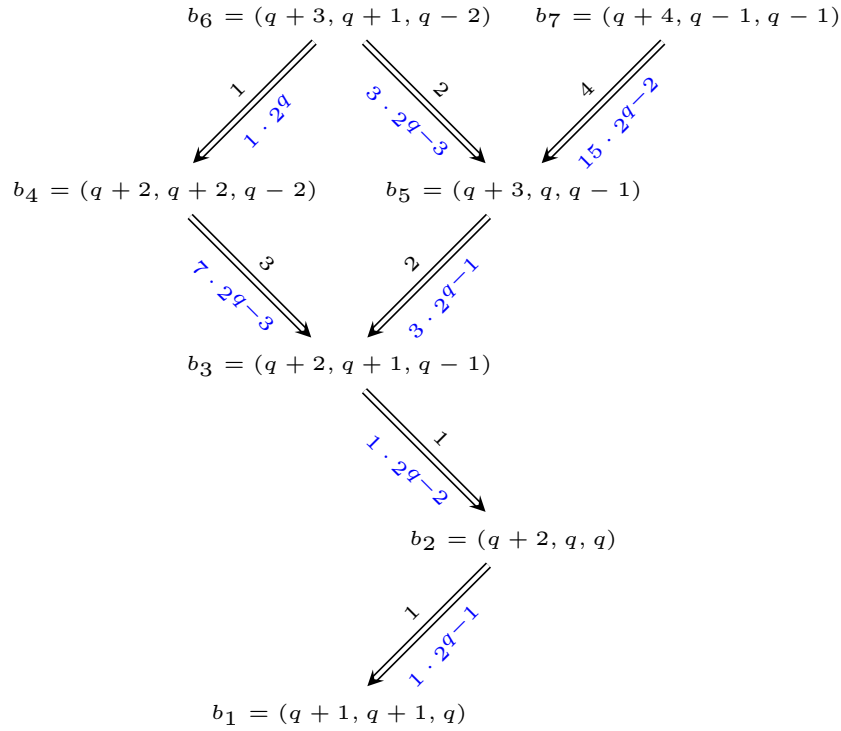


Figure 5. The bottom levels of the lattice $NPL(n, 3)$ in the case $n \equiv 2 \pmod{3}$

set of edges E . Consider the graph $K(v)$. Note that the cases $K(v) = K(q+2, q+2, q-2)$, $K(v) = K(q+3, q, q-1)$, and $K(v) = K(q+2, q+1, q-1)$ are impossible by Lemma 1.

Case 1. Assume that $K(v) = K(q+2, q, q)$ and note that, in this case, $|E| = 5$. Calculate the difference of the invariant pt :

$$\Delta pt(H, K(v)) = \Delta(K(u), K(v)) = 2^q + 7 \cdot 2^{q-3} + 2^{q-2} = 17 \cdot 2^{q-3}.$$

By Lemma 2, one can deduce that

$$17 \cdot 2^{q-3} \leq 2^5 - 1 = 31,$$

which implies $q \leq 3$, a contradiction.

Case 2. Assume that $K(v) = K(q+1, q+1, q)$. Note that, in this case, $|E| = 6$. Calculate the difference of the invariant pt :

$$\Delta pt(H, K(v)) = \Delta(K(u), K(v)) = 2^q + 7 \cdot 2^{q-3} + 2^{q-2} + 2^{q-1} = 21 \cdot 2^{q-3}.$$

By Lemma 2, one can deduce that

$$21 \cdot 2^{q-3} \leq 2^6 - 1 = 63,$$

which implies $q \leq 4$; so it suffices to check the case $q = 4$. In this case, $K(v) = K(5, 5, 4)$ and $\Delta pt(H, K(v)) = 42$. Note that, in this case, E does not induce a coordinated subgraph of type $K(5, 1)$ because such a subgraph is a one-element garland that destroys one part, a contradiction. Thus, by Lemma 3, one can obtain that there are at most $2^5 + 1 = 33$ interesting garlands, which contradicts $\Delta pt(H, K(v)) = 42$, and the proposition is proved. \square

To prove the chromatic uniqueness of the graph $K(q+4, q-1, q-1)$, we need the following two statements.

Lemma 19. *Let E be a subset of edges of some complete tripartite graph $K(v)$. If any two triangles from $\langle E \rangle$ have a common edge, then there exists an edge that is in all triangles from $\langle E \rangle$.*

Before proving this lemma, note that the condition on the graph $K(v)$ to be tripartite is necessary. Four triangles, any two of which have a common edge but no edge is common for all of them, are shown in Fig. 6.

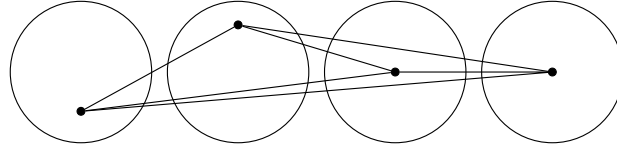


Figure 6. Four triangles, any two of which have a common edge but no edge is common for all of them.

P r o o f. Let Δ_1 and Δ_2 be two triangles from $\langle E \rangle$ that have a common edge. Let $x, y_1,$ and z be vertices of the triangle Δ_1 and let $x, y_2,$ and z be vertices of the triangle Δ_2 . Let us prove that if a triangle Δ from $\langle E \rangle$ have a common edge with the triangles Δ_1 and Δ_2 , then $xz \in \Delta$.

By contradiction, assume that $xz \notin \Delta$. The triangles Δ_1 and Δ have a common edge, without loss of generality, let it be the edge y_1z . Denote by x' the third vertex of Δ_1 and note that the vertices x and x' are in the same part of $K(v)$ because $K(v)$ is a tripartite graph. Note that, in this case, the edge set of Δ_1 is $\{x'y_1, x'z_1, y_1z_2\}$ and it has an empty intersection with the edge set of Δ_2 (see Fig. 7). □

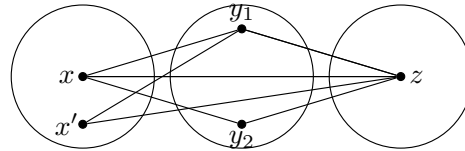


Figure 7. Three triangles in the tripartite graph that do not have a common edge

Proposition 2. *If a graph H is obtained by removing the edge set E from the graph $K(4, 4, 3)$, then the graphs H and $K(7, 2, 2)$ are not chromatically equivalent.*

P r o o f. The graph $K(7, 2, 2)$ has $7 \cdot 2 + 7 \cdot 2 + 2 \cdot 2 = 32$ edges and the graph $K(4, 4, 3)$ has $4 \cdot 4 + 4 \cdot 3 + 4 \cdot 3 = 40$ edges; consequently, $|E| = 40 - 32 = 8$.

Calculate the difference of the invariant I_3 :

$$\begin{aligned} I_3(K(7, 2, 2)) &= 7 \cdot 2 \cdot 2 = 28, \\ I_3(K(4, 4, 3)) &= 4 \cdot 4 \cdot 3 = 48, \\ 20 = \Delta I_3(K(4, 4, 3), K(7, 2, 2)) &= \Delta I_3(K(4, 4, 3), H) = \xi_1 - \xi_2 - 2\xi_3, \end{aligned}$$

$$\begin{aligned} \xi_1 &= 3e_{12} + 4e_{13} + 4e_{23} = 4|E| - e_{12}, \\ 20 &= 32 - e_{12} - \xi_2 - 2\xi_3, \\ \xi_2 + 2\xi_3 + e_{12} &= 12. \end{aligned}$$

Note that every edge is in at most four Ξ_2 -subgraphs.

The inequality $e_{12} \leq 8$ implies $\xi_2 + 2\xi_3 \geq 4$. Note that $E \not\subseteq E_{ij}$ for all nonequal i and j from 1 to 3 because, otherwise, $\xi_2 = \xi_3 = 0$, a contradiction. Assume that $e_{ij} = |E| - 1$ for some i and j . In this case, $\xi_3 = 0$ and $\xi_2 \geq 5$. Denote the single edge from $E \setminus E_{ij}$ by f . Note that all Ξ_2 -subgraphs must contain f ; therefore, $\xi_2 \leq 4$, a contradiction.

Calculate the difference of the invariant pt :

$$\begin{aligned} pt(K(7, 2, 2)) &= 2^6 + 2^1 + 2^1 - 3 = 65, \\ pt(K(4, 4, 3)) &= 2^3 + 2^3 + 2^2 - 3 = 17, \\ \Delta pt(H, K(4, 4, 3)) &= \Delta pt(K(7, 2, 2), K(4, 4, 3)) = 65 - 17 = 48. \end{aligned}$$

Note that $\langle E \rangle$ does not contain a coordinated subgraph of type $K(4, 1)$ because, otherwise, such a subgraph forms a one-element garland that destroys one part, a contradiction.

Case 1. Assume that $\xi_3 = 0$.

Case 1.1. Assume that $e_{12} = 6$. In this case, $\xi_2 = 6$.

Let edges f and g be not in E_{12} . Without loss of generality, assume that $f \in E_{23}$.

Note that any edge from E except for f and g is not adjacent to any vertex from the part V_3 . Since $|V_3| = 3$, there exists a vertex in V_3 not incident to any edge from E . Consequently, any garland cannot destroy the part V_3 .

By Lemma 3, there exist at most $2^5 + 1 = 33$ interesting garlands whose edge aggregates are subsets of E_{12} ; so it suffices to prove that the number of interesting garlands G such that $f \in E(G)$ or $g \in E(G)$ does not exceed 12 because, in this case, the total number of interesting garlands does not exceed $33 + 12 = 45 < 48$.

Case 1.1.1. Assume that edges f and g induce a Ξ_2 -subgraph. Then there exist no garland G such that $f \in E(G)$ and $g \in E(G)$. Note also that, since every edge is in at most four Ξ_2 -subgraphs and every Ξ_2 -subgraph contains f or g , each of the edges f and g is in at least three Ξ_2 -subgraphs.

Note that if f is in a garland G such that $|E(G)| > 1$, then $|G| > 1$; hence, G must destroy some part. It cannot destroy the part V_3 , so it must destroy V_1 or V_2 ; in both cases, $|E(G)| \geq 4$. Note that there exist at least two edges from E_{12} such that f induces a Ξ_2 -subgraph with each of them. This means that there exist at most four edges $h \in E_{12}$ such that f and h can be in the edge aggregate of G . Note also that $E(G)$ must contain at least three edges from E_{12} , and these three edges can be chosen in $\binom{4}{3} = 4$ ways. Therefore, f is in at most five interesting garlands. By analogy, g is in at most five interesting garlands, and the proof is complete in this case.

Case 1.1.2. Assume that edges f and g induce a coordinated subgraph of type $K(2, 1)$. Note that, in this case, each of the edges f and g is in exactly three different Ξ_2 -subgraphs. Note that if f is in a garland G that contains more than one element, then G must destroy some part. Since it cannot destroy V_3 , it must destroy V_1 or V_2 ; in both cases $|E(G)| \geq 4$.

Let $F \subseteq E_{12}$. Note that the following three statements are equivalent for the set F .

- (1) The set $F \cup \{f\}$ induces an interesting garland.
- (2) The set $F \cup \{g\}$ induces an interesting garland.
- (3) The set $F \cup \{f, g\}$ induces an interesting garland.

The equivalence of these three statements follows from the following two observations. First, if f is in some garland G , then $\{f\}$ forms an E -subgraph from G ; and the same statement holds for $\{g\}$ and $\{g, f\}$. Therefore, the sets $F \cup \{f\}$, $F \cup \{g\}$, and $F \cup \{f, g\}$ induce a garland simultaneously. Second, denote by V_f the vertex set of $\langle F \cup \{f\} \rangle$, by V_g the vertex set of $\langle F \cup \{g\} \rangle$, and by V_{gf} the vertex set of $\langle F \cup \{g, f\} \rangle$. Note that $V_f \cap (V_1 \cup V_2) = V_g \cap (V_1 \cup V_2) = V_{gf} \cap (V_1 \cup V_2)$. This

means that the sets $F \cup \{f\}$, $F \cup \{g\}$, and $F \cup \{f, g\}$ destroy the parts V_1 or V_2 simultaneously. Consequently, all of them induce interesting garlands or none of them does.

Note also that F is chosen from a three-element set formed by edges $h \in E_{12}$ such that the edges h and f do not induce a Ξ_2 -subgraph. Consequently, the set F can be chosen in at most one way. Therefore, there exist at most $3 \cdot 1 = 3$ interesting garlands G such that $f \in E(G)$ or $g \in E(G)$, and the prove is complete in this case.

Case 1.1.3. Assume that edges f and g do not induce a coordinated subgraph of type $K(2, 1)$ and do not induce a Ξ_2 -subgraph.

Note that, in this case, each of the edges f and g is in exactly three Ξ_2 -subgraphs. Note that if f is in a one-element garland G , then G is a one-edge garland. Note also that if f is in a garland G that has more than one element, then G must destroy V_1 or V_2 , and then $|E(G)| \geq 4$. Since f is in three Ξ_2 -subgraphs, the set $E(G) \setminus \{f\}$ is chosen from a five-element set of edges. Consequently, f is in at most

$$\binom{4}{3} + \binom{4}{4} = 4 + 1 = 5$$

interesting garlands that contain more than one element. Therefore, f is in at most six interesting garlands. By analogy, the edge g is in at most six interesting garlands. Therefore, the number of interesting garlands whose edge aggregates contain f or g does not exceed 12, and the lemma is proved in Case 1.1.

Case 1.2. Assume that $e_{12} \leq 5$. In this case, $\xi_2 \leq 5$.

Case 1.2.1. Assume that there exists an edge $f \in E$ that is in four Ξ_2 -subgraphs. Thus, by Lemma 12, it is in at most $2^{8-4-1} - 8 + 6 = 6$ interesting garlands. Note that the set $E \setminus \{f\}$ contains at least three Ξ_2 -subgraphs and then, by Lemma 17, there exist at most 41 interesting garlands whose edge aggregates are subsets of $E \setminus \{f\}$. Consequently, there are at most $6 + 41 = 47$ interesting garlands, a contradiction with $\Delta pt(H, K(v)) = 48$.

Case 1.2.2. Assume that there exists an edge $f \in E$ which is in exactly three Ξ_2 -subgraphs. Thus, by Lemma 12, it is in at most $2^{8-3-1} - 8 + 5 = 13$ interesting garlands. Note that $E \setminus \{f\}$ contains at least four Ξ_2 -subgraphs; so, by Lemma 18, there exist at most 34 interesting garlands whose edge aggregates are subsets of $E \setminus \{f\}$. Therefore, there exist at most $13 + 34 = 47$ interesting garlands, a contradiction with $\Delta pt(H, K(v)) = 48$.

Case 1.2.3. It remains to consider the case when every edge from E is in at most two Ξ_2 -subgraphs. Note that, since every Ξ_2 -subgraph contains two edges, $\xi_2 \leq 8$. If $e_{12} = 5$, then every Ξ_2 -subgraph must contain at least one of three edges from $E' = E \setminus E_{12}$. Consequently, some edge from E' must be in three Ξ_2 -subgraphs, a contradiction. Therefore, $e_{12} \leq 5$ and then $\xi_2 \geq 8$. Thus, $\xi_2 = 8$ and $e_{12} = 5$. Note that, in this case, every edge is in exactly two Ξ_2 -subgraphs. Since $e_{12} = 4$ and Ξ_2 -subgraph cannot contain both edges from E_{12} , every Ξ_2 -subgraph contain an edge from E_{12} .

Let F be the edge aggregate of some garland. Note that F does not contain edges of any Ξ_2 -subgraph. Consequently, every Ξ_2 -subgraph contains at least one edge from $E \setminus F$. Since every edge is in at most two Ξ_2 -subgraphs, $8 = \xi_2 \leq 2|E \setminus F|$, which implies $|E \setminus F| \geq 4$, and then $|F| \leq 4$.

Assume that there exists a four-edge garland. Denote its edge aggregate by F and define $F' = E \setminus F$. Since $|F'| = 4$, there exists at most one four-edge garland whose edge aggregate is a subset of F' . Note that every Ξ_2 -subgraph contains one edge from F and one edge from F' . Estimate the number of four-edge garlands whose edge aggregates contain an edge from F and an edge from F' . Assume that such a garland exists. Denote its edge aggregate by \tilde{F} . Define $F = \{f_1, f_2, f_3, f_4\}$ and $F' = \{f'_1, f'_2, f'_3, f'_4\}$. Let $f_1 \in \tilde{F}$ and $f'_1 \in \tilde{F}$. Without loss of generality, assume that the pairs of edges $\{f_1, f'_2\}$, $\{f_1, f'_3\}$, $\{f'_1, f_2\}$, and $\{f'_1, f_3\}$ induce Ξ_2 -subgraphs. Then

$f_4 \in \tilde{F}$. This means that every edge from F is in at most one such four-edge garland and each such garland must contain exactly two edges from F ; so the number of such garlands does not exceed $|F|/2 = 2$. Therefore, the number of four-edge garlands does not exceed 4.

Since $\langle E \rangle$ does not contain a triangle, an interesting three-edge garland must be of type $K(3, 1)$ or $K(2, 1) \dot{\cup} K(1, 1)$. Since $e_{12} = 4$, the inequality $e_{ij} \leq 4$ holds for all i and j . Note that all three edges of a garland of type $K(3, 1)$ must be in E_{ij} for some i and j . Since $\langle E \rangle$ does not contain a coordinated subgraph of type $K(4, 1)$, any two garlands of type $K(3, 1)$ cannot contain more than one common edge. Therefore, there exist at most two such garlands.

An interesting garland of type $K(2, 1) \dot{\cup} K(1, 1)$ must destroy some part of $K(v)$. Since it can destroy only a part that contains three vertices, it must destroy V_3 . Consequently, each vertex of V_3 must be incident to some edge from this garland. Therefore, its edge aggregate must be a three-element subset of $E \setminus E_{12}$. Since $|E \setminus E_{12}| = 4$, such a subset can be chosen in $\binom{4}{3} = 4$ ways; so the number of such garlands does not exceed 4.

Thus, there exist 8 one-edge garlands, at most

$$\binom{8}{2} = 28$$

interesting two-edge garlands, at most $4 + 2 = 6$ interesting garlands, and at most 4 four-edge interesting garlands. Therefore, there exist at most

$$8 + 28 + 6 + 4 = 46 < 48$$

interesting garlands. The proof is complete in Case 1.

Case 2. Assume that $\xi_3 = 1$.

Case 2.1. Assume that $e_{12} = 6$. Then $\xi_2 = 4$ and $|E_{23}| = |E_{13}| = 1$. Denote the single edge from E_{23} by e and the single edge from E_{13} by f . Note that every Ξ_2 -subgraph contains the edge e or f . If e is in three Ξ_2 -subgraphs, then $\langle E \rangle$ contains a coordinated subgraph of type $K(4, 1)$, a contradiction. By analogy, f is in at most two Ξ_2 -subgraphs. This means that 7 edges from $|E|$ form a configuration shown in Fig. 8.

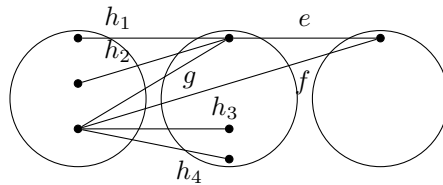


Figure 8. The case $e_{12} = 6$ and $\xi_3 = 1$

Note that any garland cannot destroy the part V_3 .

Assume that e and f are in some garland G that contains more than one element. Note that $g \in E(G)$ and $h_1, h_2, h_3, h_4 \notin E(G)$ because, otherwise, $\langle E \rangle$ contains at least two triangles. Consequently, $|E(G)| \leq 4$ and $|V(G) \cap V_i| \leq 2$ for all $i = 1, 2, 3$. Therefore, G cannot destroy any part, a contradiction. Thus, if e and f are in a garland G , then $|G| = 1$. Note that there exists only one such garland, which is a triangle.

Note that e is not in any coordinated subgraph of type $K(2, 1)$. This means that there are only two one-element garlands whose edge aggregates contain e : a triangle and a one-edge garland. Assume that e is in some garland G that contains more than one element and $f \notin E(G)$. Since G must destroy some part and cannot destroy V_3 , it destroys V_1 or V_2 ; hence, $|E(G)| \geq 4$. Note

that $f, g, h_1, h_2 \notin E(G)$, which implies $E(G) = E \setminus \{f, g, h_1, h_2\}$. Therefore, e is in at most one interesting garland that contains more than one element and does not contain f . By analogy, f is in at most one interesting garland that contains more than one element and does not contain e . Therefore, there exist at most 5 interesting garlands that contain e or f .

By Lemma 3, there exist at most 33 interesting garlands whose edge aggregates are subsets of E_{12} . Therefore, there exist at most $33 + 5 = 38$ interesting garlands, which contradicts $\Delta pt(H, K(v)) = 48$.

Case 2.2. Assume that $e_{12} \leq 5$. In this case, $\xi_2 \geq 5$. Note that any edge from the triangle cannot be in more than three Ξ_2 -subgraphs.

Assume that some edge e from the triangle is in three Ξ_2 -subgraphs. By Lemma 14, one can deduce that e is in at most $2^{8-2-3} - 8 + 3 + 3 = 6$ interesting garlands. Since $\langle E \setminus \{e\} \rangle$ contains at least three Ξ_2 -subgraphs, by Lemma 17, there exist at most 41 interesting garlands whose edge aggregates are subsets of $E \setminus \{e\}$. Therefore, there exist at most $41 + 6 = 47$ interesting garlands, a contradiction.

Assume that some edge e from the triangle is in two Ξ_2 -subgraphs. By Lemma 14, one can deduce that e is in at most $2^{8-2-2} - 8 + 2 + 3 = 13$ interesting garlands. Since $\langle E \setminus \{e\} \rangle$ contains at least three Ξ_2 -subgraphs, by Lemma 18, there exist at most 34 interesting garlands whose edge aggregates are subsets of $E \setminus \{e\}$. Therefore, there exist at most $13 + 34 = 47$ interesting garlands, a contradiction.

By Lemma 16, there exists at most one Ξ_2 -subgraph such that both its edges are not in the triangle. Taking into account that each edge of the triangle is in at most one Ξ_2 -subgraph, one can deduce that $\xi_2 \leq 3 + 1 = 4 < 5$, a contradiction.

Case 3. Consider the case $\xi_3 \geq 2$.

If $\langle E \rangle$ contains two triangles without common edges, then, by Lemma 8, the number of interesting garlands does not exceed $2^6 - 12 \cdot 8 + 58 = 26 < 48$, a contradiction. Consequently, any two triangles from $\langle E \rangle$ have a common edge and, by Lemma 19, there exists an edge e that is in all triangles. This implies $\xi_3 \leq 3$. Define $E' = E \setminus \{e\}$.

Note that there exist at most two Ξ_2 -subgraph that do not contain the edge e . Indeed, any such Ξ_2 -subgraph does not contain any edge from any triangle. Therefore, if there are at least three such subgraphs, then some two of them do not contain edges from the same triangle and, in this case, by Lemma 16, the number of interesting garlands does not exceed 46.

Let e be in k different Ξ_2 -subgraphs. Note that $k \leq 2$. Since e is in ξ_3 triangles, there exist ξ_3 pairs of edges $\{h, f\}$ such that h and f induce a Ξ_2 -subgraph in $\langle E' \rangle$ (such pairs can be generated by removing the edge e from a triangle from $\langle E \rangle$). Therefore, $\langle E' \rangle$ contains at least

$$\xi_2 + \xi_3 - k = 12 - e_{12} - 2\xi_3 + \xi_3 - 2 = 12 - k - e_{12} - \xi_3 = \xi_2'$$

different Ξ_2 -subgraphs. Note that if $e_{12} = 5$, then $\xi_3 \leq 2$; if $e_{12} \leq 4$, then $\xi_3 \leq 3$. In both cases, $e_{12} + \xi_3 \leq 7$ and this implies $\xi_2' \geq 12 - k - 7 = 5 - k$.

Case 3.1. Assume that $k = 2$. By Lemma 15, the edge e is in at most $2^{8-5} - 3 \cdot 8 + 6 + 15 = 5$ interesting garlands. The graph $\langle E' \rangle$ contains at least 3 different Ξ_2 -subgraphs, therefore, by Lemma 17, there exist at most 41 interesting garlands whose edge aggregates are subsets of E' . Thus, there exist at most $41 + 5 = 46 < 48$ interesting garlands, a contradiction.

Case 3.2. Assume that $k = 1$. By Lemma 15, the edge e is in at most $2^{8-4} - 3 \cdot 8 + 3 + 15 = 10$ interesting garlands. The subgraph $\langle E' \rangle$ contains at least four Ξ_2 -subgraphs, therefore, by Lemma 18, there exist at most 34 interesting garlands whose edge aggregates are subsets of E' . Thus, the number of interesting garlands does not exceed $10 + 34 = 44 < 48$, a contradiction.

Case 3.3. It remains to consider the case when the edge e is not in any Ξ_2 -subgraph. Note that, in this case, $\xi_2 \leq 2$.

If $e_{12} = 5$, then $\xi_3 = 2$ and $\xi_2 = 12 - 2 \cdot 2 - 5 = 3 > 2$, a contradiction.

If $e_{12} \leq 3$, then $\xi_3 \leq 3$ and $\xi_2 = 12 - (e_{12} + 2\xi_3) \geq 12 - 9 = 3 > 2$, a contradiction.

Thus, it suffices to consider the case when $e_{12} = 4$. In this case, $\xi_3 = 3$ and $\xi_2 = 2$. Consider the single edge that is not in any triangle. Note that this edge must be in two Ξ_2 -subgraphs, which is impossible. \square

Now we are ready to prove the chromatic uniqueness of the graph $K(q+4, q-1, q-1)$.

Proposition 3. *The graph $K(q+4, q-1, q-1)$ is chromatically unique for $q \geq 3$.*

P r o o f. By contradiction, assume that the graph $K(u) = K(q+4, q-1, q-1)$ is not chromatically unique. This means that there exists a graph H such that the graphs $K(u)$ and H are chromatically equivalent. Let H be a graph obtained from the graph $K(v)$ by removing some edge set E . Consider the graph $K(v)$. Note that the case $K(v) = K(q+3, q, q-1)$ is impossible by Lemma 1.

Case 1. Assume that $K(v) = K(q+3, q+1, q-2)$ and note that, in this case, $|E| = 2$. Calculate the difference of the invariant pt :

$$\Delta pt(H, K(v)) = \Delta pt(K(u), K(v)) = 15 \cdot 2^{q-2} - 3 \cdot 2^{q-3} = 27 \cdot 2^{q-3}.$$

By Lemma 2, one can deduce that $27 \cdot 2^{q-3} \leq 2^2 - 1 = 3$, which implies $q < 2$, a contradiction.

Case 2. Assume that $K(v) = K(q+2, q+2, q-2)$ and note that, in this case, $|E| = 3$. Calculate the difference of the invariant pt :

$$\Delta pt(H, K(v)) = \Delta pt(K(u), K(v)) = 15 \cdot 2^{q-2} - 3 \cdot 2^{q-3} + 2^q = 35 \cdot 2^{q-3}.$$

By Lemma 2, one can deduce that $33 \cdot 2^{q-3} \leq 2^3 - 1 = 7$, which implies $q < 2$, a contradiction.

Case 3. Assume that $K(v) = K(q+2, q+1, q-1)$ and note that, in this case, $|E| = 6$. Calculate the difference of the invariant pt :

$$\Delta pt(H, K(v)) = \Delta pt(K(u), K(v)) = 15 \cdot 2^{q-2} + 3 \cdot 2^{q-1} = 21 \cdot 2^{q-2}.$$

By Lemma 2, one can deduce that $21 \cdot 2^{q-2} \leq 2^6 - 1 = 63$, which implies $q \leq 3$, so it suffices to consider the case $q = 3$. In this case, $K(v) = K(5, 4, 2)$ and $\Delta(H, K(v)) = 42$. Calculate the difference of the invariant I_3 :

$$\begin{aligned} \Delta I_3(K(v), K(u)) &= 6(q-1), \\ \Delta I_3(K(v), H) &= \xi_1 - \xi_2 - 2\xi_3, \\ \xi_1 &= (q-1)e_{12} + (q+1)e_{13} + (q+2)e_{23} = (q-1)|E| + 2e_{13} + 3e_{23}, \\ \xi_2 + 2\xi_3 &= 2e_{13} + 3e_{23}. \end{aligned}$$

If $e_{13} = e_{23} = 0$, then $E = E_{12}$ and each active part of $K(v)$ contains at least four vertices. Therefore, by Lemma 3, there exist at most 33 interesting garlands, a contradiction. Thus, it suffices to consider the cases when $e_{13} > 0$ or $e_{23} > 0$. In both cases, $\xi_2 + 2\xi_3 \geq 2$.

Case 3.1. Assume that $\xi_3 > 0$ and edges $e_1, e_2, e_3 \in E$ induce a triangle. Consider an arbitrary garland G and define $F' = E(G) \cap \{e_1, e_2, e_3\}$. Since F' can be one of five sets (empty, one-element or equal to $\{e_1, e_2, e_3\}$), there exist at most $5 \cdot 2^{6-3} - 1 = 39 < 42$, a contradiction.

Case 3.2. Now assume that $\xi_3 = 0$. This implies $\xi_2 \geq 2$.

Case 3.2.1. Assume that there exists an edge e that is in two Ξ_2 -subgraphs. Let f and g be edges such that $\{e, g\}$ and $\{e, f\}$ induce Ξ_2 -subgraphs. Let G be a garland. Define $E' = E(G) \cap \{e, g, f\}$. Since E' can be one of five sets (empty, three one-element sets, and $\{g, f\}$), the number of garlands does not exceed $5 \cdot 2^{6-3} = 40 < 42$, a contradiction.

Case 3.2.2. Now assume that there exist two Ξ_2 -subgraphs without common edges. Let e_1, e_2, f_1 , and f_2 be edges from E such that $\{e_1, e_2\}$ and $\{f_1, f_2\}$ induce Ξ_2 -subgraphs. Let G be a garland. Define $E_e = E(G) \cap \{e_1, e_2\}$, $E_f = E(G) \cap \{f_1, f_2\}$, and $E' = E(G) \setminus \{e_1, e_2, f_1, f_2\}$. Note that $E(G) = E_e \cup E_f \cup E'$. Since E_e can be chosen in three ways, E_f can be chosen in three ways, and E' can be chosen in $2^2 = 4$ ways, the number of garlands does not exceed $3 \cdot 3 \cdot 4 = 36 < 42$, a contradiction.

Case 4. Assume that $K(v) = K(q+2, q, q)$ and note that, in this case, $|E| = 7$. Calculate the difference of the invariant pt :

$$\Delta pt(H, K(v)) = \Delta pt(K(u), K(v)) = 15 \cdot 2^{q-2} + 3 \cdot 2^{q-1} + 2^{q-2} = 22 \cdot 2^{q-2}.$$

By Lemma 2, one can deduce that $22 \cdot 2^{q-2} \leq 2^7 - 1 = 127$, which implies $q \leq 4$. So it suffices to consider the cases when $q = 4$ or $q = 3$. If $q = 4$, then $K(v) = K(6, 4, 4)$ and $\Delta pt(H, K(v)) = 88$. By Lemma 3, the number of interesting garlands does not exceed $2^6 + 1 = 65 < 88$, a contradiction. Thus, it suffices to consider the case $q = 3$. In this case, $K(v) = K(5, 3, 3)$ and $\Delta pt(H, K(v)) = 44$. Note that $\langle E \rangle$ does not contain a coordinated subgraph of type $K(5, 1)$.

Calculate the difference of the invariant I_3 :

$$\begin{aligned} \Delta I_3(K(v), K(u)) &= 7q - 4, \\ \Delta I_3(K(v), H) &= \xi_1 - \xi_2 - 2\xi_3, \\ \xi_1 &= qe_{12} + qe_{13} + (q+2)e_{23} = q|E| + 2e_{23}, \\ \xi_2 + 2\xi_3 &= 2e_{23} + 4. \end{aligned}$$

If $\xi_3 > 0$, then, by Lemma 6, the number of interesting garlands does not exceed

$$2^5 + 2^4 - 3 \cdot 7 + 13 = 32 + 16 - 21 + 13 = 40 < 44,$$

a contradiction. Therefore, $\xi_3 = 0$, which implies $\xi_2 = 2e_{23} + 4 \geq 4$. By Lemma 17, the number of interesting garlands does not exceed 41, a contradiction.

Case 5. Assume that $K(v) = K(q+1, q+1, q)$ and note that, in this case, $|E| = 8$. Calculate the difference of the invariant pt :

$$\Delta pt(H, K(v)) = \Delta pt(K(u), K(v)) = 15 \cdot 2^{q-2} + 3 \cdot 2^{q-1} + 2^{q-2} + 2^{q-1} = 24 \cdot 2^{q-2}.$$

Using Lemma 2, one can deduce that $24 \cdot 2^{q-1} \leq 2^8 - 1 = 255$, which implies $q \leq 6$; so it suffices to consider the cases when $q = 5$, $q = 4$, or $q = 3$. The case $q = 3$ is impossible by Proposition 2.

If $q = 5$, then $K(v) = K(6, 6, 5)$ and $\Delta pt(H, K(v)) = 192$. Since $\langle E \rangle$ does not contain a coordinated subgraph of type $K(8, 1)$, by Lemma 3, the number of interesting garlands does not exceed $2^7 + 1 = 129 < 192$, a contradiction.

It remains to prove the proposition in the case $q = 4$. In this case, $K(v) = K(5, 5, 4)$ and $\Delta pt(H, K(v)) = 96$.

Calculate the difference of the invariant I_3 :

$$\begin{aligned}\Delta I_3(K(v), K(u)) &= 8q - 4, \\ \Delta I_3(K(v), H) &= \xi_1 - \xi_2 - 2\xi_3, \\ \xi_1 &= qe_{12} + (q+1)e_{13} + (q+1)e_{23} = (q+1)|E| - e_{12}, \\ 8q - 4 &= 8(q+1) - e_{12} - \xi_2 - 2\xi_3, \\ 12 &= \xi_2 + 2\xi_3 + e_{12}.\end{aligned}$$

If $\xi_3 > 0$, then, by Lemma 6, the number of interesting garlands does not exceed

$$2^6 + 2^5 - 3 \cdot 8 + 13 = 85 < 96,$$

a contradiction. Thus, $\xi_3 = 0$, which implies $\xi_2 + e_{12} = 12$. Note that $e_{12} \leq 8$. If $e_{12} = 8$, then $E = E_{12}$, which implies $\xi_2 = 0$, a contradiction. Thus, $e_{12} \leq 7$ and $\xi_2 \geq 5$.

If there exist an edge e that is in at least two Ξ_2 -subgraphs, then, since $\langle E \rangle$ does not contain a coordinated subgraph of $K(6, 1)$, by Lemma 12, the edge e is in at most $2^5 - 8 + 2 + 2 = 28$ interesting garlands. By Lemma 3, the number of interesting garlands whose edge are subsets of $E \setminus \{e\}$ does not exceed $2^6 + 1 = 65$, so the total number of interesting garlands does not exceed $65 + 28 = 93 < 96$, a contradiction. Therefore, every edge is in at most one Ξ_2 -subgraphs, which implies $\xi_2 \leq 4$, a contradiction. \square

The proof of the theorem follows from Propositions 1 and 3.

7. Conclusion

In this paper, the chromatic uniqueness of a complete tripartite graph $K(n_1, n_2, n_3)$ is proved for $n_1 \geq n_2 \geq 3 \geq 2$ and $n_1 - n_3 \leq 5$. Also, some properties of the number of partitions of the vertex set of a graph G into t independent sets are established. Both problems, the chromatic uniqueness, and properties of the invariant pt , are still challenging open problems.

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