

# PRODUCTS OF ULTRAFILTERS AND MAXIMAL LINKED SYSTEMS ON WIDELY UNDERSTOOD MEASURABLE SPACES

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**Abstract:** Constructions related to products of maximal linked systems (MLSs) and MLSs on the product of widely understood measurable spaces are considered (these measurable spaces are defined as sets equipped with  $\pi$ -systems of their subsets; a  $\pi$ -system is a family closed with respect to finite intersections). We compare families of MLSs on initial spaces and MLSs on the product. Separately, we consider the case of ultrafilters. Equipping set-products with topologies, we use the box-topology and the Tychonoff product of Stone-type topologies. The properties of compactness and homeomorphism hold, respectively.

**Keywords:** Maximal linked system, Topology, Ultrafilter.

## 1. Introduction

In this investigation, properties of maximal linked systems (MLSs) and ultrafilters on widely understood measurable spaces (MSs) are considered. Every such MS is realized by equipment of a nonempty set with  $\pi$ -system of subsets of this set with “zero” and “unit” (the “zero” is an empty set, and the “unit” is our original set); a  $\pi$ -system is a family closed with respect to finite intersections. Of course, algebras, semi-algebras, topologies, and families of closed sets in topological spaces (TSs) are  $\pi$ -systems. An important variant of a  $\pi$ -system is realized by a lattice of subsets of a fixed nonempty set. A semi-algebra of sets is a  $\pi$ -system but, generally speaking, not a lattice.

We note that MLSs were considered in connection with the superextension and supercompactness problem, see [2, 16, 17, 20, 21]. In addition, MLSs on the lattice of closed sets in a TS were studied. The nonempty set of all MLSs of such type is equipped with Wallman-type topology. The supercompactness property was implemented.

In [5–7, 9, 10, 12], an analog of the superextension and supercompactness property for the space of MLSs on a  $\pi$ -system was investigated. Moreover, a Stone-type topology was also used. In addition, a bitopological space was implemented. The present study continues the above works. But here the focus is on spaces of MLSs with Stone-type topology. We consider questions related to the products of widely understood measurable spaces. In addition, representations of MLSs on the product of these MSs in terms of analogous MLSs on spaces-factors are indicated. Namely, MLSs on the product of (widely understood) MSs are limited to products of MLSs on initial spaces. This important property is complemented by a proposition of a topological nature: the properties of compactness and homeomorphism hold. In addition, the box and Tychonoff variants of topology product are considered (similar variants are used for the product of MSs). In connection with the above assumptions, we use constructions of [11, 13, 14].

## 2. General notions and notation

We use standard set-theoretic notation, including quantifiers and propositional connectives;  $\emptyset$  stands for an empty set) and  $\triangleq$  for an equality by definition. A family is a set such that all its elements are sets themselves. We adopt the axiom of choice. For every objects  $x$  and  $y$ , denote by  $\{x; y\}$  an unordered pair of  $x$  and  $y$ :  $x \in \{x; y\}$ ,  $y \in \{x; y\}$ , and  $(z = x) \vee (z = y)$  for every  $z \in \{x; y\}$ . For every object  $s$ , denote by  $\{s\} \triangleq \{s; s\}$  a singleton containing  $s$ :  $s \in \{s\}$ . In addition, sets are objects. Then, for every objects  $x$  and  $y$ , the family  $(x, y) \triangleq \{\{x\}; \{x; y\}\}$  is (see [12, Ch. II, Section 2]) the ordered pair with  $x$  as the first element and  $y$  as the second. For every ordered pair  $h$ , denote by  $\text{pr}_1(h)$  and  $\text{pr}_2(h)$  its first and second elements, respectively; thus,  $h = (\text{pr}_1(h), \text{pr}_2(h))$ .

Denote by  $\mathcal{P}(H)$  the family of all subsets of  $H$ . Let  $\mathcal{P}'(H) \triangleq \mathcal{P}(H) \setminus \{\emptyset\}$  be the family of all nonempty subsets of  $H$ . Denote by  $\text{Fin}(H)$  the family of all finite nonempty subsets of  $H$ . If  $\mathcal{H}$  is a family and  $S$  is a set, then

$$[\mathcal{H}](S) \triangleq \{H \in \mathcal{H} \mid S \subset H\} \in \mathcal{P}(\mathcal{H}).$$

For every set  $\mathbb{M}$  and a family  $\mathcal{M} \in \mathcal{P}'(\mathcal{P}(\mathbb{M}))$ , the dual family

$$\mathbf{C}_{\mathbb{M}}[\mathcal{M}] \triangleq \{\mathbb{M} \setminus M : M \in \mathcal{M}\} \in \mathcal{P}'(\mathcal{P}(\mathbb{M}))$$

is realized. If  $\mathcal{A}$  is a nonempty family and  $B$  is a set, then

$$\mathcal{A}|_B \triangleq \{A \cap B : A \in \mathcal{A}\} \in \mathcal{P}'(\mathcal{P}(B))$$

is the trace of  $A$  onto the set  $B$ . Following to [7, Section 1], if  $\mathfrak{X}$  is a nonempty family, then  $\{\cup\}(\mathfrak{X})$ ,  $\{\cap\}(\mathfrak{X})$ ,  $\{\cup\}_{\#}(\mathfrak{X})$ , and  $\{\cap\}_{\#}(\mathfrak{X})$  stand for the families of arbitrary unions, arbitrary intersections of nonempty subfamilies of  $\mathfrak{X}$ , finite unions, and finite intersections of sets from  $\mathfrak{X}$ , respectively.

*Remark 1.* In what follows, we use two types of formulas. Namely, we use expressions of type  $\{x \in X \mid \dots\}$  and expressions of type  $\{f(z) : z \in \dots\}$ . In function theory, the former is used for the preimage of a set; we have a formula corresponding to Zermelo–Fraenkel axiomatic (we first select a set  $X$ , for points of which some property  $\dots$  is postulated). The second expression corresponds logically to the image of a set. This difference is essential from point of view of bibliographic references to earlier publications of the author. Therefore, we use two variants of separator character:  $|$  (vertical line) in the first case and  $:$  (colon) in the second. This stipulation is important for the constructions that follow.

For sets  $A$  and  $B$ , we denote by  $B^A$  (see [19, Ch. II, § 6]) the set of all mappings (functions) from  $A$  to  $B$ ; values of mappings are denoted in traditional way. If  $A$  and  $B$  are sets,  $f \in B^A$ , and  $C \in \mathcal{P}(A)$ , then  $f^1(C) \triangleq \{f(x) : x \in C\} \in \mathcal{P}(B)$  and  $(f|C) \in B^C$  is, by definition, the restriction of the mapping  $f$  to the set  $C$ :  $(f|C)(x) \triangleq f(x) \quad \forall x \in C$ . For mappings, index form is often used (a family with index, see [22, Ch. I, I.1]).

In what follows,  $\mathbb{R}$  is the real line,  $\mathbb{N} \triangleq \{1; 2; \dots\} \in \mathcal{P}'(\mathbb{R})$ , and  $\overline{1, n} \triangleq \{k \in \mathbb{N} \mid k \leq n\}$  for  $n \in \mathbb{N}$ . We assume that elements of  $\mathbb{N}$ , i.e., positive integer natural numbers are not sets. Therefore, for every set  $H$  and  $n \in \mathbb{N}$ , instead of  $H^{\overline{1, n}}$ , we use the more traditional notation  $H^n$  for the set of all mappings from  $\overline{1, n}$  to  $H$ ; thus,  $H^n$  is the set of all processions  $(h_i)_{i \in \overline{1, n}} : \overline{1, n} \rightarrow H$ .

**Special families.** Until the end of this section, we fix a nonempty set  $\mathbf{I}$ . The elements of  $\mathcal{P}'(\mathcal{P}(\mathbf{I}))$  are nonempty families of subsets of  $\mathbf{I}$ . Define the family of all  $\pi$ -systems of subsets of  $\mathbf{I}$

with “zero” and “unit”:

$$\pi[\mathbf{I}] \triangleq \{\mathcal{I} \in \mathcal{P}'(\mathcal{P}(\mathbf{I})) \mid (\emptyset \in \mathcal{I}) \& (\mathbf{I} \in \mathcal{I}) \& (A \cap B \in \mathcal{I} \ \forall A \in \mathcal{I} \ \forall B \in \mathcal{I})\}. \quad (2.1)$$

Of course,  $\mathcal{P}(\mathbf{I}) \in \pi[\mathbf{I}]$ . Consider a very useful notion of semi-algebra of sets. For  $\mathcal{L} \in \pi[\mathbf{I}]$ ,  $A \in \mathcal{P}(\mathbf{I})$ , and  $n \in \mathbb{N}$ , we introduce finite partitions of  $A$  by sets of  $\mathcal{L}$ :

$$\Delta_n(A, \mathcal{L}) \triangleq \left\{ (L_i)_{i \in \overline{1, n}} \in \mathcal{L}^n \mid \left( A = \bigcup_{i=1}^n L_i \right) \& (L_p \cap L_q = \emptyset \ \forall p \in \overline{1, n} \ \forall q \in \overline{1, n} \setminus \{p\}) \right\}.$$

The family of all semi-algebras of subsets of  $\mathbf{I}$  is defined as follows:

$$\Pi[\mathbf{I}] \triangleq \{\mathcal{L} \in \pi[\mathbf{I}] \mid \forall L \in \mathcal{L} \ \exists n \in \mathbb{N} : \Delta_n(\mathbf{I} \setminus L, \mathcal{L}) \neq \emptyset\}. \quad (2.2)$$

In addition, we introduce yet another type of  $\pi$ -systems; this type is important in questions of interconnection between ultrafilters and MLSs. Namely,

$$\begin{aligned} \pi_*^\sharp[\mathbf{I}] &\triangleq \{\mathcal{I} \in \pi[\mathbf{I}] \mid \forall I_1 \in \mathcal{I} \ \forall I_2 \in \mathcal{I} \ \forall I_3 \in \mathcal{I} \\ &((I_1 \cap I_2 \neq \emptyset) \& (I_2 \cap I_3 \neq \emptyset) \& (I_1 \cap I_3 \neq \emptyset)) \implies (I_1 \cap I_2 \cap I_3 \neq \emptyset)\}. \end{aligned}$$

Of course, very general constructions are connected with lattices. The family of all lattices of subsets of  $\mathbf{I}$  with “zero” and “unit” is

$$(\text{LAT})_0[\mathbf{I}] \triangleq \{\mathcal{I} \in \pi[\mathbf{I}] \mid A \cup B \in \mathcal{I} \ \forall A \in \mathcal{I} \ \forall B \in \mathcal{I}\}, \quad (2.3)$$

We introduce the family

$$(\text{alg})[\mathbf{I}] \triangleq \{\mathcal{A} \in (\text{LAT})_0[\mathbf{I}] \mid \mathbf{I} \setminus A \in \mathcal{A} \ \forall A \in \mathcal{A}\} \quad (2.4)$$

of all algebras of subsets of  $\mathbf{I}$ . For  $\mathcal{A} \in (\text{alg})[\mathbf{I}]$ ,  $(\mathbf{I}, \mathcal{A})$  is an MS with algebra of sets. Moreover,

$$(\text{top})[\mathbf{I}] \triangleq \left\{ \tau \in (\text{LAT})_0[\mathbf{I}] \mid \bigcup_{G \in \mathcal{G}} G \in \tau \ \forall \mathcal{G} \in \mathcal{P}'(\tau) \right\} \quad (2.5)$$

is the family of all topologies on  $\mathbf{I}$  and

$$(\text{clos})[\mathbf{I}] \triangleq \{\mathbf{C}_\mathbf{I}[\tau] : \tau \in (\text{top})[\mathbf{I}]\} \in \mathcal{P}'((\text{LAT})_0[\mathbf{I}]) \quad (2.6)$$

is the family of all closed topologies [1, Ch.4, § 1] on  $\mathbf{I}$ . So, (2.4)–(2.6) are important types of lattices (see (2.3)). Semi-algebras (see (2.2)) are, generally speaking, not lattices: if  $\mathcal{L} \in \Pi[\mathbf{I}]$ , then it is possible that  $\mathcal{L} \notin (\text{LAT})_0[\mathbf{I}]$ .

**Elements of topology.** We consider the families  $(\text{BAS})[\mathbf{I}]$  and  $(\text{p-BAS})[\mathbf{I}]$  of all open bases and subbases on  $\mathbf{I}$ , respectively; this notation correspond to [9, Section 2] (see also [7, Section 2]). Of course,  $\{\cup\}(\beta) \in (\text{top})[\mathbf{I}]$  for  $\beta \in (\text{BAS})[\mathbf{I}]$ ; moreover,  $\{\cap\}_\sharp(\chi) \in (\text{BAS})[\mathbf{I}]$  for  $\chi \in (\text{p-BAS})[\mathbf{I}]$ . Note that (see (2.1))

$$\pi[\mathbf{I}] \subset (\text{BAS})[\mathbf{I}]; \quad (2.7)$$

therefore,  $\{\cup\}(\mathcal{L}) \in (\text{top})[\mathbf{I}]$  is defined for  $\mathcal{L} \in \pi[\mathbf{I}]$ . If  $\tau \in (\text{top})[\mathbf{I}]$ , then

$$(\tau - \text{BAS})_0[\mathbf{I}] \triangleq \{\beta \in (\text{BAS})[\mathbf{I}] \mid \tau = \{\cup\}(\beta)\}.$$

Moreover,

$$(\text{p-BAS})_0[\mathbf{I}; \tau] \triangleq \{\chi \in (\text{p-BAS})[\mathbf{I}] \mid \{\cap\}_\sharp(\chi) \in (\tau - \text{BAS})_0[\mathbf{I}]\}.$$

Thus, we have introduced open bases and subbases of the specific TS  $(\mathbf{I}, \tau)$ .

**Linkedness.** If  $\mathcal{J} \in \mathcal{P}'(\mathcal{P}(\mathbf{I}))$ , then we suppose that

$$\langle \mathcal{J} - \text{link} \rangle[\mathbf{I}] \triangleq \{ \mathcal{E} \in \mathcal{P}'(\mathcal{J}) \mid \Sigma_1 \cap \Sigma_2 \neq \emptyset \quad \forall \Sigma_1 \in \mathcal{E} \quad \forall \Sigma_2 \in \mathcal{E} \}. \quad (2.8)$$

Elements of the family (2.8) and only they are linked subfamilies of  $\mathcal{J}$ . As a corollary,

$$\langle \mathcal{J} - \text{link} \rangle_0[\mathbf{I}] \triangleq \{ \mathcal{E} \in \langle \mathcal{J} - \text{link} \rangle[\mathbf{I}] \mid \forall \mathcal{S} \in \langle \mathcal{J} - \text{link} \rangle[\mathbf{I}] \quad (\mathcal{E} \subset \mathcal{S}) \implies (\mathcal{E} = \mathcal{S}) \} \quad (2.9)$$

is the family of all maximal linked subfamilies of  $\mathcal{J}$ . We call every family of (2.9) an MLS (on  $\mathcal{J}$ ). In what follows, for our goals, it suffices to consider the case  $\mathcal{J} \in \pi[\mathbf{I}]$ . Therefore, until the end of this section, suppose that  $\mathcal{J} \in \pi[\mathbf{I}]$ . Now, we note only several simple properties. So,  $\{\Sigma\} \in \langle \mathcal{J} - \text{link} \rangle[\mathbf{I}]$  for  $\Sigma \in \mathcal{J} \setminus \{\emptyset\}$ . Then, by the Zorn lemma,  $\langle \mathcal{J} - \text{link} \rangle_0[\mathbf{I}] \neq \emptyset$ . Moreover,

$$\langle \mathcal{J} - \text{link} \rangle_0[\mathbf{I}] = \{ \mathcal{E} \in \langle \mathcal{J} - \text{link} \rangle[\mathbf{I}] \mid \forall J \in \mathcal{J} \quad (J \cap \Sigma \neq \emptyset \quad \forall \Sigma \in \mathcal{E}) \implies (J \in \mathcal{E}) \}.$$

Finally, note that, for  $\mathcal{E} \in \langle \mathcal{J} - \text{link} \rangle_0[\mathbf{I}]$ , we have

$$([\mathcal{J}](\Sigma) \subset \mathcal{E} \quad \forall \Sigma \in \mathcal{E}) \& (\mathbf{I} \in \mathcal{E}). \quad (2.10)$$

More detailed information on the properties of MLSs can be found in [5–7, 9–12]. Now we introduce some constructions for a Stone-type topology. If  $J \in \mathcal{J}$ , then

$$\langle \mathcal{J} - \text{link} \rangle_0^0[\mathbf{I}|J] \triangleq \{ \mathcal{E} \in \langle \mathcal{J} - \text{link} \rangle_0[\mathbf{I}] \mid J \in \mathcal{E} \} \in \mathcal{P}(\langle \mathcal{J} - \text{link} \rangle_0[\mathbf{I}]). \quad (2.11)$$

The sets (2.11) define an open subbase. More precisely, the subbase

$$\hat{\mathcal{C}}_0^*[\mathbf{I}; \mathcal{J}] \triangleq \{ \langle \mathcal{J} - \text{link} \rangle_0^0[\mathbf{I}|J] : J \in \mathcal{J} \} \in (\text{p-BAS})[\langle \mathcal{J} - \text{link} \rangle_0[\mathbf{I}]]$$

generates the following topology of Stone type:

$$\mathbb{T}_*[\mathbf{I}|\mathcal{J}] \triangleq \{ \cup \} \{ \{ \cap \} \# (\hat{\mathcal{C}}_0^*[\mathbf{I}; \mathcal{J}]) \} \in (\text{top})[\langle \mathcal{J} - \text{link} \rangle_0[\mathbf{I}]]. \quad (2.12)$$

In addition,  $(\langle \mathcal{J} - \text{link} \rangle_0[\mathbf{I}], \mathbb{T}_*[\mathbf{I}|\mathcal{J}])$  is a zero-dimensional  $T_2$ -space.

### 3. Generalized Cartesian products

In this sections, we recall some constructions connected with Cartesian products and generalized Cartesian products. We note also some notions connected with family products.

If  $X$  and  $Y$  are nonempty sets,  $\mathcal{X} \in \mathcal{P}'(\mathcal{P}(X))$ , and  $\mathcal{Y} \in \mathcal{P}'(\mathcal{P}(Y))$ , then

$$\mathcal{X}\{\times\}\mathcal{Y} \triangleq \{ \text{pr}_1(z) \times \text{pr}_2(z) : z \in \mathcal{X} \times \mathcal{Y} \} \in \mathcal{P}'(\mathcal{P}(X \times Y)) \quad (3.1)$$

( $\mathcal{X} \times \mathcal{Y}$  is the usual product of  $\mathcal{X}$  and  $\mathcal{Y}$ , i.e., the set of ordered pairs); (3.1) is the simplest variant of the constructions used below. It is easy to verify the property

$$\mathcal{X}\{\times\}\mathcal{Y} \in \pi[X \times Y] \quad \forall \mathcal{X} \in \pi[X] \quad \forall \mathcal{Y} \in \pi[Y]. \quad (3.2)$$

We consider  $(X \times Y, \mathcal{X}\{\times\}\mathcal{Y})$  as the product of the MSs  $(X, \mathcal{X})$  and  $(Y, \mathcal{Y})$ .

Now we recall notions connected with generalized Cartesian products. If  $\mathbb{X}$  and  $\mathbb{Y}$  are nonempty sets and  $(Y_x)_{x \in \mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$ , then (by the axiom of choice)

$$\prod_{x \in \mathbb{X}} Y_x \triangleq \{f \in \mathbb{Y}^{\mathbb{X}} \mid f(x) \in Y_x \ \forall x \in \mathbb{X}\} \in \mathcal{P}'(\mathbb{Y}^{\mathbb{X}}). \quad (3.3)$$

In connection with (3.3), note that, for every nonempty sets  $\mathbb{X}$ ,  $\tilde{\mathbb{Y}}$ , and  $\hat{\mathbb{Y}}$  and a mapping  $(Y_x)_{x \in \mathbb{X}} \in \mathcal{P}'(\tilde{\mathbb{Y}})^{\mathbb{X}} \cap \mathcal{P}'(\hat{\mathbb{Y}})^{\mathbb{X}}$ , we have

$$\{f \in \tilde{\mathbb{Y}}^{\mathbb{X}} \mid f(x) \in Y_x \ \forall x \in \mathbb{X}\} = \{f \in \hat{\mathbb{Y}}^{\mathbb{X}} \mid f(x) \in Y_x \ \forall x \in \mathbb{X}\}. \quad (3.4)$$

In what follows, in constructions of type (3.3) we take into account (3.4). If  $\mathbb{X}$  and  $\mathbb{Y}$  are nonempty sets and  $(Y_x)_{x \in \mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$ , then

$$\prod_{x \in \mathbb{X}} \mathcal{P}'(\mathcal{P}(Y_x)) = \{(\mathcal{Y}_x)_{x \in \mathbb{X}} \in \mathcal{P}'(\mathcal{P}(\mathbb{Y}))^{\mathbb{X}} \mid \mathcal{Y}_s \in \mathcal{P}'(\mathcal{P}(Y_s)) \ \forall s \in \mathbb{X}\};$$

moreover, if  $(\mathcal{E}_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} \mathcal{P}'(\mathcal{P}(Y_x))$ , then

$$\odot_{x \in \mathbb{X}} \mathcal{E}_x \triangleq \left\{ \prod_{x \in \mathbb{X}} \Sigma_x : (\Sigma_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} \mathcal{E}_x \right\}. \quad (3.5)$$

We consider the family (3.5) as a box product of the families  $\mathcal{E}_x$ ,  $x \in \mathbb{X}$ . Here, we note the natural analogy with the base of the known box topology (see [18, Ch. 3]).

If  $\mathbb{H}$  is a set, then we suppose that

$$(\text{Fam})[\mathbb{H}] \triangleq \{\mathcal{H} \in \mathcal{P}'(\mathcal{P}(\mathbb{H})) \mid \mathbb{H} \in \mathcal{H}\};$$

of course,  $\mathcal{P}(\mathbb{H}) \in (\text{Fam})[\mathbb{H}]$ ; moreover,  $(\text{alg})[\mathbb{H}] \subset \Pi[\mathbb{H}] \subset \pi[\mathbb{H}] \subset (\text{Fam})[\mathbb{H}]$  and, by (2.5),  $(\text{top})[\mathbb{H}] \subset (\text{Fam})[\mathbb{H}]$ . As a corollary, for nonempty sets  $\mathbb{X}$  and  $\mathbb{Y}$ , a mapping  $(Y_x)_{x \in \mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$ , and a mapping  $(\mathcal{F}_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} (\text{Fam})[Y_x]$ , we obtain

$$\begin{aligned} \otimes_{x \in \mathbb{X}} \mathcal{F}_x &\triangleq \left\{ H \in \mathcal{P}\left(\prod_{x \in \mathbb{X}} Y_x\right) \mid \exists (\mathbb{F}_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} \mathcal{F}_x : \right. \\ &\left. (H = \prod_{x \in \mathbb{X}} F_x) \& (\exists K \in \text{Fin}(\mathbb{X}) : \mathbb{F}_s = Y_s \ \forall s \in \mathbb{X} \setminus K) \right\}. \end{aligned} \quad (3.6)$$

In connection with (3.5), note that, for every nonempty sets  $\mathbb{X}$  and  $\mathbb{Y}$ , a mapping  $(Y_x)_{x \in \mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$ , and a mapping  $(\mathcal{Y}_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} \pi[Y_x]$ , we have

$$\odot_{x \in \mathbb{X}} \mathcal{Y}_x = \left\{ \prod_{x \in \mathbb{X}} \Sigma_x : (\Sigma_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} \mathcal{Y}_x \right\} \in \pi\left[\prod_{x \in \mathbb{X}} Y_x\right]. \quad (3.7)$$

In connection with (3.6), note that, for the above  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $(Y_x)_{x \in \mathbb{X}}$ , and  $(\mathcal{Y}_x)_{x \in \mathbb{X}}$ , we have

$$\begin{aligned} \otimes_{x \in \mathbb{X}} \mathcal{Y}_x &= \left\{ H \in \mathcal{P}\left(\prod_{x \in \mathbb{X}} Y_x\right) \mid \exists (\mathbb{F}_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} \mathcal{Y}_x : (H = \prod_{x \in \mathbb{X}} \mathbb{F}_x) \right. \\ &\left. \& (\exists K \in \text{Fin}(\mathbb{X}) : \mathbb{F}_s = Y_s \ \forall s \in \mathbb{X} \setminus K) \right\} \in \pi\left[\prod_{x \in \mathbb{X}} Y_x\right]. \end{aligned} \quad (3.8)$$

Note useful particular cases of (3.7) and (3.8): for nonempty sets  $\mathbb{X}$  and  $\mathbb{Y}$  and mappings  $(Y_x)_{x \in \mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$  and  $(\tau_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} (\text{top})[Y_x]$ , we have

$$\left( \bigodot_{x \in \mathbb{X}} \tau_x \in \pi \left[ \prod_{x \in \mathbb{X}} Y_x \right] \right) \& \left( \bigotimes_{x \in \mathbb{X}} \tau_x \in \pi \left[ \prod_{x \in \mathbb{X}} Y_x \right] \right). \quad (3.9)$$

Using (2.7) in (3.9), we obtain two variants of topological equipment:

$$\left( \mathbf{t}_{\odot}[(\tau_x)_{x \in \mathbb{X}}] \triangleq \{\cup\} \left( \bigodot_{x \in \mathbb{X}} \tau_x \right) \in (\text{top}) \left[ \prod_{x \in \mathbb{X}} Y_x \right] \right) \& \left( \mathbf{t}_{\otimes}[(\tau_x)_{x \in \mathbb{X}}] \triangleq \{\cup\} \left( \bigotimes_{x \in \mathbb{X}} \tau_x \right) \in (\text{top}) \left[ \prod_{x \in \mathbb{X}} Y_x \right] \right). \quad (3.10)$$

Namely, by (3.10), we obtain the following two TSs:

$$\left( \prod_{x \in \mathbb{X}} Y_x, \mathbf{t}_{\odot}[(\tau_x)_{x \in \mathbb{X}}] \right), \quad \left( \prod_{x \in \mathbb{X}} Y_x, \mathbf{t}_{\otimes}[(\tau_x)_{x \in \mathbb{X}}] \right);$$

thus, we obtain the box TS and the Tychonoff product. Of course, topologies (3.10) are comparable. Moreover, for every nonempty sets  $\mathbb{X}$  and  $\mathbb{Y}$  and mappings  $(Y_x)_{x \in \mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$  and  $(\mathcal{I}_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} \pi[Y_x]$ , we have

$$\bigotimes_{x \in \mathbb{X}} \mathcal{I}_x \subset \bigodot_{x \in \mathbb{X}} \mathcal{I}_x. \quad (3.11)$$

From (3.11), the comparability of topologies (3.10) follows, since

$$\prod_{x \in \mathbb{X}} (\text{top})[Y_x] \subset \prod_{x \in \mathbb{X}} \pi[Y_x].$$

Thus, for every nonempty sets  $\mathbb{X}$  and  $\mathbb{Y}$  and mappings  $(Y_x)_{x \in \mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$  and  $(\tau_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} (\text{top})[Y_x]$ , we have

$$\mathbf{t}_{\otimes}[(\tau_x)_{x \in \mathbb{X}}] \subset \mathbf{t}_{\odot}[(\tau_x)_{x \in \mathbb{X}}].$$

#### 4. Ultrafilters and maximal linked systems

In this section, we fix a nonempty set  $E$  and a  $\pi$ -system  $\mathcal{L} \in \pi[E]$ . Recall the notions of filter and ultrafilter on this  $\pi$ -system. So,

$$\mathbb{F}^*(\mathcal{L}) \triangleq \{ \mathcal{F} \in \mathcal{P}'(\mathcal{L} \setminus \{\emptyset\}) \mid (A \cap B \in \mathcal{F} \ \forall A \in \mathcal{F} \ \forall B \in \mathcal{F}) \& ([\mathcal{L}](F) \subset \mathcal{F} \ \forall F \in \mathcal{F}) \}$$

is the set of all filters on  $\mathcal{L}$ . Hence (see [7, Section 2]),

$$\begin{aligned} \mathbb{F}_0^*(\mathcal{L}) &\triangleq \{ \mathcal{U} \in \mathbb{F}^*(\mathcal{L}) \mid \forall \mathcal{F} \in \mathbb{F}^*(\mathcal{L}) \ (\mathcal{U} \subset \mathcal{F}) \implies (\mathcal{U} = \mathcal{F}) \} \\ &= \{ \mathcal{U} \in \mathbb{F}^*(\mathcal{L}) \mid \forall L \in \mathcal{L} \ (L \cap U \neq \emptyset \ \forall U \in \mathcal{U}) \implies (L \in \mathcal{U}) \}. \end{aligned}$$

We recall that  $\mathbb{F}_0^*(\mathcal{L}) \neq \emptyset$  (this is a simplest corollary of the Zorn Lemma). If  $L \in \mathcal{L}$ , then

$$\Phi_{\mathcal{L}}(L) \triangleq \{ \mathcal{U} \in \mathbb{F}_0^*(\mathcal{L}) \mid L \in \mathcal{U} \} = \{ \mathcal{U} \in \mathbb{F}_0^*(\mathcal{L}) \mid L \cap U \neq \emptyset \ \forall U \in \mathcal{U} \}. \quad (4.1)$$

Using (4.1), we introduce the following  $\pi$ -system:

$$(\mathbb{U}\mathbb{F})[E; \mathcal{L}] \triangleq \{ \Phi_{\mathcal{L}}(L) : L \in \mathcal{L} \} \in \pi[\mathbb{F}_0^*(\mathcal{L})]. \quad (4.2)$$

From (2.7) and (4.2), the property  $(\mathbb{UF})[E; \mathcal{L}] \in (\text{BAS})[\mathbb{F}_0^*(\mathcal{L})]$  follows and, as a corollary,

$$\mathbf{T}_{\mathcal{L}}^*[E] \triangleq \{\cup\}((\mathbb{UF})[E; \mathcal{L}]) \in (\text{top})[\mathbb{F}_0^*(\mathcal{L})]. \quad (4.3)$$

In connection with (4.3), note that  $(\mathbb{F}_0^*(\mathcal{L}), \mathbf{T}_{\mathcal{L}}^*[E])$  is a zero-dimensional  $T_2$ -space, see [3]. Thus,

$$(\mathbb{UF})[E; \mathcal{L}] \in (\mathbf{T}_{\mathcal{L}}^*[E] - \text{BAS})_0[\mathbb{F}_0^*(\mathcal{L})].$$

In what follows, we use the inclusion  $\mathbb{F}_0^*(\mathcal{L}) \subset \langle \mathcal{L} - \text{link} \rangle_0[E]$ , see [8, (3.2)]. Now, we recall one general property (see [8, (4.2)]):

$$(\langle \mathcal{L} - \text{link} \rangle_0[E] = \mathbb{F}_0^*(\mathcal{L})) \iff (\mathcal{L} \in \pi_*^\sharp[E]). \quad (4.4)$$

In this connection, note that (see [8, (3.12)]), in the general case of  $\mathcal{L}$ , we have

$$\mathbf{T}_{\mathcal{L}}^*[E] = \mathbb{T}_* \langle E | \mathcal{L} \rangle |_{\mathbb{F}_0^*(\mathcal{L})}. \quad (4.5)$$

In connection with (4.4), we note [8, (4.3)] where supercompactness conditions for a topology of Wallman type were considered. Moreover, in the general case of  $\mathcal{L} \in \pi[E]$ , we have the following representation [8, (4.1)]:

$$\langle \mathcal{L} - \text{link} \rangle_0[E] \setminus \mathbb{F}_0^*(\mathcal{L}) = \{\mathcal{E} \in \langle \mathcal{L} - \text{link} \rangle_0[E] \mid \exists \Sigma_1 \in \mathcal{E} \exists \Sigma_2 \in \mathcal{E} \exists \Sigma_3 \in \mathcal{E} : \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 = \emptyset\}.$$

Therefore, we obtain the following useful equality:

$$\mathbb{F}_0^*(\mathcal{L}) = \{\mathcal{E} \in \langle \mathcal{L} - \text{link} \rangle_0[E] \mid \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \emptyset \ \forall \Sigma_1 \in \mathcal{E} \ \forall \Sigma_2 \in \mathcal{E} \ \forall \Sigma_3 \in \mathcal{E}\}. \quad (4.6)$$

It is easily to verify that

$$\mathbb{F}_0^*(\mathcal{L}) \in \mathbf{C}_{\langle \mathcal{L} - \text{link} \rangle_0[E]}[\mathbb{T}_* \langle E | \mathcal{L} \rangle]. \quad (4.7)$$

By (4.5) and (4.7), we conclude that  $(\mathbb{F}_0^*(\mathcal{L}), \mathbf{T}_{\mathcal{L}}^*[E])$  is a closed subspace of  $(\langle \mathcal{L} - \text{link} \rangle_0[E], \mathbb{T}_* \langle E | \mathcal{L} \rangle)$ .

## 5. The case of product of two widely understood measurable spaces

In this section, we fix nonempty sets  $X$  and  $Y$ . In addition, we fix two  $\pi$ -systems  $\mathcal{X} \in \pi[X]$  and  $\mathcal{Y} \in \pi[Y]$ . We recall that (see (3.1))

$$\mathcal{A}\{\times\}\mathcal{B} \triangleq \{\text{pr}_1(z) \times \text{pr}_2(z) : z \in \mathcal{A} \times \mathcal{B}\}$$

for  $\mathcal{A} \in \mathcal{P}'(\mathcal{P}(X))$  and  $\mathcal{B} \in \mathcal{P}'(\mathcal{P}(Y))$ ; of course,  $\mathcal{A}\{\times\}\mathcal{B} \in \mathcal{P}'(\mathcal{P}(X \times Y))$ . Note that  $X \times Y \neq \emptyset$  and

$$\mathcal{X}\{\times\}\mathcal{Y} = \{\text{pr}_1(z) \times \text{pr}_2(z) : z \in \mathcal{X} \times \mathcal{Y}\} \in \pi[X \times Y]. \quad (5.1)$$

**Proposition 1.** *For  $\mathcal{A} \in \langle \mathcal{X} - \text{link} \rangle[X]$  and  $\mathcal{B} \in \langle \mathcal{Y} - \text{link} \rangle[Y]$ , we have*

$$\mathcal{A}\{\times\}\mathcal{B} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link} \rangle[X \times Y].$$

The proof follows from the definitions.

Below, we use constructions of [11, § 7]. We recall these constructions very briefly. So (see [11, Proposition 17]),

$$\forall H \in (\mathcal{X}\{\times\}\mathcal{Y}) \setminus \{\emptyset\} \ \exists! z \in (\mathcal{X} \setminus \{\emptyset\}) \times (\mathcal{Y} \setminus \{\emptyset\}) : \quad H = \text{pr}_1(z) \times \text{pr}_2(z). \quad (5.2)$$

Using (5.2), we introduce the mappings

$$(\varphi_1 \in (\mathcal{X} \setminus \{\emptyset\})^{(\mathcal{X}\{\times\}\mathcal{Y}) \setminus \{\emptyset\}}) \& (\varphi_2 \in (\mathcal{Y} \setminus \{\emptyset\})^{(\mathcal{X}\{\times\}\mathcal{Y}) \setminus \{\emptyset\}}),$$

for which  $S = \varphi_1(S) \times \varphi_2(S) \quad \forall S \in (\mathcal{X}\{\times\}\mathcal{Y}) \setminus \{\emptyset\}$ . By (2.8), we obtain

$$\langle \mathcal{J} - \text{link} \rangle[\mathbf{I}] \subset \mathcal{P}'(\mathcal{J} \setminus \{\emptyset\}) \quad (5.3)$$

for every nonempty set  $\mathbf{I}$  and  $\mathcal{J} \in \pi[\mathbf{I}]$ . Then, by (5.1) and (5.3), we have

$$\langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link} \rangle[X \times Y] \subset \mathcal{P}'((\mathcal{X}\{\times\}\mathcal{Y}) \setminus \{\emptyset\}).$$

Then, by [11, Proposition 18], for  $\mathcal{E} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link} \rangle[X \times Y]$ , we obtain

$$\left( (\varphi_1)^1(\mathcal{E}) \in \langle \mathcal{X} - \text{link} \rangle[X] \right) \& \left( (\varphi_2)^1(\mathcal{E}) \in \langle \mathcal{Y} - \text{link} \rangle[Y] \right) \quad (5.4)$$

and, by [11, Proposition 19], the following inclusion holds:

$$\mathcal{E} \subset (\varphi_1)^1(\mathcal{E})\{\times\}(\varphi_2)^1(\mathcal{E}). \quad (5.5)$$

From (2.9), (5.4), (5.5), and Proposition 1, we find (see [11, Propositions 20–21]) that, for

$$\begin{aligned} \mathcal{E} &\in \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link} \rangle_0[X \times Y], \\ \mathcal{E} &= (\varphi_1)^1(\mathcal{E})\{\times\}(\varphi_2)^1(\mathcal{E}), \end{aligned} \quad (5.6)$$

where  $(\varphi_1)^1(\mathcal{E}) \in \langle \mathcal{X} - \text{link} \rangle_0[X]$  and  $(\varphi_2)^1(\mathcal{E}) \in \langle \mathcal{Y} - \text{link} \rangle_0[Y]$ . Moreover,

$$\begin{aligned} \forall \mathcal{A} \in \langle \mathcal{X} - \text{link} \rangle_0[X] \quad \forall \mathcal{B} \in \langle \mathcal{Y} - \text{link} \rangle_0[Y] \\ \mathcal{A}\{\times\}\mathcal{B} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link} \rangle_0[X \times Y], \end{aligned} \quad (5.7)$$

see [11, Proposition 22]. As a corollary, from (5.6) and (5.7), we obtain

$$\langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link} \rangle_0[X \times Y] = \{ \text{pr}_1(z)\{\times\}\text{pr}_2(z) : z \in \langle \mathcal{X} - \text{link} \rangle_0[X] \times \langle \mathcal{Y} - \text{link} \rangle_0[Y] \} \quad (5.8)$$

(see [11, Theorem 2]). So, MLSs on the product  $(X \times Y, \mathcal{X}\{\times\}\mathcal{Y})$  are exhausted by products of MLSs on  $(X, \mathcal{X})$  and  $(Y, \mathcal{Y})$ . Note that it is possible to use that MLSs in (5.8). For arbitrary linked families, the property similar to (5.8) is, generally speaking, incorrect.

*Example 1.* Assume that  $X = Y = \overline{1, 3}$ ; thus,  $X = Y$  is a three-element set:  $1 \in X$ ,  $2 \in X$ , and  $3 \in X$ . Suppose that  $\mathcal{X} = \mathcal{P}(X)$  and  $\mathcal{Y} = \mathcal{P}(Y)$ ; of course,  $\mathcal{X} = \mathcal{Y}$ . Now, we introduce the linked family  $\mathcal{E}$  by the rule  $X \times \{2\} \in \mathcal{E}$ ,  $\{2\} \times Y \in \mathcal{E}$ ,  $\{(2, 2)\} \in \mathcal{E}$ , and the family  $\mathcal{E}$  does not contain any other sets. So,  $\mathcal{E}$  is a specific three-element family. Of course,  $\{(2, 2)\} = \{2\} \times \{2\}$ . We have the obvious inclusion

$$\mathcal{E} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link} \rangle[X \times Y].$$

However,

$$\mathcal{E} \neq \mathcal{A}\{\times\}\mathcal{B} \quad \forall \mathcal{A} \in \langle \mathcal{X} - \text{link} \rangle[X] \quad \forall \mathcal{B} \in \langle \mathcal{Y} - \text{link} \rangle[Y].$$

Indeed, let  $\mathcal{E} = \mathcal{A}\{\times\}\mathcal{B}$  for some  $\mathcal{A} \in \langle \mathcal{X} - \text{link} \rangle[X]$  and  $\mathcal{B} \in \langle \mathcal{Y} - \text{link} \rangle[Y]$ . Then

$$(X \times \{2\} \in \mathcal{A}\{\times\}\mathcal{B}) \& (\{2\} \times Y \in \mathcal{A}\{\times\}\mathcal{B}).$$

Using (5.2), we find that  $X \in \mathcal{A}$  and  $Y \in \mathcal{B}$ . Then,  $X \times Y \in \mathcal{A}\{\times\}\mathcal{B}$ . But  $X \times Y \notin \mathcal{E}$ . The obtained contradiction proves the required property:  $\mathcal{E}$  does not have a rectangular structure.  $\square$

Note that, by (5.7), we have

$$\mathcal{U}_1\{\times\}\mathcal{U}_2 \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link} \rangle_0[X \times Y] \quad \forall \mathcal{U}_1 \in \mathbb{F}_0^*(\mathcal{X}) \quad \forall \mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{Y}).$$



**Proposition 2.** *If  $\mathcal{U}_1 \in \mathbb{F}_0^*(\mathcal{X})$  and  $\mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{Y})$ , then  $\mathcal{U}_1\{\times\}\mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y})$ .*

*P r o o f.* Fix  $\mathcal{U}_1 \in \mathbb{F}_0^*(\mathcal{X})$  and  $\mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{Y})$ . Then, in particular,  $\mathcal{U}_1 \in \langle \mathcal{X} - \text{link} \rangle_0[X]$  and  $\mathcal{U}_2 \in \langle \mathcal{Y} - \text{link} \rangle_0[Y]$ . By (5.7), we have

$$\mathcal{U}_1\{\times\}\mathcal{U}_2 = \{\text{pr}_1(z) \times \text{pr}_2(z) : z \in \mathcal{U}_1 \times \mathcal{U}_2\} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link} \rangle_0[X \times Y]. \quad (5.9)$$

Let  $\Gamma \in \mathcal{U}_1\{\times\}\mathcal{U}_2$ ,  $\Lambda \in \mathcal{U}_1\{\times\}\mathcal{U}_2$ , and let  $T \in \mathcal{U}_1\{\times\}\mathcal{U}_2$ . Using (5.9), we choose

$$(\Gamma_1 \in \mathcal{U}_1) \& (\Gamma_2 \in \mathcal{U}_2) \& (\Lambda_1 \in \mathcal{U}_1) \& (\Lambda_2 \in \mathcal{U}_2) \& (T_1 \in \mathcal{U}_1) \& (T_2 \in \mathcal{U}_2)$$

with the following properties:

$$(\Gamma = \Gamma_1 \times \Gamma_2) \& (\Lambda = \Lambda_1 \times \Lambda_2) \& (T = T_1 \times T_2). \quad (5.10)$$

By (4.6), we obtain the following obvious statements:

$$(\Gamma_1 \cap \Lambda_1 \cap T_1 \neq \emptyset) \& (\Gamma_2 \cap \Lambda_2 \cap T_2 \neq \emptyset). \quad (5.11)$$

Let  $\alpha \in \Gamma_1 \cap \Lambda_1 \cap T_1$  and  $\beta \in \Gamma_2 \cap \Lambda_2 \cap T_2$  (we use (5.11)). Then, by (5.10),  $(\alpha, \beta) \in \Gamma \cap \Lambda \cap T$ . Since the choice of  $\Gamma$ ,  $\Lambda$ , and  $T$  was arbitrary, the required inclusion  $\mathcal{U}_1\{\times\}\mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y})$  follows from (4.6) and (5.9).  $\square$

**Proposition 3.** *If  $\mathcal{U} \in \mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y})$ , then  $\exists \mathcal{A} \in \mathbb{F}_0^*(\mathcal{X}) \quad \exists \mathcal{B} \in \mathbb{F}_0^*(\mathcal{Y}) : \mathcal{U} = \mathcal{A}\{\times\}\mathcal{B}$ .*

*P r o o f.* Fix  $\mathcal{U} \in \mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y})$ . Then, by (4.6), we have

$$\mathcal{U} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link} \rangle_0[X \times Y] \quad (5.12)$$

and the following property:

$$A \cap B \cap C \neq \emptyset \quad \forall A \in \mathcal{U} \quad \forall B \in \mathcal{U} \quad \forall C \in \mathcal{U}. \quad (5.13)$$

From (5.8) and (5.12), we conclude that  $\mathcal{U} = \mathcal{U}_1\{\times\}\mathcal{U}_2$  for some  $\mathcal{U}_1 \in \langle \mathcal{X} - \text{link} \rangle_0[X]$  and  $\mathcal{U}_2 \in \langle \mathcal{Y} - \text{link} \rangle_0[Y]$ . In addition (see (2.10)),  $X \in \mathcal{U}_1$  and  $Y \in \mathcal{U}_2$ .

Consider an MLS  $\mathcal{U}_1$ . For this, we fix  $M \in \mathcal{U}_1$ ,  $N \in \mathcal{U}_1$ , and  $T \in \mathcal{U}_1$ . Then, by the choice of  $\mathcal{U}_1$  and  $\mathcal{U}_2$ , we have

$$(M \times Y \in \mathcal{U}) \& (N \times Y \in \mathcal{U}) \& (T \times Y \in \mathcal{U}), \quad (5.14)$$

(see (2.10)). From (5.13) and (5.14), we obtain  $M \cap N \cap T \neq \emptyset$ . Since the choice of  $M$ ,  $N$ , and  $T$  was arbitrary, the inclusion  $\mathcal{U}_1 \in \mathbb{F}_0^*(\mathcal{X})$  is obtained (see (4.6)). The inclusion  $\mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{Y})$  is established similarly.  $\square$

**Theorem 1.** *The following equality holds:*

$$\mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y}) = \{\text{pr}_1(z)\{\times\}\text{pr}_2(z) : z \in \mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y})\}.$$

The proof reduces to immediate combination of Propositions 2 and 3. Finally, we note an important property of topological character (see [13, Theorem 5.1]). We recall that, by (3.1) and (3.2),

$$\begin{aligned} \mathbb{T}_* \langle X | \mathcal{X} \rangle \{\times\} \mathbb{T}_* \langle Y | \mathcal{Y} \rangle &= \{\text{pr}_1(z) \times \text{pr}_2(z) : z \in \mathbb{T}_* \langle X | \mathcal{X} \rangle \times \mathbb{T}_* \langle Y | \mathcal{Y} \rangle\} \\ &\in \pi[\langle \mathcal{X} - \text{link} \rangle_0[X] \times \langle \mathcal{Y} - \text{link} \rangle_0[Y]]; \end{aligned}$$

then, by (2.7), the natural topology

$$\mathbb{T}_*\langle X|\mathcal{X}\rangle \otimes \mathbb{T}_*\langle Y|\mathcal{Y}\rangle \triangleq \{\cup\}(\mathbb{T}_*\langle X|\mathcal{X}\rangle\{\times\}\mathbb{T}_*\langle Y|\mathcal{Y}\rangle) \in (\text{top})[\langle \mathcal{X} - \text{link}\rangle_0[X] \times \langle \mathcal{Y} - \text{link}\rangle_0[Y]]$$

of the product of Stone-type TSs is realized. Moreover, the following Stone-type topology is defined:

$$\mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle \in (\text{top})[\langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link}\rangle_0[X \times Y]].$$

Then, by [13, Theorem 5.1], the mapping

$$z \longmapsto \text{pr}_1(z)\{\times\}\text{pr}_2(z) : \langle \mathcal{X} - \text{link}\rangle_0[X] \times \langle \mathcal{Y} - \text{link}\rangle_0[Y] \longrightarrow \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link}\rangle_0[X \times Y] \quad (5.15)$$

is a homeomorphism from the TS

$$(\langle \mathcal{X} - \text{link}\rangle_0[X] \times \langle \mathcal{Y} - \text{link}\rangle_0[Y], \mathbb{T}_*\langle X|\mathcal{X}\rangle \otimes \mathbb{T}_*\langle Y|\mathcal{Y}\rangle) \quad (5.16)$$

onto the TS  $(\langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link}\rangle_0[X \times Y], \mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle)$ .

Note that, by (4.7), we have

$$\mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y}) \in \mathbf{C}_{\langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link}\rangle_0[X \times Y]}[\mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle].$$

Moreover, using (4.5), we obtain

$$\mathbf{T}_{\mathcal{X}\{\times\}\mathcal{Y}}^*[X \times Y] = \mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle|_{\mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y})}. \quad (5.17)$$

So, ultrafilters of  $\pi$ -system  $\mathcal{X}\{\times\}\mathcal{Y}$  form a closed subspace of TSs homeomorphic to (5.16). Theorem 1 reveals the structure of this subspace.

## 6. Infinite products of maximal linked systems, 1

Unless otherwise stated, in what follows, nonempty sets  $X$  and  $\mathbf{E}$  and a mapping  $(E_x)_{x \in X} \in \mathcal{P}'(\mathbf{E})^X$  are fixed (for  $x \in X$ , we denote by  $E_x$  a nonempty subset of  $\mathbf{E}$ ). Define the set

$$\mathbb{E} \triangleq \prod_{x \in X} E_x = \{f \in \mathbf{E}^X \mid f(x) \in E_x \forall x \in X\} \in \mathcal{P}'(\mathbf{E}^X) \quad (6.1)$$

(hereinafter, the axiom of choice is used). Finally, we fix

$$(\mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \pi[E_x]. \quad (6.2)$$

We obtain (see (6.2)) the following two variants of  $\pi$ -systems:

$$\begin{aligned} \bigotimes_{x \in X} \mathcal{L}_x = \left\{ H \in \mathcal{P}(\mathbb{E}) \mid \exists (L_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x : (H = \prod_{x \in X} L_x) \& (\exists K \in \text{Fin}(X) : \right. \\ \left. L_s = E_s \forall s \in X \setminus K) \right\} \in \pi[\mathbb{E}], \end{aligned} \quad (6.3)$$

$$\bigodot_{x \in X} \mathcal{L}_x = \left\{ \prod_{x \in X} L_x : (L_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x \right\} \in \pi[\mathbb{E}], \quad (6.4)$$

$$\bigotimes_{x \in X} \mathcal{L}_x \subset \bigodot_{x \in X} \mathcal{L}_x \quad (6.5)$$

(we use [8, (6.4)–(6.5)]); in connection with (6.3)–(6.5), we recall (3.6)–(3.8). So, we have two comparable  $\pi$ -systems on  $\mathbb{E}$ .

Now, we note one simple property:

$$\begin{aligned} \forall (A_x)_{x \in X} \in \mathcal{P}'(\mathbf{E})^X \quad \forall (B_x)_{x \in X} \in \mathcal{P}'(\mathbf{E})^X \\ \left( \prod_{x \in X} A_x = \prod_{x \in X} B_x \right) \iff (A_x = B_x \quad \forall x \in X). \end{aligned} \quad (6.6)$$

Moreover, we note that

$$\begin{aligned} \forall (H_x^{(1)})_{x \in X} \in \mathcal{P}(\mathbf{E})^X \quad \forall (H_x^{(2)})_{x \in X} \in \mathcal{P}(\mathbf{E})^X \\ \left( \prod_{x \in X} H_x^{(1)} \right) \cap \left( \prod_{x \in X} H_x^{(2)} \right) = \prod_{x \in X} (H_x^{(1)} \cap H_x^{(2)}). \end{aligned} \quad (6.7)$$

The property (6.7) assumes a natural development; now, we note only that

$$\begin{aligned} \forall (H_x^{(1)})_{x \in X} \in \mathcal{P}(\mathbf{E})^X \quad \forall (H_x^{(2)})_{x \in X} \in \mathcal{P}(\mathbf{E})^X \quad \forall (H_x^{(3)})_{x \in X} \in \mathcal{P}(\mathbf{E})^X \\ \left( \prod_{x \in X} H_x^{(1)} \right) \cap \left( \prod_{x \in X} H_x^{(2)} \right) \cap \left( \prod_{x \in X} H_x^{(3)} \right) = \prod_{x \in X} (H_x^{(1)} \cap H_x^{(2)} \cap H_x^{(3)}). \end{aligned} \quad (6.8)$$

By (6.6), an obvious corollary is realized; namely,

$$\begin{aligned} \forall H \in \left( \bigodot_{x \in X} \mathcal{P}(E_x) \right) \setminus \{\emptyset\} \quad \exists! (\Sigma_x)_{x \in X} \in \prod_{x \in X} \mathcal{P}'(E_x) \\ H = \prod_{x \in X} \Sigma_x. \end{aligned} \quad (6.9)$$

Using (6.9), we define a mapping

$$\mathbf{P} : \left( \bigodot_{x \in X} \mathcal{P}(E_x) \right) \setminus \{\emptyset\} \longrightarrow \prod_{x \in X} \mathcal{P}'(E_x)$$

by the following rule: if  $H \in \left( \bigodot_{x \in X} \mathcal{P}(E_x) \right) \setminus \{\emptyset\}$ , then  $\mathbf{P}(H) \in \prod_{x \in X} \mathcal{P}'(E_x)$  is a mapping such that

$$H = \prod_{\chi \in X} \mathbf{P}(H)(\chi). \quad (6.10)$$

We can use the variant  $H = \prod_{x \in X} \Sigma_x$ , where  $(\Sigma_x)_{x \in X} \in \prod_{x \in X} \mathcal{P}'(E_x)$ . In addition, by (6.9), we have

$$\Sigma_\chi = \mathbf{P} \left( \prod_{x \in X} \Sigma_x \right) (\chi) \quad \forall (\Sigma_x)_{x \in X} \in \prod_{x \in X} \mathcal{P}'(E_x) \quad \forall \chi \in X. \quad (6.11)$$

Now, note the following obvious inclusions:

$$\left( \prod_{x \in X} (\mathcal{L}_x \setminus \{\emptyset\}) \subset \prod_{x \in X} \mathcal{P}'(E_x) \right) \& \left( \left( \bigodot_{x \in X} \mathcal{L}_x \right) \setminus \{\emptyset\} \subset \left( \bigodot_{x \in X} \mathcal{P}(E_x) \right) \setminus \{\emptyset\} \right). \quad (6.12)$$

Now, for  $\chi \in X$ , we define  $\mathbf{P}_\chi : \left( \bigodot_{x \in X} \mathcal{P}(E_x) \right) \setminus \{\emptyset\} \longrightarrow \mathcal{P}'(E_\chi)$  by the natural rule

$$\mathbf{P}_\chi(H) \triangleq \mathbf{P}(H)(\chi) \quad \forall H \in \left( \bigodot_{x \in X} \mathcal{P}(E_x) \right) \setminus \{\emptyset\}. \quad (6.13)$$

Of course, (6.13) defines the corresponding projection mapping. From (6.11) and (6.13), for  $\chi \in X$  and  $(\Sigma_x)_{x \in X} \in \prod_{x \in X} \mathcal{P}'(E_x)$ , we obtain

$$\mathbf{P}_\chi \left( \prod_{x \in X} \Sigma_x \right) = \Sigma_\chi. \quad (6.14)$$

From (6.12) and (6.14), we, in particular, obtain

$$\mathbf{P}_\chi \left( \prod_{x \in X} L_x \right) = L_\chi \quad \forall (L_x)_{x \in X} \in \prod_{x \in X} (\mathcal{L}_x \setminus \{\emptyset\}) \quad \forall \chi \in X.$$

Using the notion of the set image, we suppose that  $\forall \mathcal{H} \in \mathcal{P}(\langle \bigodot_{x \in X} \mathcal{P}(E_x) \rangle \setminus \{\emptyset\}) \quad \forall \chi \in X$

$$\mathbf{P}_\chi^1(\mathcal{H}) \stackrel{\Delta}{=} (\mathbf{P}_\chi)^1(\mathcal{H}). \quad (6.15)$$

Then, the following obvious property holds: if  $\mathcal{H} \in \mathcal{P}(\langle \bigodot_{x \in X} \mathcal{L}_x \rangle \setminus \{\emptyset\})$  and  $\chi \in X$ , then

$$\mathbf{P}_\chi^1(\mathcal{H}) \in \mathcal{P}(\mathcal{L}_\chi \setminus \{\emptyset\}). \quad (6.16)$$

We can use a natural combination of (5.3) and (6.16): a linked system can be used as  $\mathcal{H}$ . In addition, by [13, Proposition 3.2], we have

$$\mathbf{P}_\chi^1(\mathcal{E}) \in \langle \mathcal{L}_\chi - \text{link} \rangle [E_\chi] \quad \forall \mathcal{E} \in \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle [\mathbb{E}] \quad \forall \chi \in X.$$

As a corollary, for  $\mathcal{E} \in \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle [\mathbb{E}]$ , we obtain the mapping

$$(\mathbf{P}_x^1(\mathcal{E}))_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle [E_x]. \quad (6.17)$$

**Proposition 4.** *If  $(\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle [E_x]$ , then  $\bigodot_{x \in X} \mathcal{E}_x \in \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle [\mathbb{E}]$ .*

This proposition corresponds to [13, Proposition 3.1]. To prove Proposition 4, it suffices to use (6.7) (and the axiom of choice). From (6.17) and Proposition 4, we obtain

$$\bigodot_{x \in X} \mathbf{P}_x^1(\mathcal{E}) \in \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle [\mathbb{E}] \quad \forall \mathcal{E} \in \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle [\mathbb{E}]. \quad (6.18)$$

Note an obvious analog of (5.5); namely, for  $\mathcal{E} \in \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0 [\mathbb{E}]$ , we have

$$\mathcal{E} \subset \bigodot_{x \in X} \mathbf{P}_x^1(\mathcal{E});$$

therefore (see (2.9) and (6.18)), by [13, Proposition 3.4], we obtain

$$\mathcal{E} = \bigodot_{x \in X} \mathbf{P}_x^1(\mathcal{E}). \quad (6.19)$$

In connection with (6.19), note that, by [13, Proposition 3.5], we have

$$\mathbf{P}_\chi^1(\mathcal{E}) \in \langle \mathcal{L}_\chi - \text{link} \rangle_0 [E_\chi] \quad \forall \mathcal{E} \in \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0 [\mathbb{E}] \quad \forall \chi \in X.$$

Then, (6.17) is supplemented by the following statement:

$$(\mathbf{P}_x^1(\mathcal{E}))_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0 [E_x] \quad \forall \mathcal{E} \in \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0 [\mathbb{E}]. \quad (6.20)$$

Moreover, by [13, Proposition 3.6], we obtain the following property:

$$\bigodot_{x \in X} \mathcal{E}_x \in \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0 [\mathbb{E}] \quad \forall (\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0 [E_x].$$

By (6.19) and (6.20), the following basic statement (see [13, Theorem 3.1]) holds:

$$\langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}] = \left\{ \bigodot_{x \in X} \mathcal{E}_x : (\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \right\}. \quad (6.21)$$

In (6.21), we have a natural analog of (5.8). In connection with (6.21), we note that

$$\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) = \left\{ (\mathcal{U}_x)_{x \in X} \in \mathcal{P}'(\mathcal{P}(\mathbf{E}))^X \mid \mathcal{U}_s \in \mathbb{F}_0^*(\mathcal{L}_s) \ \forall s \in X \right\} \subset \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x]. \quad (6.22)$$

Then, by (6.21) and (6.22), we obtain

$$\bigodot_{x \in X} \mathcal{U}_x \in \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}] \quad \forall (\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x). \quad (6.23)$$

**Proposition 5.** *If  $(\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)$ , then  $\bigodot_{x \in X} \mathcal{U}_x \in \mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x)$ .*

*P r o o f.* Fix  $(\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)$ . Then, for  $\chi \in X$ , we obtain

$$\mathcal{U}_\chi \in \mathbb{F}_0^*(\mathcal{L}_\chi). \quad (6.24)$$

Recall (4.6) and (6.4). Then, by (4.6) and (6.23), we have

$$\begin{aligned} & \left( \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \emptyset \quad \forall \Sigma_1 \in \bigodot_{x \in X} \mathcal{U}_x \quad \forall \Sigma_2 \in \bigodot_{x \in X} \mathcal{U}_x \quad \forall \Sigma_3 \in \bigodot_{x \in X} \mathcal{U}_x \right) \\ & \implies \left( \bigodot_{x \in X} \mathcal{U}_x \in \mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x) \right). \end{aligned} \quad (6.25)$$

Let  $\mathbb{A} \in \bigodot_{x \in X} \mathcal{U}_x$ ,  $\mathbb{B} \in \bigodot_{x \in X} \mathcal{U}_x$ , and let  $\mathbb{C} \in \bigodot_{x \in X} \mathcal{U}_x$ . Then, by (3.7), for some

$$\left( (\mathbb{A}_x)_{x \in X} \in \prod_{x \in X} \mathcal{U}_x \right) \& \left( (\mathbb{B}_x)_{x \in X} \in \prod_{x \in X} \mathcal{U}_x \right) \& \left( (\mathbb{C}_x)_{x \in X} \in \prod_{x \in X} \mathcal{U}_x \right),$$

we obtain the following equalities:

$$\left( \mathbb{A} = \prod_{x \in X} \mathbb{A}_x \right) \& \left( \mathbb{B} = \prod_{x \in X} \mathbb{B}_x \right) \& \left( \mathbb{C} = \prod_{x \in X} \mathbb{C}_x \right). \quad (6.26)$$

From (6.22), for  $x \in X$ , we obtain the inclusions  $\mathbb{A}_x \in \mathcal{P}(\mathbf{E})$ ,  $\mathbb{B}_x \in \mathcal{P}(\mathbf{E})$ , and  $\mathbb{C}_x \in \mathcal{P}(\mathbf{E})$ . Then, by (6.8) and (6.26)

$$\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} = \prod_{x \in X} (\mathbb{A}_x \cap \mathbb{B}_x \cap \mathbb{C}_x). \quad (6.27)$$

In addition, for  $x \in X$ , we obtain  $\mathbb{A}_x \in \mathcal{U}_x$ ,  $\mathbb{B}_x \in \mathcal{U}_x$ , and  $\mathbb{C}_x \in \mathcal{U}_x$ ; then, by (4.6) and (6.24)  $\mathbb{A}_x \cap \mathbb{B}_x \cap \mathbb{C}_x \neq \emptyset$ . So,

$$(\mathbb{A}_x \cap \mathbb{B}_x \cap \mathbb{C}_x)_{x \in X} \in \mathcal{P}'(\mathbf{E})^X.$$

Using (6.27) (and the axiom of choice), we obtain  $\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} \neq \emptyset$ . Since the choice of  $\mathbb{A}$ ,  $\mathbb{B}$ , and  $\mathbb{C}$  was arbitrary, it is established that the premise of implication (6.25) is true. So, we obtain the required property

$$\bigodot_{x \in X} \mathcal{U}_x \in \mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x).$$

□

**Proposition 6.** *If  $\mathcal{U} \in \mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x)$ , then  $\exists (\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) : \mathcal{U} = \bigodot_{x \in X} \mathcal{U}_x$ .*

*P r o o f.* Fix  $\mathcal{U} \in \mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x)$ . Then, in particular,

$$\mathcal{U} \in \left\langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \right\rangle_0[\mathbb{E}].$$

By (6.21),  $\mathcal{U} = \bigodot_{x \in X} \mathcal{E}_x$ , where

$$(\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x].$$

By formula (4.6), we get

$$\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \emptyset \quad \forall \Sigma_1 \in \mathcal{U} \quad \forall \Sigma_2 \in \mathcal{U} \quad \forall \Sigma_3 \in \mathcal{U}. \quad (6.28)$$

Let  $\chi \in X$ . Then,  $\mathcal{E}_\chi \in \langle \mathcal{L}_\chi - \text{link} \rangle_0[E_\chi]$ . Therefore, by formula (4.6), we get

$$(\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \emptyset \quad \forall \Sigma_1 \in \mathcal{E}_\chi \quad \forall \Sigma_2 \in \mathcal{E}_\chi \quad \forall \Sigma_3 \in \mathcal{E}_\chi) \implies (\mathcal{E}_\chi \in \mathbb{F}_0^*(\mathcal{L}_\chi)). \quad (6.29)$$

Choose arbitrary  $A \in \mathcal{E}_\chi$ ,  $B \in \mathcal{E}_\chi$ , and  $C \in \mathcal{E}_\chi$ . By (5.3),  $A \in \mathcal{P}'(\mathbf{E})$ ,  $B \in \mathcal{P}'(\mathbf{E})$ , and  $C \in \mathcal{P}'(\mathbf{E})$ . Now, we introduce  $(\tilde{A}_x)_{x \in X} \in \mathcal{P}'(\mathbf{E})^X$  by the rule

$$(\tilde{A}_\chi \triangleq A) \& (\tilde{A}_x \triangleq E_x \quad \forall x \in X \setminus \{\chi\}).$$

Similarly, we introduce  $(\tilde{B}_x)_{x \in X} \in \mathcal{P}'(\mathbf{E})^X$  by the rule

$$(\tilde{B}_\chi \triangleq B) \& (\tilde{B}_x \triangleq E_x \quad \forall x \in X \setminus \{\chi\}).$$

Finally, define  $(\tilde{C}_x)_{x \in X} \in \mathcal{P}'(\mathbf{E})^X$  by the rule

$$(\tilde{C}_\chi \triangleq C) \& (\tilde{C}_x \triangleq E_x \quad \forall x \in X \setminus \{\chi\}).$$

Then, by (6.8), we obtain the following obvious equality:

$$\left( \prod_{x \in X} \tilde{A}_x \right) \cap \left( \prod_{x \in X} \tilde{B}_x \right) \cap \left( \prod_{x \in X} \tilde{C}_x \right) = \prod_{x \in X} (\tilde{A}_x \cap \tilde{B}_x \cap \tilde{C}_x). \quad (6.30)$$

Note that, by (2.10),  $(\tilde{A}_x \in \mathcal{E}_x) \& (\tilde{B}_x \in \mathcal{E}_x) \& (\tilde{C}_x \in \mathcal{E}_x)$  for  $x \in X$ . Therefore,

$$\left( (\tilde{A}_x)_{x \in X} \in \prod_{x \in X} \mathcal{E}_x \right) \& \left( (\tilde{B}_x)_{x \in X} \in \prod_{x \in X} \mathcal{E}_x \right) \& \left( (\tilde{C}_x)_{x \in X} \in \prod_{x \in X} \mathcal{E}_x \right).$$

By the choice of  $(\mathcal{E}_x)_{x \in X}$ , we obtain (see (3.7))

$$\left( \prod_{x \in X} \tilde{A}_x \in \mathcal{U} \right) \& \left( \prod_{x \in X} \tilde{B}_x \in \mathcal{U} \right) \& \left( \prod_{x \in X} \tilde{C}_x \in \mathcal{U} \right).$$

As a corollary, by (6.28), we have the following important statement:

$$\left( \prod_{x \in X} \tilde{A}_x \right) \cap \left( \prod_{x \in X} \tilde{B}_x \right) \cap \left( \prod_{x \in X} \tilde{C}_x \right) \neq \emptyset.$$

Then, from (6.30), we obtain  $\tilde{A}_x \cap \tilde{B}_x \cap \tilde{C}_x \neq \emptyset$  for  $x \in X$ . In particular,  $\tilde{A}_x \cap \tilde{B}_x \cap \tilde{C}_x \neq \emptyset$ . As a corollary,  $A \cap B \cap C \neq \emptyset$ . Since the choice of  $A$ ,  $B$ , and  $C$  was arbitrary, the following property holds:

$$\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \emptyset \quad \forall \Sigma_1 \in \mathcal{E}_X \quad \forall \Sigma_2 \in \mathcal{E}_X \quad \forall \Sigma_3 \in \mathcal{E}_X.$$

From (6.29), we obtain  $\mathcal{E}_X \in \mathbb{F}_0^*(\mathcal{L}_X)$ . Since the choice of  $X$  was arbitrary,

$$\mathcal{E}_x \in \mathbb{F}_0^*(\mathcal{L}_x) \quad \forall x \in X.$$

As a corollary, by the choice of  $(\mathcal{E}_x)_{x \in X}$ , we obtain

$$(\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) : \mathcal{U} = \bigodot_{x \in X} \mathcal{E}_x.$$

□

**Theorem 2.** *The following equality is true:*

$$\mathbb{F}_0^*\left(\bigodot_{x \in X} \mathcal{L}_x\right) = \left\{ \bigodot_{x \in X} \mathcal{U}_x : (\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) \right\}.$$

The proof immediately follows from Propositions 5 and 6. Returning to (6.21), we note that

$$\mathbf{f} \triangleq \left( \bigodot_{x \in X} \mathcal{E}_x \right)_{(\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0 [E_x]} \in \left\langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \right\rangle_0 [\mathbb{E}]^{\prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0 [E_x]} \quad (6.31)$$

is a surjection. Moreover (see (2.11)), by [14, Proposition 4.3], for  $(L_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x$ , we have

$$\mathbf{f}^{-1} \left( \left\langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \right\rangle_0^0 [\mathbb{E} \mid \prod_{x \in X} L_x] \right) = \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0^0 [E_x \mid L_x]. \quad (6.32)$$

Moreover, the following set-product is defined:

$$\prod_{x \in X} \hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] = \{ (\mathbb{H}_x)_{x \in X} \in \mathcal{P}(\mathcal{P}'(\mathcal{P}(\mathbf{E})))^X \mid \mathbb{H}_x \in \hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] \forall x \in X \}.$$

In addition (see Section 2),  $\hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] \in \mathcal{P}'(\mathcal{P}(\langle \mathcal{L}_x - \text{link} \rangle_0 [E_x]))$  for  $x \in X$ . Then, by (3.5), we have

$$\bigodot_{x \in X} \hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] = \left\{ \prod_{x \in X} \mathbb{H}_x : (\mathbb{H}_x)_{x \in X} \in \prod_{x \in X} \hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] \right\} \in \mathcal{P}' \left( \mathcal{P} \left( \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0 [E_x] \right) \right);$$

thus, the box product of the families  $\hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x]$ ,  $x \in X$ , is defined. Moreover, we have the property

$$\hat{\mathfrak{C}}_0^* \left[ \mathbb{E}; \bigodot_{x \in X} \mathcal{L}_x \right] \in (\text{p-BAS}) \left[ \left\langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \right\rangle_0 [\mathbb{E}] \right].$$

From (6.32), we obtain the following statement:

$$\mathbf{f}^{-1}(\mathbb{H}) \in \bigodot_{x \in X} \hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] \quad \forall \mathbb{H} \in \hat{\mathfrak{C}}_0^* \left[ \mathbb{E}; \bigodot_{x \in X} \mathcal{L}_x \right]. \quad (6.33)$$

Now, we recall that (see (2.12)), for  $x \in X$ ,

$$\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle \in (\text{top})[\langle \mathcal{L}_x - \text{link} \rangle_0[E_x]] : \hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] \subset \mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle. \quad (6.34)$$

Then,

$$(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X} \in \prod_{x \in X} (\text{top})[\langle \mathcal{L}_x - \text{link} \rangle_0[E_x]].$$

By (3.9),

$$\bigodot_{x \in X} \mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle = \left\{ \prod_{x \in X} G_x : (G_x)_{x \in X} \in \prod_{x \in X} \mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle \right\} \in \pi \left[ \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \right]$$

is used as an open base for the corresponding box topology:

$$\mathfrak{t}_\odot[(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}] = \{\cup\}(\bigodot_{x \in X} \mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle) \in (\text{top}) \left[ \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \right].$$

Moreover, by (6.34), we obtain

$$\bigodot_{x \in X} \hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] \subset \bigodot_{x \in X} \mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle \subset \mathfrak{t}_\odot[(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}]. \quad (6.35)$$

On the other hand, by (2.12), the following inclusion holds:

$$\hat{\mathfrak{C}}_0^*[\mathbb{E}; \bigodot_{x \in X} \mathcal{L}_x] \in (\text{p-BAS})_0 \left[ \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}]; \mathbb{T}_* \langle \mathbb{E} | \bigodot_{x \in X} \mathcal{L}_x \rangle \right]. \quad (6.36)$$

Therefore, from (6.33) and (6.35), we find that  $\mathbf{f}$  is a continuous mapping in the sense of topologies

$$\mathfrak{t}_\odot[(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}], \quad \mathbb{T}_* \langle \mathbb{E} | \bigodot_{x \in X} \mathcal{L}_x \rangle; \quad (6.37)$$

we use [15, Proposition 1.4.1]. So, we established the continuity of the mapping (6.31). In addition, the space-product of the families  $\langle \mathcal{L}_x - \text{link} \rangle_0[E_x]$ ,  $x \in X$ , is equipped with box topology. Moreover, note that  $\mathbf{f}$  (6.31) is a bijection from

$$\prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x]$$

onto  $\langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}]$ ; see [14, Proposition 5.2]. As a result,  $\mathbf{f}$  (6.31) is a continuous bijection, i.e., condensation in the sense of topologies (6.37). So, the TS

$$\left( \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x], \mathfrak{t}_\odot[(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}] \right)$$

condenses on the following space of Stone type:

$$\left( \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}], \mathbb{T}_* \langle \mathbb{E} | \bigodot_{x \in X} \mathcal{L}_x \rangle \right). \quad (6.38)$$

In addition, by (4.7), we obtain

$$\mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x) \in \mathbf{C}_{\langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}]} \left[ \mathbb{T}_* \langle \mathbb{E} | \bigodot_{x \in X} \mathcal{L}_x \rangle \right].$$

Theorem 2 reveals the structure of the set  $\mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x)$ . By (4.5), we have

$$\mathbf{T}_{\bigodot_{x \in X} \mathcal{L}_x}^*[\mathbb{E}] = \mathbb{T}_* \langle \mathbb{E} | \bigodot_{x \in X} \mathcal{L}_x \rangle |_{\mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x)};$$

thus, ultrafilters of the  $\pi$ -system  $\bigodot_{x \in X} \mathcal{L}_x$  form a closed subspace of the space (6.38).



## 7. Infinite products of maximal linked systems, 2

We use the notation of the previous section:  $X$ ,  $\mathbb{E}$ ,  $(E_x)_{x \in X}$ , and  $\mathbb{E}$ . By (3.8), (6.3), and (6.5), we have

$$\begin{aligned} \bigotimes_{x \in X} \mathcal{L}_x &= \left\{ \Lambda \in \mathcal{P}(\mathbb{E}) \mid \exists (L_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x : \right. \\ &\left. \left( \Lambda = \prod_{x \in X} L_x \right) \& (\exists K \in \text{Fin}(X) : L_s = E_s \ \forall s \in X \setminus K) \right\} \in \pi[\mathbb{E}] \cap \mathcal{P}\left( \bigodot_{x \in X} \mathcal{L}_x \right). \end{aligned} \quad (7.1)$$

Consider a widely understood MS

$$\left( \mathbb{E}, \bigotimes_{x \in X} \mathcal{L}_x \right) : \bigotimes_{x \in X} \mathcal{L}_x \subset \bigodot_{x \in X} \mathcal{L}_x. \quad (7.2)$$

Note that (see (2.10)) the following inclusion is true:

$$\prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \subset \prod_{x \in X} (\text{Fam})[E_x].$$

Therefore, by [13, (4.5), Proposition 4.1], we obtain

$$\begin{aligned} \bigotimes_{x \in X} \mathcal{E}_x &= \left\{ H \in \mathcal{P}(\mathbb{E}) \mid \exists (\Sigma_x)_{x \in X} \in \prod_{x \in X} \mathcal{E}_x : \right. \\ &\left. \left( H = \prod_{x \in X} \Sigma_x \right) \& (\exists K \in \text{Fin}(X) : \Sigma_s = E_s \ \forall s \in X \setminus K) \right\} \\ &\in \left\langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \right\rangle_0[\mathbb{E}] \quad \forall (\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x]. \end{aligned} \quad (7.3)$$

We recall that (see [13, Theorem 4.1]) the following equality is true:

$$\left\langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \right\rangle_0[\mathbb{E}] = \left\{ \bigotimes_{x \in X} \mathcal{E}_x : (\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \right\} \quad (7.4)$$

By (6.10), (6.12), (6.13), and (7.2), we have

$$\Lambda = \prod_{x \in X} \mathbf{P}_x(\Lambda) \quad \forall \Lambda \in \left( \bigotimes_{x \in X} \mathcal{L}_x \right) \setminus \{\emptyset\}.$$

We use notation (6.15) for the image operation. Then, by [13, Propostion 4.2], we have

$$\left( \mathbf{P}_x^1(\mathcal{E}) \right)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \quad \forall \mathcal{E} \in \left\langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \right\rangle_0[\mathbb{E}] \quad (7.5)$$

(we use (5.3)). In this connection, we use the following useful property:

$$\left( \mathcal{E}_x \right)_{x \in X} = \left( \mathbf{P}_\chi^1 \left( \bigotimes_{x \in X} \mathcal{E}_x \right) \right)_{\chi \in X} \quad \forall (\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x]; \quad (7.6)$$

in (7.6), we use (7.4), (7.5), and [14, Proposition 6.1]. Now, we recall (6.22); hence (see (7.4)),

$$\bigotimes_{x \in X} \mathcal{U}_x \in \left\langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \right\rangle_0[\mathbb{E}] \quad \forall (\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x). \quad (7.7)$$

Moreover, by (7.1), the general constructions imply the following obvious inclusion:

$$\mathbb{F}_0^*(\bigotimes_{x \in X} \mathcal{L}_x) \subset \langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}]. \quad (7.8)$$

In what follows, we consider questions related to a representation of ultrafilters on  $\bigotimes_{x \in X} \mathcal{L}_x$  as products (7.7). In this connection, we recall [4]. But, in the present constructions, we use a scheme based on (4.6).

**Proposition 7.** *If  $(\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)$ , then  $\bigotimes_{x \in X} \mathcal{U}_x \in \mathbb{F}_0^*(\bigotimes_{x \in X} \mathcal{L}_x)$ .*

*P r o o f.* Fix

$$(\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x).$$

Then, by (7.7), we have

$$\bigotimes_{x \in X} \mathcal{U}_x \in \langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}]. \quad (7.9)$$

The inclusion  $\mathcal{U}_x \in \mathbb{F}_0^*(\mathcal{L}_x)$  holds for  $x \in X$ ; therefore,

$$\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 = (\Sigma_1 \cap \Sigma_2) \cap \Sigma_3 \neq \emptyset \quad \forall \Sigma_1 \in \mathcal{U}_x \quad \forall \Sigma_2 \in \mathcal{U}_x \quad \forall \Sigma_3 \in \mathcal{U}_x \quad (7.10)$$

(we use the axioms of filter). Moreover, by (4.6) and (7.9), we obtain the following implication:

$$\begin{aligned} & \left( \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \emptyset \quad \forall \Sigma_1 \in \bigotimes_{x \in X} \mathcal{U}_x \quad \forall \Sigma_2 \in \bigotimes_{x \in X} \mathcal{U}_x \quad \forall \Sigma_3 \in \bigotimes_{x \in X} \mathcal{U}_x \right) \\ & \implies \left( \bigotimes_{x \in X} \mathcal{U}_x \in \mathbb{F}_0^*(\bigotimes_{x \in X} \mathcal{L}_x) \right). \end{aligned} \quad (7.11)$$

Now, we choose arbitrary sets

$$\left( \mathbb{A} \in \bigotimes_{x \in X} \mathcal{U}_x \right) \& \left( \mathbb{B} \in \bigotimes_{x \in X} \mathcal{U}_x \right) \& \left( \mathbb{C} \in \bigotimes_{x \in X} \mathcal{U}_x \right). \quad (7.12)$$

Using (7.3), (7.8), (7.9), and (7.12), we obtain

$$\left( \mathbb{A} \in \mathcal{P}(\mathbb{E}) \right) \& \left( \mathbb{B} \in \mathcal{P}(\mathbb{E}) \right) \& \left( \mathbb{C} \in \mathcal{P}(\mathbb{E}) \right).$$

In addition, for some  $(\tilde{\mathbb{A}}_x)_{x \in X} \in \prod_{x \in X} \mathcal{U}_x$ , we have

$$\left( \mathbb{A} = \prod_{x \in X} \tilde{\mathbb{A}}_x \right) \& \left( \exists K \in \text{Fin}(X) : \tilde{\mathbb{A}}_s = E_s \quad \forall s \in X \setminus K \right). \quad (7.13)$$

Similarly, for some  $(\tilde{\mathbb{B}}_x)_{x \in X} \in \prod_{x \in X} \mathcal{U}_x$ , we have

$$\left( \mathbb{B} = \prod_{x \in X} \tilde{\mathbb{B}}_x \right) \& \left( \exists K \in \text{Fin}(X) : \tilde{\mathbb{B}}_s = E_s \quad \forall s \in X \setminus K \right).$$

Finally, for some  $(\tilde{\mathbb{C}}_x)_{x \in X} \in \prod_{x \in X} \mathcal{U}_x$ , we obtain

$$\left( \mathbb{C} = \prod_{x \in X} \tilde{\mathbb{C}}_x \right) \& \left( \exists K \in \text{Fin}(X) : \tilde{\mathbb{C}}_s = E_s \quad \forall s \in X \setminus K \right). \quad (7.14)$$

Then,  $\tilde{\mathbb{A}}_x \in \mathcal{U}_x$ ,  $\tilde{\mathbb{B}}_x \in \mathcal{U}_x$ , and  $\tilde{\mathbb{C}}_x \in \mathcal{U}_x$  for  $x \in X$ . Therefore, by (7.10), for  $x \in X$ , we have

$$\tilde{\mathbb{A}}_x \cap \tilde{\mathbb{B}}_x \cap \tilde{\mathbb{C}}_x \neq \emptyset; \quad (7.15)$$

as a corollary,  $\tilde{\mathbb{A}}_x \cap \tilde{\mathbb{B}}_x \cap \tilde{\mathbb{C}}_x \in \mathcal{P}'(\mathbf{E})$ . Then, by (7.15), we have

$$\prod_{x \in X} (\tilde{\mathbb{A}}_x \cap \tilde{\mathbb{B}}_x \cap \tilde{\mathbb{C}}_x) \neq \emptyset \quad (7.16)$$

(we use the axiom of choice). In addition,  $(\tilde{\mathbb{A}}_x)_{x \in X} \in \mathcal{P}(\mathbf{E})^X$ ,  $(\tilde{\mathbb{B}}_x)_{x \in X} \in \mathcal{P}(\mathbf{E})^X$ , and  $(\tilde{\mathbb{C}}_x)_{x \in X} \in \mathcal{P}(\mathbf{E})^X$ . Then, by (6.8) and (7.13)–(7.14), we have

$$\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} = \prod_{x \in X} (\tilde{\mathbb{A}}_x \cap \tilde{\mathbb{B}}_x \cap \tilde{\mathbb{C}}_x).$$

From (7.16), the property  $\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} \neq \emptyset$  follows. Since the choice of  $\mathbb{A}$ ,  $\mathbb{B}$ , and  $\mathbb{C}$  was arbitrary (see (7.12)), the premise of implication (7.11) is true. As a corollary, we obtain

$$\bigotimes_{x \in X} \mathcal{U}_x \in \mathbb{F}_0^* \left( \bigotimes_{x \in X} \mathcal{L}_x \right).$$

□

**Proposition 8.** *If  $\mathcal{U} \in \mathbb{F}_0^* \left( \bigotimes_{x \in X} \mathcal{L}_x \right)$ , then*

$$\exists (\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) : \mathcal{U} = \bigotimes_{x \in X} \mathcal{U}_x. \quad (7.17)$$

*P r o o f.* Fix  $\mathcal{U} \in \mathbb{F}_0^* \left( \bigotimes_{x \in X} \mathcal{L}_x \right)$ . Then, by (7.8), we have

$$\mathcal{U} \in \langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}]. \quad (7.18)$$

Therefore, from (7.4) and (7.18), we find that, for some mapping

$$(\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x],$$

the following equality holds:

$$\mathcal{U} = \bigotimes_{x \in X} \mathcal{E}_x. \quad (7.19)$$

In addition,  $\mathcal{E}_x \in \langle \mathcal{L}_x - \text{link} \rangle_0[E_x]$  for  $x \in X$ . Fix  $\chi \in X$ ; then  $\mathcal{E}_\chi \in \langle \mathcal{L}_\chi - \text{link} \rangle_0[E_\chi]$ . By (4.6), we obtain the following implication:

$$(\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \emptyset \quad \forall \Sigma_1 \in \mathcal{E}_\chi \quad \forall \Sigma_2 \in \mathcal{E}_\chi \quad \forall \Sigma_3 \in \mathcal{E}_\chi) \implies (\mathcal{E}_\chi \in \mathbb{F}_0^*(\mathcal{L}_\chi)). \quad (7.20)$$

Choose arbitrary sets  $A \in \mathcal{E}_\chi$ ,  $B \in \mathcal{E}_\chi$ , and  $C \in \mathcal{E}_\chi$ . Using (2.10), we introduce

$$(\tilde{A}_x)_{x \in X} \in \prod_{x \in X} \mathcal{E}_x$$

by the following rule:  $\tilde{A}_\chi \triangleq A$  and  $\tilde{A}_x \triangleq E_x$  for  $x \in X \setminus \{\chi\}$ . We obtain

$$\mathbb{A} \triangleq \prod_{x \in X} \tilde{A}_x \in \mathcal{P}(\mathbf{E}). \quad (7.21)$$

Therefore, for  $\mathbb{A}$  (7.21), we find that  $\exists(\Sigma_x)_{x \in X} \in \prod_{x \in X} \mathcal{E}_x$  :

$$\left( \mathbb{A} = \prod_{x \in X} \Sigma_x \right) \& (\exists K \in \text{Fin}(X) : \Sigma_s = E_s \quad \forall s \in X \setminus K).$$

Then, by (3.6), (3.8), and (7.19), we conclude that  $\mathbb{A} \in \mathcal{U}$ . Introduce (see (2.10)) a mapping

$$(\tilde{B}_x)_{x \in X} \in \prod_{x \in X} \mathcal{E}_x$$

by the rule:  $\tilde{B}_\chi \triangleq B$  and  $\tilde{B}_x \triangleq E_x$  for  $x \in X \setminus \{\chi\}$ . Then

$$\mathbb{B} \triangleq \prod_{x \in X} \tilde{B}_x \in \mathcal{P}(\mathbb{E}). \quad (7.22)$$

So,  $\mathbb{B}$  (7.22) is a set, for which  $\exists(\Sigma_x)_{x \in X} \in \prod_{x \in X} \mathcal{E}_x$  :

$$\left( \mathbb{B} = \prod_{x \in X} \Sigma_x \right) \& (\exists K \in \text{Fin}(X) : \Sigma_s = E_s \quad \forall s \in X \setminus K).$$

As a result, we conclude that (see (3.6) and (7.19))  $\mathbb{B} \in \mathcal{U}$ . Finally, we introduce (see (2.10)) a mapping

$$(\tilde{C}_x)_{x \in X} \in \prod_{x \in X} \mathcal{E}_x$$

by the following rule:  $\tilde{C}_\chi \triangleq C$  and  $\tilde{C}_x \triangleq E_x$  for  $x \in X \setminus \{\chi\}$ . Then

$$\mathbb{C} \triangleq \prod_{x \in X} \tilde{C}_x \in \mathcal{P}(\mathbb{E}) \quad (7.23)$$

is a set, for which, by (7.23),  $\exists(\Sigma_x)_{x \in X} \in \prod_{x \in X} \mathcal{E}_x$  :

$$\left( \mathbb{C} = \prod_{x \in X} \Sigma_x \right) \& (\exists K \in \text{Fin}(X) : \Sigma_s = E_s \quad \forall s \in X \setminus K).$$

From (3.6) and (7.19), we conclude that  $\mathbb{C} \in \mathcal{U}$ . So,  $\mathbb{A} \in \mathcal{U}$ ,  $\mathbb{B} \in \mathcal{U}$ , and  $\mathbb{C} \in \mathcal{U}$ . By the choice of  $\mathcal{U}$ , we have (see (4.6)) the property

$$\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} \neq \emptyset. \quad (7.24)$$

But  $(\tilde{A}_x)_{x \in X} \in \mathcal{P}(\mathbf{E})^X$ ,  $(\tilde{B}_x)_{x \in X} \in \mathcal{P}(\mathbf{E})^X$ , and  $(\tilde{C}_x)_{x \in X} \in \mathcal{P}(\mathbf{E})^X$ ; therefore (see (6.8) and (7.21)–(7.23)),

$$\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} = \prod_{x \in X} (\tilde{A}_x \cap \tilde{B}_x \cap \tilde{C}_x).$$

Then, by (7.24), we obtain  $\tilde{A}_x \cap \tilde{B}_x \cap \tilde{C}_x \neq \emptyset \quad \forall x \in X$ . In particular,

$$A \cap B \cap C = A_\chi \cap B_\chi \cap C_\chi \neq \emptyset.$$

Since the choice of  $A$ ,  $B$ , and  $C$  was arbitrary, we obtain

$$\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \emptyset \quad \forall \Sigma_1 \in \mathcal{E}_\chi \quad \forall \Sigma_2 \in \mathcal{E}_\chi \quad \forall \Sigma_3 \in \mathcal{E}_\chi.$$

Then (see (7.20))  $\mathcal{E}_\chi \in \mathbb{F}_0^*(\mathcal{L}_\chi)$ . Since the choice of  $\chi$  was arbitrary, it is established that  $\mathcal{E}_x \in \mathbb{F}_0^*(\mathcal{L}_x) \quad \forall x \in X$ . So,

$$(\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x).$$

Using (7.19), we obtain the required statement (7.17).  $\square$

**Theorem 3.** *The following equality holds:*

$$\mathbb{F}_0^*(\bigotimes_{x \in X} \mathcal{L}_x) = \left\{ \bigotimes_{x \in X} \mathcal{U}_x : (\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) \right\}.$$

The proof immediately follows from Propositions 7 and 8. In connection with Theorem 3, we recall constructions of [4].

Following to [14], we introduce the following natural mapping:

$$\mathbf{g} \triangleq \left( \bigotimes_{x \in X} \mathcal{E}_x \right)_{(\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x]} \in \left\langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \right\rangle_0[\mathbb{E}]^{\prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x]}. \quad (7.25)$$

So,

$$\mathbf{g} : \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \longrightarrow \left\langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \right\rangle_0[\mathbb{E}];$$

in addition,

$$\mathbf{g}((\mathcal{E}_x)_{x \in X}) = \bigotimes_{x \in X} \mathcal{E}_x \quad \forall (\mathcal{E}_x)_{x \in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x]. \quad (7.26)$$

The properties of  $\mathbf{g}$  (see (7.25), (7.26)) were considered in [14]. Now we will restrict ourselves to listing them. Note that

$$\begin{aligned} \bigotimes_{x \in X} \hat{\mathcal{C}}_0^*[E_x; \mathcal{L}_x] = & \left\{ \mathbb{C} \in \mathcal{P} \left( \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \right) \mid \exists (\mathbb{F}_x)_{x \in X} \in \prod_{x \in X} \hat{\mathcal{C}}_0^*[E_x; \mathcal{L}_x] : \right. \\ & \left. (\mathbb{C} = \prod_{x \in X} \mathbb{F}_x) \& (\exists K \in \text{Fin}(X) : \mathbb{F}_s = \langle \mathcal{L}_s - \text{link} \rangle_0[E_s] \quad \forall s \in X \setminus K) \right\} \end{aligned} \quad (7.27)$$

(in (7.27), we use (3.6) and take into account that, for  $x \in X$ ,

$$\langle \mathcal{L}_x - \text{link} \rangle_0^0[E_x|E_x] = \langle \mathcal{L}_x - \text{link} \rangle_0[E_x],$$

see [7, (4.7)]). Then, by [14, Proposition 6.2], we obtain

$$\mathbf{g}^{-1}(\mathbf{H}) \in \bigotimes_{x \in X} \hat{\mathcal{C}}_0^*[E_x; \mathcal{L}_x] \quad \forall \mathbf{H} \in \hat{\mathcal{C}}_0^*[\mathbb{E}; \bigotimes_{x \in X} \mathcal{L}_x]. \quad (7.28)$$

Now, we recall (6.34). As a corollary, the following  $\pi$ -system is defined:

$$\begin{aligned} \bigotimes_{x \in X} \mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle = & \left\{ \mathbb{H} \in \mathcal{P} \left( \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \right) \mid \exists (\mathbb{B}_x)_{x \in X} \in \prod_{x \in X} \mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle : \right. \\ & \left. (\mathbb{H} = \prod_{x \in X} \mathbb{B}_x) \& (\exists K \in \text{Fin}(X) : \mathbb{B}_s = \langle \mathcal{L}_s - \text{link} \rangle_0[E_s] \quad \forall s \in X \setminus K) \right\} \\ & \in \pi \left[ \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \right]; \end{aligned} \quad (7.29)$$

we use (3.6) and (3.9). By means of (2.7), (3.10), and (7.29), the topology

$$\mathbf{t}_\otimes[(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}] = \{\cup\} \left( \bigotimes_{x \in X} \mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle \right) \in (\text{top}) \left[ \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \right] \quad (7.30)$$

is defined. From (6.34) and (7.30), we obtain

$$\bigotimes_{x \in X} \hat{\mathcal{C}}_0^*[E_x; \mathcal{L}_x] \subset \bigotimes_{x \in X} \mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle \subset \mathbf{t}_\otimes[(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}]. \quad (7.31)$$

Therefore, by (7.28) and (7.31), we have the following property:

$$\mathbf{g}^{-1}(\mathbf{H}) \in \mathbf{t}_{\otimes}[(\mathbb{T}_*\langle E_x | \mathcal{L}_x \rangle)_{x \in X}] \quad \forall \mathbf{H} \in \hat{\mathbf{C}}_0^*[\mathbb{E}; \bigotimes_{x \in X} \mathcal{L}_x]. \quad (7.32)$$

Using (6.36) and (7.32), we obtain the following important property:  $\mathbf{g}$  (7.25) is a continuous mapping in the sense of TS

$$\left( \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x], \mathbf{t}_{\otimes}[(\mathbb{T}_*\langle E_x | \mathcal{L}_x \rangle)_{x \in X}] \right), \quad \left( \langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}], \mathbb{T}_*\langle \mathbb{E} | \bigotimes_{x \in X} \mathcal{L}_x \rangle \right); \quad (7.33)$$

we use [15, Proposition 1.4.1]. Now, we recall [14, Proposition 6.4] that  $\mathbf{g}$  is a bijection from  $\prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x]$  onto  $\langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}]$  (in this connection, we recall (7.6)). In addition, we recall the following useful statement [14, Proposition 6.5]:

$$\mathbf{g}^1(\mathbb{H}) \in \hat{\mathbf{C}}_0^*[\mathbb{E}; \bigotimes_{x \in X} \mathcal{L}_x] \quad \forall \mathbb{H} \in \bigotimes_{x \in X} \mathbf{C}_0^*[E_x; \mathcal{L}_x].$$

By means of this property, the following important statement was established in [14, Proposition 7.1]:  $\mathbf{g}$  is an open mapping in the sense of TS (7.33). So, we obtain the following basic statement (see [14, Theorem 7.1]).

**Theorem 4.** *The mapping  $\mathbf{g}$  (7.25) is a homeomorphism from the TS*

$$\left( \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x], \mathbf{t}_{\otimes}[(\mathbb{T}_*\langle E_x | \mathcal{L}_x \rangle)_{x \in X}] \right)$$

onto the TS

$$\left( \langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}], \mathbb{T}_*\langle \mathbb{E} | \bigotimes_{x \in X} \mathcal{L}_x \rangle \right).$$

From (4.7), we obtain

$$\mathbb{F}_0^*(\bigotimes_{x \in X} \mathcal{L}_x) \in \mathbf{C}_{\langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}]}[\mathbb{T}_*\langle \mathbb{E} | \bigotimes_{x \in X} \mathcal{L}_x \rangle].$$

Theorem 3 reveals the structure of the set  $\mathbb{F}_0^*(\bigotimes_{x \in X} \mathcal{L}_x)$ . By (4.5), we have

$$\mathbf{T}_{\bigotimes_{x \in X} \mathcal{L}_x}^*[\mathbb{E}] = \mathbb{T}_*\langle \mathbb{E} | \bigotimes_{x \in X} \mathcal{L}_x \rangle |_{\mathbb{F}_0^*(\bigotimes_{x \in X} \mathcal{L}_x)}.$$

Thus, ultrafilters of our  $\pi$ -system  $\bigotimes_{x \in X} \mathcal{L}_x$  form a closed subspace of the second TS in (7.33).

## 8. Some corollaries for ultrafilter spaces

In this section, we consider some statements related to products of spaces with topologies of type (4.3). But, at first, we note general properties connected with subspaces of TSs.

For every TS  $(\mathbf{X}, \tau)$ ,  $\mathbf{X} \neq \emptyset$ , and  $(\mathbf{Y}, \vartheta)$ ,  $\mathbf{Y} \neq \emptyset$ , denote by  $C(\mathbf{X}, \tau, \mathbf{Y}, \vartheta)$  the set of all mappings from  $\mathbf{Y}^{\mathbf{X}}$  continuous with respect to the topologies  $\tau$  and  $\vartheta$ . Similarly, for nonempty sets  $\mathbf{X}$  and  $\mathbf{Y}$ , let

$$\mathbf{Y}_{(*)}^{\mathbf{X}} \triangleq \{f \in \mathbf{Y}^{\mathbf{X}} \mid f^1(\mathbf{X}) = \mathbf{Y}\}$$

be the set of all surjections from  $\mathbf{X}$  onto  $\mathbf{Y}$ , let

$$(\text{bi})[\mathbf{X}; \mathbf{Y}] \triangleq \{f \in \mathbf{Y}_{(*)}^{\mathbf{X}} \mid \forall x_1 \in \mathbf{X} \forall x_2 \in \mathbf{X} (f(x_1) = f(x_2)) \implies (x_1 = x_2)\}$$

be the set of all bijections from  $\mathbf{X}$  onto  $\mathbf{Y}$ ; finally, for  $\tau_1 \in (\text{top})[\mathbf{X}]$  and  $\tau_2 \in (\text{top})[\mathbf{Y}]$ , let

$$C_{\mathbf{c}}^0(\mathbf{X}, \tau_1, \mathbf{Y}, \tau_2) \triangleq C(\mathbf{X}, \tau_1, \mathbf{Y}, \tau_2) \cap (\text{bi})[\mathbf{X}; \mathbf{Y}] \quad (8.1)$$

be the set of all condensations from  $(\mathbf{X}, \tau_1)$  onto  $(\mathbf{Y}, \tau_2)$ . We note yet another important notion: for every TS  $(\mathbf{X}, \tau_1)$ ,  $\mathbf{X} \neq \emptyset$ , and  $(\mathbf{Y}, \tau_2)$ ,  $\mathbf{Y} \neq \emptyset$ , let

$$C_{\text{op}}(\mathbf{X}, \tau_1, \mathbf{Y}, \tau_2) \triangleq \{f \in C(\mathbf{X}, \tau_1, \mathbf{Y}, \tau_2) \mid f^1(G) \in \tau_2 \forall G \in \tau_1\}$$

be the set of all open mappings from  $(\mathbf{X}, \tau_1)$  in  $(\mathbf{Y}, \tau_2)$ . Then

$$(\text{Hom})[\mathbf{X}; \tau_1; \mathbf{Y}; \tau_2] \triangleq C_{\text{op}}(\mathbf{X}, \tau_1, \mathbf{Y}, \tau_2) \cap (\text{bi})[\mathbf{X}; \mathbf{Y}] \in \mathcal{P}(C_{\mathbf{c}}^0(\mathbf{X}, \tau_1, \mathbf{Y}, \tau_2))$$

is the set (possibly empty) of all homeomorphisms from  $(\mathbf{X}, \tau_1)$  onto  $(\mathbf{Y}, \tau_2)$ . Now, we note several simple general properties.

- (1) If  $(\mathbf{X}, \tau_1)$ ,  $\mathbf{X} \neq \emptyset$ , and  $(\mathbf{Y}, \tau_2)$ ,  $\mathbf{Y} \neq \emptyset$ , are two TS,  $f \in C(\mathbf{X}, \tau_1, \mathbf{Y}, \tau_2)$ , and  $A \in \mathcal{P}'(\mathbf{X})$ , then  $f^1(A) \in \mathcal{P}'(\mathbf{Y})$  and

$$(f|A) \in C(A, \tau_1|_A, f^1(A), \tau_2|_{f^1(A)}).$$

- (2) If  $\mathbf{X}$  and  $\mathbf{Y}$  are nonempty sets,  $f \in (\text{bi})[\mathbf{X}; \mathbf{Y}]$ , and  $A \in \mathcal{P}'(\mathbf{X})$ , then  $(f|A) \in (\text{bi})[A; f^1(A)]$ .

Immediate combination of (1) and (2) implies the following properties.

- (3) If  $(\mathbf{X}, \tau_1)$ ,  $\mathbf{X} \neq \emptyset$ , and  $(\mathbf{Y}, \tau_2)$ ,  $\mathbf{Y} \neq \emptyset$ , are two TS,  $f \in C_{\mathbf{c}}^0(\mathbf{X}, \tau_1, \mathbf{Y}, \tau_2)$ , and  $A \in \mathcal{P}'(\mathbf{X})$ , then  $(f|A) \in C_{\mathbf{c}}^0(A, \tau_1|_A, f^1(A), \tau_2|_{f^1(A)})$ .

- (4) If  $(\mathbf{X}, \tau_1)$ ,  $\mathbf{X} \neq \emptyset$ , and  $(\mathbf{Y}, \tau_2)$ ,  $\mathbf{Y} \neq \emptyset$ , are two TS,  $f \in (\text{Hom})[\mathbf{X}; \tau_1; \mathbf{Y}; \tau_2]$ , and  $A \in \mathcal{P}'(\mathbf{X})$ , then

$$(f|A) \in (\text{Hom})[A; \tau_1|_A, f^1(A), \tau_2|_{f^1(A)}].$$

Now, we note some statements on the structure of a subspace of the product of TSs. If  $(\mathbf{X}, \tau_1)$ ,  $\mathbf{X} \neq \emptyset$ , and  $(\mathbf{Y}, \tau_2)$ ,  $\mathbf{Y} \neq \emptyset$ , are two TS, then, similarly to Section 5, in what follows, we suppose that

$$\tau_1 \otimes \tau_2 \triangleq \{\cup\}(\tau_1 \{\times\} \tau_2). \quad (8.2)$$

Note that (3.6) and (8.2) should be distinguished; in (8.2), we consider a topology. Then, using [15, Proposition 2.3.2], for every TS  $(\mathbf{X}, \tau_1)$ ,  $\mathbf{X} \neq \emptyset$ , and  $(\mathbf{Y}, \tau_2)$ ,  $\mathbf{Y} \neq \emptyset$ , and sets  $A \in \mathcal{P}'(\mathbf{X})$  and  $B \in \mathcal{P}'(\mathbf{Y})$ , we obtain

$$(\tau_1 \otimes \tau_2)|_{A \times B} = \tau_1|_A \otimes \tau_2|_B. \quad (8.3)$$

Moreover, if  $\mathbf{X}$  and  $\mathbf{Y}$  are nonempty sets,  $(Y_x)_{x \in \mathbf{X}} \in \mathcal{P}'(\mathbf{Y})^{\mathbf{X}}$ ,  $(\tau_x)_{x \in \mathbf{X}} \in \prod_{x \in \mathbf{X}} (\text{top})[Y_x]$ , and  $(A_x)_{x \in \mathbf{X}} \in \prod_{x \in \mathbf{X}} \mathcal{P}'(Y_x)$ , then

$$\mathbf{t}_{\odot}[(\tau_x)_{x \in \mathbf{X}}] \prod_{x \in \mathbf{X}} A_x = \mathbf{t}_{\odot}[(\tau_x|_{A_x})_{x \in \mathbf{X}}]; \quad (8.4)$$

of course, we keep in mind that, in the case under consideration,

$$(A_x)_{x \in \mathbf{X}} \in \mathcal{P}'(\mathbf{Y})^{\mathbf{X}}, \quad (\tau_x|_{A_x})_{x \in \mathbf{X}} \in \prod_{x \in \mathbf{X}} (\text{top})[A_x], \quad \prod_{x \in \mathbf{X}} A_x \in \mathcal{P}'\left(\prod_{x \in \mathbf{X}} Y_x\right).$$

In (8.4), we have an analogy with [15, Proposition 2.3.2] (an obvious verification of (8.4) we omit). Finally, for every nonempty sets  $\mathbb{X}$  and  $\mathbb{Y}$ , mappings  $(Y_x)_{x \in \mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$ ,  $(\tau_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} (\text{top})[Y_x]$ , and  $(A_x)_{x \in \mathbb{X}} \in \prod_{x \in \mathbb{X}} \mathcal{P}'(Y_x)$ , we have

$$\mathbf{t}_{\otimes}[(\tau_x)_{x \in \mathbb{X}}] |_{\prod_{x \in \mathbb{X}} A_x} = \mathbf{t}_{\otimes}[(\tau_x|_{A_x})_{x \in \mathbb{X}}]. \quad (8.5)$$

Now, we consider some topological properties for products of ultrafilter spaces. We begin with the simplest case.

**The case of product of two ultrafilter spaces.** In this subsection, we fix nonempty sets  $X$  and  $Y$ . In addition, we fix  $\pi$ -systems  $\mathcal{X} \in \pi[X]$  and  $\mathcal{Y} \in \pi[Y]$ . Then

$$\begin{aligned} \mathbf{T}_{\mathcal{X}}^*[X] &\in (\text{top})[\mathbb{F}_0^*(\mathcal{X})] & \text{and} & & \mathbf{T}_{\mathcal{Y}}^*[Y] &\in (\text{top})[\mathbb{F}_0^*(\mathcal{Y})]; \\ \mathbb{F}_0^*(\mathcal{X}) &\in \mathcal{P}'(\langle \mathcal{X} - \text{link} \rangle_0[X]) & \text{and} & & \mathbb{F}_0^*(\mathcal{Y}) &\in \mathcal{P}'(\langle \mathcal{Y} - \text{link} \rangle_0[Y]). \end{aligned}$$

We recall (4.5):

$$(\mathbf{T}_{\mathcal{X}}^*[X] = \mathbb{T}_* \langle X | \mathcal{X} \rangle |_{\mathbb{F}_0^*(\mathcal{X})}) \& (\mathbf{T}_{\mathcal{Y}}^*[Y] = \mathbb{T}_* \langle Y | \mathcal{Y} \rangle |_{\mathbb{F}_0^*(\mathcal{Y})}). \quad (8.6)$$

By (8.2), the following topology is defined:

$$\mathbf{T}_{\mathcal{X}}^*[X] \otimes \mathbf{T}_{\mathcal{Y}}^*[Y] \in (\text{top})[\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y})].$$

Using (8.3) and (8.6), we obtain

$$\mathbf{T}_{\mathcal{X}}^*[X] \otimes \mathbf{T}_{\mathcal{Y}}^*[Y] = (\mathbb{T}_* \langle X | \mathcal{X} \rangle \otimes \mathbb{T}_* \langle Y | \mathcal{Y} \rangle) |_{\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y})}, \quad (8.7)$$

where

$$\mathbb{T}_* \langle X | \mathcal{X} \rangle \otimes \mathbb{T}_* \langle Y | \mathcal{Y} \rangle \in (\text{top})[\langle \mathcal{X} - \text{link} \rangle_0[X] \times \langle \mathcal{Y} - \text{link} \rangle_0[Y]].$$

The mapping (5.15) is a homeomorphism. Finally, we recall (5.17). Now, we note that

$$z \mapsto \text{pr}_1(z) \{ \times \} \text{pr}_2(z) : \quad \mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y}) \longrightarrow \mathbb{F}_0^*(\mathcal{X} \{ \times \} \mathcal{Y}) \quad (8.8)$$

is defined correctly (see Theorem 1). In addition, the mapping (8.8) is a restriction of (5.15) to the set  $\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y})$ . To make this and subsequent statements shorter, we introduce new notation. In this subsection, denote by  $\mathbf{u}$  and  $\mathbf{v}$  the mappings (5.15) and (8.8), respectively. Then,

$$\mathbf{v} = (\mathbf{u} |_{\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y})}). \quad (8.9)$$

Moreover, by Theorem 1 and (5.15), we obtain

$$\mathbb{F}_0^*(\mathcal{X} \{ \times \} \mathcal{Y}) = \mathbf{u}^1(\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y})). \quad (8.10)$$

**Theorem 5.** *The mapping (8.8) is a homeomorphism in the sense of topologies (8.7) and*

$$\mathbf{T}_{\mathcal{X} \{ \times \} \mathcal{Y}}^*[X \times Y] : \mathbf{v} \in (\text{Hom})[\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y}); \mathbf{T}_{\mathcal{X}}^*[X] \otimes \mathbf{T}_{\mathcal{Y}}^*[Y]; \mathbb{F}_0^*(\mathcal{X} \{ \times \} \mathcal{Y}); \mathbf{T}_{\mathcal{X} \{ \times \} \mathcal{Y}}^*[X \times Y]].$$

*P r o o f.* We use (8.9) and (8.10) in constructions connected with (4). For this, we note that (see Section 5)

$$\begin{aligned} \mathbf{u} \in (\text{Hom})[\langle \mathcal{X} - \text{link} \rangle_0[X] \times \langle \mathcal{Y} - \text{link} \rangle_0[Y]; \mathbb{T}_* \langle X | \mathcal{X} \rangle \otimes \mathbb{T}_* \langle Y | \mathcal{Y} \rangle; \langle \mathcal{X} \{ \times \} \mathcal{Y} - \text{link} \rangle_0[X \times Y]; \\ \mathbb{T}_* \langle X \times Y | \mathcal{X} \{ \times \} \mathcal{Y} \rangle]. \end{aligned}$$



Consider (4) with the following specific definitions:

$$\begin{aligned} \mathbf{X} &= \langle \mathcal{X} - \text{link} \rangle_0[X] \times \langle \mathcal{Y} - \text{link} \rangle_0[Y], \quad \tau_1 = \mathbb{T}_* \langle X | \mathcal{X} \rangle \otimes \mathbb{T}_* \langle Y | \mathcal{Y} \rangle, \\ \mathbf{Y} &= \langle \mathcal{X} \{ \times \} \mathcal{Y} - \text{link} \rangle_0[X \times Y], \quad \tau_2 = \mathbb{T}_* \langle X \times Y | \mathcal{X} \{ \times \} \mathcal{Y} \rangle, \\ f &= \mathbf{u}, \quad A = \mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y}). \end{aligned} \quad (8.11)$$

Using (4), (8.9) and (8.11), we obtain

$$\begin{aligned} \mathbf{v} \in (\text{Hom}) \left[ \mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y}); (\mathbb{T}_* \langle X | \mathcal{X} \rangle \otimes \mathbb{T}_* \langle Y | \mathcal{Y} \rangle) \Big|_{\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y})}, \mathbf{u}^1(\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y})); \right. \\ \left. \mathbb{T}_* \langle X \times Y | \mathcal{X} \{ \times \} \mathcal{Y} \rangle \Big|_{\mathbf{u}^1(\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y}))} \right]. \end{aligned} \quad (8.12)$$

Then, we use (4.5), (8.7), (8.9), and (8.10). We have the chain of equalities

$$\mathbf{T}_{\mathcal{X} \{ \times \} \mathcal{Y}}^* [X \times Y] = \mathbb{T}_* \langle X \times Y | \mathcal{X} \{ \times \} \mathcal{Y} \rangle \Big|_{\mathbb{F}_0^*(\mathcal{X} \{ \times \} \mathcal{Y})} = \mathbb{T}_* \langle X \times Y | \mathcal{X} \{ \times \} \mathcal{Y} \rangle \Big|_{\mathbf{u}^1(\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y}))}.$$

Using (8.7), (8.9), (8.10), and (8.12), we obtain the required property of  $\mathbf{v}$ .  $\square$

**The case of box topology on the product of ultrafilter spaces.** In this and subsequent subsections, we use nonempty sets  $X$  and  $\mathbf{E}$  and the mapping  $(E_x)_{x \in X} \in \mathcal{P}'(\mathbf{E})^X$  defined in Section 6. Moreover, we follow (6.1) for the set  $\mathbb{E}$ . In what follows, we fix  $(\mathcal{L}_x)_{x \in X}$  (6.2). Then, by (4.3), we have

$$(\mathbf{T}_{\mathcal{L}_x}^* [E_x])_{x \in X} \in \prod_{x \in X} (\text{top})[\mathbb{F}_0^*(\mathcal{L}_x)]. \quad (8.13)$$

In addition,

$$\mathbb{F}_0^*(\mathcal{L}_x) \in \mathcal{P}'(\langle \mathcal{L}_x - \text{link} \rangle_0[E_x]) \quad \forall x \in X.$$

Therefore,

$$(\mathbb{F}_0^*(\mathcal{L}_x))_{x \in X} \in \prod_{x \in X} \mathcal{P}'(\langle \mathcal{L}_x - \text{link} \rangle_0[E_x]).$$

Using (4.5), we obtain

$$\mathbf{T}_{\mathcal{L}_x}^* [E_x] = \mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle \Big|_{\mathbb{F}_0^*(\mathcal{L}_x)} \quad \forall x \in X. \quad (8.14)$$

From (3.10) and (8.13), the following property is extracted:

$$\mathbf{t}_{\odot}[(\mathbf{T}_{\mathcal{L}_x}^* [E_x])_{x \in X}] \in (\text{top}) \left[ \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) \right]. \quad (8.15)$$

We recall that, by Proposition 5, the mapping

$$(\mathcal{U}_x)_{x \in X} \longmapsto \bigodot_{x \in X} \mathcal{U}_x : \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) \longrightarrow \mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x) \quad (8.16)$$

is defined correctly. By (6.31), this mapping (8.16) is a restriction of (6.31) to the set  $\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)$ . For brevity, we denote the mapping (8.16) by  $\mathbf{w}$ . By (6.22), we have

$$\mathbf{w} \stackrel{\Delta}{=} \left( \bigodot_{x \in X} \mathcal{U}_x \right) (\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) \in \mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x)^{\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)}. \quad (8.17)$$

Then

$$\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) \in \mathcal{P}' \left( \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \right)$$

and

$$\mathbf{w} = (\mathbf{f} | \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)). \quad (8.18)$$

Moreover, we recall that, by (6.31) and Theorem 2,

$$\mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x) = \mathbf{f}^1(\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)). \quad (8.19)$$

**Theorem 6.** *The mapping (8.16) is a condensation in the sense of topologies (8.15) and  $\mathbf{T}_{\mathbb{E}}^*[\bigodot_{x \in X} \mathcal{L}_x]$ :*

$$\mathbf{w} \in C_{\mathbf{c}}^0\left(\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x), \mathbf{t}_{\odot}[(\mathbf{T}_{\mathcal{L}_x}^*[E_x])_{x \in X}], \mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x), \mathbf{T}_{\odot_{x \in X} \mathcal{L}_x}^*[\mathbb{E}]\right). \quad (8.20)$$

*P r o o f.* We use (8.17)–(8.19) in constructions connected with (3). For this, we recall that (see Section 6)

$$\mathbf{f} \in C_{\mathbf{c}}^0\left(\prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x], \mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x | \mathcal{L}_x \rangle)_{x \in X}], \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}], \mathbb{T}_*\langle \mathbb{E} | \bigodot_{x \in X} \mathcal{L}_x \rangle\right).$$

Now, we use (3) with the following specific definitions:

$$\begin{aligned} \mathbf{X} &= \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x], \quad \tau_1 = \mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x | \mathcal{L}_x \rangle)_{x \in X}], \\ \mathbf{Y} &= \langle \bigodot_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}], \quad \tau_2 = \mathbb{T}_*\langle \mathbb{E} | \bigodot_{x \in X} \mathcal{L}_x \rangle, \\ f &= \mathbf{f}, \quad A = \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x). \end{aligned} \quad (8.21)$$

Then, by (3), (8.18), and (8.21), we obtain

$$\mathbf{w} \in C_{\mathbf{c}}^0\left(\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x), \mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x | \mathcal{L}_x \rangle)_{x \in X}] | \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x), \mathbf{f}^1(\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)), \mathbb{T}_*\langle \mathbb{E} | \bigodot_{x \in X} \mathcal{L}_x \rangle | \mathbf{f}^1(\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x))\right).$$

By (8.19), the following inclusion holds:

$$\mathbf{w} \in C_{\mathbf{c}}^0\left(\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x), \mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x | \mathcal{L}_x \rangle)_{x \in X}] | \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x), \mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x), \mathbb{T}_*\langle \mathbb{E} | \bigodot_{x \in X} \mathcal{L}_x \rangle | \mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x)\right). \quad (8.22)$$

Now, we use (8.4) with the following specific definitions:

$$\mathbf{X} = X, \quad \mathbf{Y} = \mathcal{P}'(\mathcal{P}(\mathbf{E})). \quad (8.23)$$

Using (8.23), we also suppose that

$$(Y_x)_{x \in \mathbf{X}} = (\langle \mathcal{L}_x - \text{link} \rangle_0[E_x])_{x \in X}, \quad (\tau_x)_{x \in \mathbf{X}} = (\mathbb{T}_*\langle E_x | \mathcal{L}_x \rangle)_{x \in X}, \quad (A_x)_{x \in \mathbf{X}} = (\mathbb{F}_0^*(\mathcal{L}_x))_{x \in X}. \quad (8.24)$$

In this connection (see (8.23) and (8.24)), we recall that, by (2.8) and (2.9), the following chain of inclusions holds:

$$\langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \subset \langle \mathcal{L}_x - \text{link} \rangle[E_x] \subset \mathcal{P}'(\mathcal{L}_x) \subset \mathcal{P}'(\mathcal{P}(E_x)) \subset \mathcal{P}'(\mathcal{P}(\mathbf{E})) = \mathbf{Y};$$

moreover,  $\langle \mathcal{L}_x - \text{link} \rangle_0[E_x] \neq \emptyset$  for  $x \in X$ . Therefore (see (8.23)),

$$(\langle \mathcal{L}_x - \text{link} \rangle_0[E_x])_{x \in X} \in \mathcal{P}'(\mathbf{Y})^X.$$

This corresponds to the conditions for (8.4). Then, from (8.4), (8.23), and (8.24), we have

$$\mathbf{t}_\odot[(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}] | \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) = \mathbf{t}_\odot[(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X} | \mathbb{F}_0^*(\mathcal{L}_x)]_{x \in X} = \mathbf{t}_\odot[(\mathbf{T}_{\mathcal{L}_x}^*[E_x])_{x \in X}]; \quad (8.25)$$

of course, in (8.25), we use (4.5). Moreover, by (4.5), we have

$$\mathbb{T}_* \langle \mathbb{E} | \bigodot_{x \in X} \mathcal{L}_x \rangle | \mathbb{F}_0^*(\bigodot_{x \in X} \mathcal{L}_x) = \mathbf{T}_{\bigodot_{x \in X} \mathcal{L}_x}[\mathbb{E}]. \quad (8.26)$$

From (8.22), (8.25), and (8.26), we obtain (8.20).  $\square$

**The case of generalized Cartesian product of ultrafilter spaces.** We follow the previous subsection (see also Sections 6 and 7), using  $X$ ,  $\mathbf{E}$ ,  $(E_x)_{x \in X}$ ,  $\mathbb{E}$ , and  $(\mathcal{L}_x)_{x \in X}$ . Of course, we use (8.13)–(8.14). Then, by (3.10) and (8.13), we have

$$\mathbf{t}_\otimes[(\mathbf{T}_{\mathcal{L}_x}^*[E_x])_{x \in X}] \in (\text{top})[\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)]. \quad (8.27)$$

From Proposition 7, we conclude that

$$(\mathcal{U}_x)_{x \in X} \mapsto \bigotimes_{x \in X} \mathcal{U}_x : \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) \longrightarrow \mathbb{F}_0^*(\bigotimes_{x \in X} \mathcal{L}_x) \quad (8.28)$$

is a restriction of the mapping  $\mathbf{g}$  (7.25) to the set  $\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)$ . We denote this mapping (8.28) by  $\mathbf{r}$  for brevity; so,

$$\mathbf{r} \triangleq (\bigotimes_{x \in X} \mathcal{U}_x)_{(\mathcal{U}_x)_{x \in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)} \in \mathbb{F}_0^*(\bigotimes_{x \in X} \mathcal{L}_x)^{\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)}. \quad (8.29)$$

Similar to (8.18), we obtain the following equality:

$$\mathbf{r} = (\mathbf{g} | \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)). \quad (8.30)$$

Moreover, note that, by (7.25) and Theorem 3, we have

$$\mathbb{F}_0^*(\bigotimes_{x \in X} \mathcal{L}_x) = \mathbf{g}^1(\prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x)). \quad (8.31)$$

**Theorem 7.** *The mapping (8.28) is a homeomorphism in the sense of topologies (8.27) and  $\mathbf{T}_{\bigotimes_{x \in X} \mathcal{L}_x}^*[\mathbb{E}]$ :*

$$\mathbf{r} \in (\text{Hom}) \left[ \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x); \mathbf{t}_\otimes[(\mathbf{T}_{\mathcal{L}_x}^*[E_x])_{x \in X}]; \mathbb{F}_0^*(\bigotimes_{x \in X} \mathcal{L}_x); \mathbf{T}_{\bigotimes_{x \in X} \mathcal{L}_x}^*[\mathbb{E}] \right]. \quad (8.32)$$

*P r o o f.* We use (8.29)–(8.31) in constructions connected with (4). For this, we note that, by Theorem 4,

$$\mathbf{g} \in (\text{Hom}) \left[ \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0[E_x]; \mathbf{t}_\otimes[(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}]; \langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \rangle_0[\mathbb{E}]; \mathbb{T}_* \langle \mathbb{E} | \bigotimes_{x \in X} \mathcal{L}_x \rangle \right].$$

Now, we use (4) with

$$\begin{aligned} \mathbf{X} &= \prod_{x \in X} \langle \mathcal{L}_x - \text{link} \rangle_0 [E_x], \quad \tau_1 = \mathbf{t}_\otimes [(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}], \\ \mathbf{Y} &= \langle \bigotimes_{x \in X} \mathcal{L}_x - \text{link} \rangle_0 [\mathbb{E}], \quad \tau_2 = \mathbb{T}_* \langle \mathbb{E} | \bigotimes_{x \in X} \mathcal{L}_x \rangle, \\ f &= \mathbf{g}, \quad A = \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x). \end{aligned} \tag{8.33}$$

Then, by (4), (8.30), and (8.31), we obtain (see (8.33)) the following property:

$$\begin{aligned} \mathbf{r} \in (\text{Hom}) \left[ \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x); \mathbf{t}_\otimes [(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}] \middle| \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x); \right. \\ \left. \mathbf{g}^1 \left( \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x); \mathbb{T}_* \langle \mathbb{E} | \bigotimes_{x \in X} \mathcal{L}_x \rangle \middle| \mathbf{g}^1 \left( \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) \right) \right) \right]. \end{aligned}$$

Using (8.31), we get the obvious inclusion:

$$\begin{aligned} \mathbf{r} \in (\text{Hom}) \left[ \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x); \mathbf{t}_\otimes [(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}] \middle| \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x); \mathbb{F}_0^* \left( \bigotimes_{x \in X} \mathcal{L}_x \right); \right. \\ \left. \mathbb{T}_* \langle \mathbb{E} | \bigotimes_{x \in X} \mathcal{L}_x \rangle \middle| \mathbb{F}_0^* \left( \bigotimes_{x \in X} \mathcal{L}_x \right) \right]. \end{aligned} \tag{8.34}$$

Now, using (4.5) and (8.34), we obtain

$$\mathbf{r} \in (\text{Hom}) \left[ \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x); \mathbf{t}_\otimes [(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}] \middle| \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x); \mathbb{F}_0^* \left( \bigotimes_{x \in X} \mathcal{L}_x \right); \mathbf{T}_* \bigotimes_{x \in X} \mathcal{L}_x [\mathbb{E}] \right]. \tag{8.35}$$

In what follows, we use (8.5). In addition, we suppose that, in (8.5),

$$(\mathbb{X} = X) \& (\mathbb{Y} = \mathcal{P}'(\mathcal{P}(\mathbf{E}))). \tag{8.36}$$

Using (8.36), we suppose that, in (8.5),

$$\begin{aligned} (Y_x)_{x \in \mathbb{X}} &= (\langle \mathcal{L}_x - \text{link} \rangle_0 [E_x])_{x \in X}, \quad (\tau_x)_{x \in \mathbb{X}} = (\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}, \\ (A_x)_{x \in \mathbb{X}} &= (\mathbb{F}_0^*(\mathcal{L}_x))_{x \in X}. \end{aligned}$$

Then we obtain the following chain of equalities:

$$\mathbf{t}_\otimes [(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}] \middle| \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) = \mathbf{t}_\otimes [(\mathbb{T}_* \langle E_x | \mathcal{L}_x \rangle)_{x \in X}] \middle| \mathbb{F}_0^*(\mathcal{L}_x)_{x \in X} = \mathbf{t}_\otimes [(\mathbf{T}_{\mathcal{L}_x}^* [E_x])_{x \in X}].$$

Therefore, by (8.35), the following inclusion holds:

$$\mathbf{r} \in (\text{Hom}) \left[ \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x); \mathbf{t}_\otimes [(\mathbf{T}_{\mathcal{L}_x}^* [E_x])_{x \in X}] \middle| \mathbb{F}_0^* \left( \bigotimes_{x \in X} \mathcal{L}_x \right); \mathbf{T}_* \bigotimes_{x \in X} \mathcal{L}_x [\mathbb{E}] \right].$$

So, the property (8.32) is established.  $\square$

## 9. Conclusion

In this paper, some questions related to the structure of ultrafilters and MLSs on products of widely understood MSs were considered. In this connection, two basic directions were developed: the direction connected with representations for ultrafilter and MLSs on the products of MSs (set-theoretical direction) and (topological) direction connected with topological relations between TSs of Stone type arising under consideration of topology products (in the box and Cartesian variants) and topologies on the sets of ultrafilters and MLSs for the product of the corresponding measurable structures. In the first direction, the following property is established: ultrafilters and MLSs on products of MSs are exhausted by products of ultrafilters and MLSs, respectively. In the second direction, important properties of homeomorphism and compaction were obtained. In addition, the compaction property is established for the box products of TSs. In the case of the generalized Cartesian product, the homeomorphism property holds. This comparison shows the better character of Tychonoff's product of TSs compared to box TSs.

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