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HANKEL DETERMINANT OF CERTAIN ORDERS FOR SOME SUBCLASSES OF HOLOMORPHIC FUNCTIONS

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Abstract: In this paper, we are introducing certain subfamilies of holomorphic functions and making an attempt to obtain an upper bound (UB) to the second and third order Hankel determinants by applying certain lemmas, Toeplitz determinants, for the normalized analytic functions belong to these classes, defined on the open unit disc in the complex plane. For one of the inequality, we have obtained sharp bound.

Keywords: Holomorphic function, Upper bound, Hankel determinant, Positive real function.

1. Introduction

Let $\mathcal{A}$ represent a family of mappings f of the type

$$f(z) = z + \sum_{t=2}^{\infty} a_t z^t$$

in the open unit disc

$$\mathcal{U} = \{z \in \mathbb{C} : 1 > |z|\},$$

and $\mathcal{S}$ is the subfamily of $\mathcal{A}$, possessing univalent (schlicht) mappings. Pommerenke [17] characterized the r^{th}-Hankel determinant of order n, for f with $r,\ n \in \mathbb{N}$, namely

$$H_{r,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix} \quad (a_1 = 1). \tag{1.1}$$

The Fekete–Szegö functional [7] is obtained for $r = 2$ and $n = 1$ in (1.1), denoted by $H_{2,1}(f)$. Further, sharp bounds to the functional $|H_{2,2}(f)|$, obtained for $r = 2$ and $n = 2$ in (1.1), are called as Hankel determinant of order two, given by

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

In recent years, the estimation of an upper bound (UB) to $|H_{2,2}(f)|$ was studied by many authors. The exact estimates of $|H_{2,2}(f)|$ for the functions namely, bounded turning, starlike and convex functions, each one is a subfamily of $\mathcal{S}$, symbolized as $\mathcal{R}$, S^* and $\mathcal{K}$ respectively and fulfilling the conditions

$$\operatorname{Re} f'(z) > 0, \quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$$

in the unit disc $\mathcal{U}$, were proved by Janteng et al. [9, 10] and the derived bounds are $4/9$, 1 and $1/8$ respectively. Choosing $r = 2$ and $n = p+1$ in (1.1), we obtain Hankel determinant of second order for the p-valent function (see [20]), given by

$$H_{2,(p+1)}(f) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2.$$

The case $r = 3$ seems to be much tough than $r = 2$. Few papers were devoted for the study of third order Hankel determinant denoted as $H_{3,1}(f)$, with $r = 3$ and $n = 1$ in (1.1), namely

$$H_{3,1}(f) = \begin{vmatrix} a_1 = 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

Calculating the determinant, we have

$$H_{3,1}(f) = a_1(a_3a_5 - a_4^2) + a_2(a_3a_4 - a_2a_5) + a_3(a_2a_4 - a_3^2). \tag{1.2}$$

The concept of estimation of an upper bound for $H_{3,1}(f)$ was firstly introduced and studied by Babalola [3], who tried to estimate this functional in the classes $\mathcal{R}$, S^* and $\mathcal{K}$, his results are as follows

(i) $f \in S^* \Rightarrow |H_{3,1}(f)| \leq 16$;

(ii) $f \in \mathcal{K} \Rightarrow |H_{3,1}(f)| \leq 0.714$;

(iii) $f \in \mathcal{R} \Rightarrow |H_{3,1}(f)| \leq 0.742$.

As a result of the paper by Babalola [3], mach research associated with the Hankel determinant of order 3 and 4, for specific subfamilies of holomorphic functions have been done (see [1–5, 11, 12, 15, 18, 19]). Motivated by the results obtained by the indicated authors, here we make an attempt to derive an upper bound to $|H_{2,3}(f)| = a_3a_5 - a_4^2$, $|H_{3,1}(f)|$, when f belongs to the following new subfamilies of holomorphic functions.

Definition 1. *A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{R}_b(\alpha)$, where $b \neq 0$ is a real number with α $(0 \leq \alpha < 1)$, if it satisfies the condition*

$$\operatorname{Re}\left(1 - \frac{2}{b} + \frac{2}{b}f'(z)\right) > \alpha, \quad z \in \mathcal{U}.$$

It is observed that for $b = 2$ and for the values $b = 2$, $\alpha = 0$, we have $\mathcal{R}(\alpha)$, the class consisting of functions whose derivative has positive real part of order α $(0 \leq \alpha < 1)$ and $\mathcal{R}$ respectively.

Definition 2. *A function $f(z) \in \mathcal{A}$ is said to be in the class $S_b^*(\alpha)$, where b is a non-zero real number with α $(0 \leq \alpha < 1)$, if it satisfies the condition*

$$\operatorname{Re}\left(1 - \frac{2}{b} + \frac{2}{b}\left(\frac{zf'(z)}{f(z)}\right)\right) > \alpha, \quad z \in \mathcal{U}.$$

For the values $b = 2$ and $b = 2,\ \alpha = 0$, $S_b^*(\alpha)$ reduces to $S^*(\alpha)$, class consisting of starlike functions of order $\alpha\,(0 \leq \alpha < 1)$ and S^* respectively.

Definition 3. *A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{K}_b(\alpha)$, where $b \neq 0$ is a real number with $\alpha\,(0 \leq \alpha < 1)$, if it satisfies the condition*

$$\mathrm{Re}\left(1 - \frac{2}{b} + \frac{2}{b}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \alpha, \quad z \in \mathcal{U}.$$

In particular for $b = 2$ and for the values $b = 2,\ \alpha = 0$, $\mathcal{K}_b(\alpha)$ reduces to $\mathcal{K}(\alpha)$, the class consisting of convex functions of order $\alpha\,(0 \leq \alpha < 1)$ and $\mathcal{K}$ respectively.

In proving our results, the following sharp estimates are needed, which are in the form of Lemmas hold good for functions possessing positive real part. Define the collection $\mathcal{P}$ of all functions g, each one called as Carathéodory function [6] of the form

$$g(z) = 1 + \sum_{t=1}^{\infty} c_t z^t,$$

which is holomorphic in $\mathcal{U}$ and $\mathrm{Re}\, g(z) > 0$ for $z \in \mathcal{U}$.

Lemma 1 [8]. *If $g \in \mathcal{P}$, then the estimate $|c_i - \mu c_j c_{i-j}| \leq 2$ holds for $i, j \in \mathbb{N}$, with $i > j$ and $\mu \in [0, 1]$.*

Lemma 2 [14]. *If $g \in \mathcal{P}$, then the estimate $|c_i - c_j c_{i-j}| \leq 2$ holds for $i, j \in \mathbb{N}$, with $i > j$.*

Lemma 3 [16]. *If $g \in \mathcal{P}$, then $|c_t| \leq 2$, for $t \in \mathbb{N}$, equality occurs for the function*

$$h(z) = \frac{1+z}{1-z}, \quad z \in \mathcal{U}.$$

Lemma 4 [21]. *If $g \in \mathcal{P}$, then $|c_2 c_4 - c_3^2| \leq 4 - 1/2 \cdot |c_2|^2 + 1/4 \cdot |c_2|^3$.*

In order to procure our results, we adopt the procedure framed through Libera and Zlotkiewicz [13].

2. Main results

Theorem 1. *If*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}_b(\alpha),$$

where b is any real number with $0 < b \leq 1/(1-\alpha)$, for $0 \leq \alpha < 1$ then

$$|H_{3,1}(f)| \leq \frac{41 b^2 (1-\alpha)^2}{240}.$$

P r o o f. For

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}_b(\alpha),$$

by virtue of Definition 1, we have

$$\frac{b(1-\alpha) + 2\left\{f'(z) - 1\right\}}{b(1-\alpha)} = g(z) \Leftrightarrow b(1-\alpha) + 2\left\{f'(z) - 1\right\} = b(1-\alpha) g(z). \tag{2.1}$$

Using the series representations for $f'(z)$ and $g(z)$ in (2.1), after simplifying, we get

$$a_n = \frac{tc_{n-1}}{2n}, \quad \text{where} \quad t = b(1-\alpha), \quad n \geq 2. \tag{2.2}$$

Putting the values of a_i, for $i \in \{2,3,4,5\}$ from (2.2), in $H_{3,1}(f)$, given in (1.2), we have

$$H_{3,1}(f) = t^2 \left[\frac{c_2c_4}{60} - \frac{tc_2^3}{216} - \frac{c_3^2}{64} - \frac{tc_1^2c_4}{160} + \frac{tc_1c_2c_3}{96}\right]. \tag{2.3}$$

On grouping the terms in the expression (2.3), we obtain

$$\begin{aligned} H_{3,1}(f) = t^2 \Bigg[& \frac{tc_4(c_2-c_1^2)}{160} - \frac{c_3}{64}\left(c_3 - \frac{tc_1c_2}{2}\right) + \frac{tc_2(c_4-c_2^2)}{216} \\ & - \frac{c_2}{192}\left(c_4 - \frac{tc_1c_3}{2}\right) + \frac{(189-94t)c_2c_4}{8640}\Bigg]. \end{aligned} \tag{2.4}$$

Applying the triangle inequality in (2.4), we get

$$\begin{aligned} \left|H_{3,1}(f)\right| \leq t^2 \Bigg[& \frac{t|c_4||(c_2-c_1^2)|}{160} + \frac{|c_3|}{64}\left|c_3 - \frac{tc_1c_2}{2}\right| + \frac{t|c_2||c_4-c_2^2|}{216} \\ & + \frac{|c_2|}{192}\left|c_4 - \frac{tc_1c_3}{2}\right| + \frac{(189-94t)|c_2||c_4|}{8640}\Bigg]. \end{aligned} \tag{2.5}$$

Upon using the Lemmas 1–3 in the inequality (2.5), we obtain

$$|H_{3,1}(f)| \leq \frac{41t^2}{240} = \frac{41b^2(1-\alpha)^2}{240}. \tag{2.6}$$

□

Remark 1. Choosing $b = 2$ and $\alpha = 0$ in the inequality (2.6), it coincides with the result obtained by Zaprawa [22].

Theorem 2. *If*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}_b(\alpha),$$

where b is any real number with $0 < b \leq 1/(1-\alpha)$, for $0 \leq \alpha < 1$ then $|H_{2,3}(f)| \leq b^2(1-\alpha)^2/15$.

P r o o f. Substituting the values of a_3, a_4, and a_5 from (2.2) in $H_{2,3}(f)$, we have

$$\begin{aligned} H_{2,3}(f) = a_3a_5 - a_4^2 = t^2\left[\frac{c_2c_4}{60} - \frac{c_3^2}{64}\right] &= t^2\left[\frac{c_2c_4}{60} - \frac{c_2c_4}{64} + \frac{c_2c_4}{64} - \frac{c_3^2}{64}\right] \\ = t^2\left[\frac{c_2c_4 - c_3^2}{64} + \frac{c_2c_4}{960}\right], & \quad \text{where} \quad t = b(1-\alpha). \end{aligned} \tag{2.7}$$

Applying the triangle inequality in (2.7) and then using the Lemmas 3 and 4, after simplifying, we get

$$|H_{2,3}(f)| = |a_3a_5 - a_4^2| \leq \frac{b^2(1-\alpha)^2}{15}. \tag{2.8}$$

□

Remark 2. Choosing $b = 2$ and $\alpha = 0$ in the inequality (2.8), it coincides with the result obtained by Zaprawa [21]. At this stage, the inequality in (2.8) becomes sharp for the function

$$g(z) = \frac{1+z^2}{1-z^2}.$$

Theorem 3. *If*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_b^*(\alpha),$$

where b is any real number with $0 < b \leq 1/(1-\alpha)$, for $0 \leq \alpha < 1$ then

$$|H_{3,1}(f)| \leq \left[\frac{b(1-\alpha)}{12}\right]^2 [34 + b(1-\alpha)].$$

P r o o f. For

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_b^*(\alpha),$$

from the Definition 2, we have

$$\frac{\{b(1-\alpha)-2\} f(z) + 2zf'(z)}{b(1-\alpha)f(z)} = g(z) \Leftrightarrow \{b(1-\alpha)-2\} f(z) + 2zf'(z) = b(1-\alpha)f(z)g(z) \tag{2.9}$$

Replacing $f(z)$, $f'(z)$ and $g(z)$ with their equivalent series expressions in (2.9) and applying the same procedure as we carried in Theorem 1, we obtain

$$\begin{aligned} a_2 &= \frac{tc_1}{2}, \quad a_3 = \frac{t}{8}\left(2c_2 + tc_1^2\right), \quad a_4 = \frac{t}{48}\left(8c_3 + 6tc_1c_2 + t^2c_1^3\right), \\ a_5 &= \frac{t}{384}\left(48c_4 + 32tc_1c_3 + 12tc_2^2 + 12t^2c_1^2c_2 + t^3c_1^4\right), \quad \text{where} \quad t = b(1-\alpha). \end{aligned} \tag{2.10}$$

Substituting the values of a_2, a_3, a_4, and a_5 from (2.10) in the functional given in (1.2), we get

$$\begin{aligned} H_{3,1}(f) = \left(\frac{t}{94}\right)^2 \Big[&-t^4c_1^6 + 6t^3c_1^4c_2 + 32t^2c_1^3c_3 - 36t^2c_1^2c_2^2 - 144tc_1^2c_4 \\ &+192tc_1c_2c_3 - 72tc_2^3 + 288c_2c_4 - 256c_3^2\Big]. \end{aligned} \tag{2.11}$$

On grouping the terms in (2.11), we have

$$\begin{aligned} H_{3,1}(f) = \left(\frac{t}{94}\right)^2 \Bigg[&160\left(c_2 - \frac{tc_1^2}{2}\right)\left(c_4 - \frac{tc_2^2}{2}\right) + 8t\left(c_2 - \frac{tc_1^2}{2}\right)^3 + \\ &128\left(c_2 - \frac{tc_1^2}{2}\right)\left(c_4 - \frac{tc_1c_3}{2}\right) - 256\left(c_3 - \frac{8tc_1c_2}{16}\right)^2\Bigg]. \end{aligned} \tag{2.12}$$

On applying the triangle inequality in (2.12), we obtain

$$\begin{aligned} |H_{3,1}(f)| \leq \left(\frac{t}{94}\right)^2 \Bigg[&160\left|c_2 - \frac{tc_1^2}{2}\right|\left|c_4 - \frac{tc_2^2}{2}\right| + 8t\left|c_2 - \frac{tc_1^2}{2}\right|^3 + \\ &128\left|c_2 - \frac{tc_1^2}{2}\right|\left|c_4 - \frac{tc_1c_3}{2}\right| + 256\left|c_3 - \frac{8tc_1c_2}{16}\right|^2\Bigg]. \end{aligned}$$

Further, the above inequality simplifies to

$$|H_{3,1}(f)| \leq \left(\frac{t}{12}\right)^2 [34+t] = \left[\frac{b(1-\alpha)}{12}\right]^2 [34+b(1-\alpha)]. \tag{2.13}$$

$\square$

Remark 3. Choosing $b = 2$ and $\alpha = 0$ in the inequality (2.13), we see that it coincides with that of Zaprawa [22].

Theorem 4. *If*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_b(\alpha),$$

where b is any real number with $0 < b \leq 1/(1-\alpha)$, $0 \leq \alpha < 1$ then

$$|H_{3,1}(f)| \leq \left[\frac{b(1-\alpha)}{12\sqrt{15}}\right]^2 [33+8b(1-\alpha)].$$

P r o o f. For

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_b(\alpha),$$

from Definition 3, we have

$$\frac{\{b(1-\alpha)-2\} f(z) + 2zf'(z)}{b(1-\alpha)f(z)} = g(z) \Leftrightarrow \{b(1-\alpha)-2\} f(z) + 2zf'(z) \ = b(1-\alpha)f(z)g(z).$$

Applying the same procedure as we did in Theorem 1, we obtain

$$a_2 = \frac{tc_1}{4}, \quad a_3 = \frac{t}{24}\left(2c_2 + tc_1^2\right), \quad a_4 = \frac{t}{192}\left(8c_3 + 6tc_1c_2 + t^2c_1^3\right),$$

$$a_5 = \frac{t}{1920}\left(48c_4 + 32tc_1c_3 + 12tc_2^2 + 12t^2c_1^2c_2 + t^3c_1^4\right), \quad \text{where} \quad t = b(1-\alpha).$$

Further, we have

$$H_{3,1}(f) = \frac{t^2}{552960}\Big[-t^4c_1^6 + 12t^3c_1^4c_2 + 48t^2c_1^3c_3 - 84t^2c_1^2c_2^2 - 288tc_1^2c_4 \\ +288tc_1c_2c_3 - 32tc_2^3 + 1152c_2c_4 - 960c_3^2\Big].$$

On grouping the suitable terms in the above expression, we have

$$H_{3,1}(f) = \frac{t^2}{552960}\Bigg[64t\Big(c_2 - \frac{tc_1^2}{4}\Big)^3 + 384c_4\Big(c_2 - \frac{tc_1^2}{2}\Big) + 576c_2\Big(c_4 - \frac{tc_2^2}{2}\Big) \\ +192\Big(c_2 - \frac{tc_1^2}{2}\Big)\Big(c_4 - \frac{tc_1c_3}{2}\Big) - 960c_3\Big(c_3 - \frac{2tc_1c_2}{5}\Big) + 192tc_2^2\Big(c_2 - \frac{3tc_1^2}{16}\Big)\Bigg]. \tag{2.14}$$

Applying the triangle inequality and then the Lemmas 1–3 in (2.14), we get

$$|H_{3,1}(f)| \leq \left[\frac{t}{12\sqrt{15}}\right]^2 [33+8t] = \left[\frac{b(1-\alpha)}{12\sqrt{15}}\right]^2 [33+8b(1-\alpha)]. \tag{2.15}$$

Remark 4. Choosing $b = 2$ and $\alpha = 0$ in the inequality (2.15), we see that it coincides with the result obtained by Zaprawa [22].

3. Conclusion

The upper bounds to the fourth order Hankel determinants for all the above defined subclasses of analytic functions were derived.

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