ON AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH FIXED MEAN VALUE[1]

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Abstract: Let T_n^+ be the set of nonnegative trigonometric polynomials τ_n of degree n that are strictly positive at zero. For $0 \le \alpha \le 2\pi/(n{+}2)$, we find the minimum of the mean value of polynomial $(\cos\alpha - \cos x)\tau_n(x)/\tau_n(0)$ over $\tau_n \in T_n^+$ on the period $[-\pi, \pi)$.

Key words: Trigonometric polynomials, Extremal problem.

Let T_n be the space of trigonometric polynomials of degree n with real coefficients, and let T_n^+ be the set of nonnegative polynomials from T_n that are strictly positive at zero. For a real α we define

$$\chi_n(\alpha) = \inf_{\tau_n \in T_n^+} \frac{1}{2\pi\tau_n(0)} \int_{-\pi}^{\pi} \tau_n(x)(\cos\alpha - \cos x)\,dx. \tag{1}$$

In 1915, Fejér [4] (see also [2, vol. 2, Sec. 6, Problem 52]) proved the following statement.

Fejér's Theorem. *Let the polynomial* $\tau_n(x) = a_0 + \sum\limits_{\nu=1}^{n} (a_\nu \cos\nu x + b_\nu \sin\nu x)$ *belong to the set* T_n^+. *Then*

$$\sqrt{a_1^2 + b_1^2} \le 2a_0 \cos\frac{\pi}{n+2}. \tag{2}$$

This inequality turns into the equality for the polynomial

$$t_n(x) = \Big(\cos\frac{n+2}{2}x\Big)^2 \Big/ \Big(\cos x - \cos\frac{\pi}{n+2}\Big)^2. \tag{3}$$

This theorem is equivalent to the statement that

$$\chi_n\big(\pi/(n+2)\big) = 0. \tag{4}$$

For $0 \le \alpha < \pi$, put

$$Q_{(n+3)/2,\alpha}(x) = \Big(\sin\frac{n+1}{2}\alpha\,\sin\frac{n+3}{2}x - \sin\frac{n+3}{2}\alpha\,\sin\frac{n+1}{2}x\Big)\Big/\sin\frac{\alpha}{2}, \quad 0 < \alpha < \pi, \tag{5}$$

$$Q_{(n+3)/2,0}(x) = \lim_{\alpha\to 0} Q_{(n+3)/2,\alpha}(x) = (n+1)\sin\frac{n+3}{2}x - (n+3)\sin\frac{n+1}{2}x.$$

In this paper we prove the following result.

[1]The paper was originally published in a hard accessible collection of articles *Approximation of Functions by Polynomials and Splines* (The Ural Scientific Center of the Academy of Sciences of the USSR, Sverdlovsk, 1985), p. 15–22 (in Russian).

Theorem. *Let n be a nonnegative integer and $0 \le \alpha \le 2\pi/(n+2)$. Then* (1) *takes the value*

$$\chi_n(\alpha) = \frac{\left(\sin\frac{n+3}{2}\alpha - \sin\frac{n+1}{2}\alpha\right)(1-\cos\alpha)}{(n+3)\sin\frac{n+1}{2}\alpha - (n+1)\sin\frac{n+3}{2}\alpha}, \quad 0 < \alpha \le \frac{2\pi}{n+2}, \tag{6}$$

$$\chi_n(0) = \lim_{\alpha\to 0}\chi_n(\alpha) = \frac{6}{(n+1)(n+2)(n+3)},$$

and the infimum is attained for the polynomial

$$\tau_{n,\alpha}(x) = \left(\frac{Q_{(n+3)/2,\alpha}(x)}{(\cos x - \cos\alpha)\sin(x/2)}\right)^2, \tag{7}$$

where $Q_{(n+3)/2,\alpha}$ is given by (5).

Note that $\chi_n(\alpha) \ge 0$ for $0 \le \alpha \le \pi/(n+2)$ and $\chi_n(\alpha) \le 0$ for $\pi/(n+2) \le \alpha \le 2\pi/(n+2)$. First we prove two auxiliary statements. Set $\alpha_0 = \pi$, $\alpha_1 = 2\pi/3$, and for $n \ge 2$ let α_n be the first positive root of the equation

$$\left(\sin\frac{n+3}{2}x\right)\Big/\sin\frac{n+1}{2}x = c_n, \quad c_{2m} = -1, \quad c_{2m-1} = -\frac{m+1}{m}. \tag{8}$$

It is easy to see that for $r \ge 2$ we have

$$\alpha_{2r-2} = \pi/r, \quad 2\pi/(2r+1) < \alpha_{2r-1} < \pi/r. \tag{9}$$

Lemma 1. *If n is a nonnegative integer and $0 < \alpha < \alpha_n$, then the function $Q_{(n+3)/2,\alpha}$ defined by* (5) *has exactly $[(n+5)/2]$ zeros $x_0 = 0 < x_1 = \alpha < x_2 < x_3 < \ldots < x_{[(n+3)/2]}$ in the interval $[0,\pi]$. For each polynomial $\tau_{n+1} \in T_{n+1}$ we have*

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\tau_{n+1}(x)\,dx = {} & \frac{\sin\frac{n+1}{2}\alpha - \sin\frac{n+3}{2}\alpha}{(n+3)\sin\frac{n+1}{2}\alpha - (n+1)\sin\frac{n+3}{2}\alpha}\,\tau_n(0) \\ & + \sum_{k=1}^{[(n+3)/2]} g_{n+1}(x_k)\big(\tau_{n+1}(x_k) + \tau_{n+1}(-x_k)\big),\end{aligned} \tag{10}$$

where

$$\begin{aligned} g_{2r-1}(x) &= \frac{\sin x}{2r\sin x - \sin 2rx}, \\ g_{2r}(x) &= \begin{cases} \dfrac{\sin x}{2\big(r\sin x - \sin rx\cos(r+1)x\big)}, & x \ne \pi, \\ \dfrac{\sin r\alpha + \sin(r+1)\alpha}{4\big(r\sin(r+1)\alpha + (r+1)\sin r\alpha\big)}, & x = \pi. \end{cases}\end{aligned} \tag{11}$$

Moreover, the numbers $\left(\frac{2\pi}{n+2} - \alpha\right)g_{n+1}(x_{[(n+3)/2]})$, $g_{n+1}(x_k)$, $1 \le k \le [(n+1)/2]$, are nonnegative.

P r o o f. First we consider the case when $n = 0$ and $0 < \alpha < \pi$. The function $Q_{3/2,\alpha}(x) = 2\sin(x/2)(\cos x - \cos\alpha)$ has two zeros $x_0 = 0$, $x_1 = \alpha$ in the interval $[0,\pi]$. We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\tau_1(x)\,dx = \frac{\cos\alpha}{\cos\alpha - 1}\,\tau_1(0) + \frac{\tau_1(\alpha) + \tau_1(-\alpha)}{2(1-\cos\alpha)}, \quad \tau_1 \in T_1,$$

since this formula is valid for the polynomials 1, $\sin x$, $\cos x$ and thus the lemma follows for $n = 0$.

Now let $n = 1$ and $0 < \alpha < 2\pi/3$. Then the function $Q_{2,\alpha}(x) = 4\cos\frac{\alpha}{2}\sin x(\cos x - \cos\alpha)$ has three zeros $x_0 = 0$, $x_1 = \alpha$, $x_2 = \pi$ in the interval $[0,\pi]$. The quadrature formula

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\tau_2(x)\,dx = \frac{1-2\cos\alpha}{4(1-\cos\alpha)}\,\tau_2(0) + \frac{\tau_2(\alpha)+\tau_2(-\alpha)}{2(1-\cos\alpha)} + \frac{1+2\cos\alpha}{8(1+\cos\alpha)}\big(\tau_2(\pi)+\tau_2(-\pi)\big), \quad \tau_2 \in T_2,$$

holds, for it holds for the polynomials $\sin x$, $\sin 2x$, $(1+\cos x)(\cos\alpha-\cos x)$, $1-\cos 2x$, $(1-\cos x)(\cos\alpha-\cos x)$ which generate the space T_2. This proves the lemma for $n=1$.

Next we consider the case of an odd $n=2r-1$, $r\ge 2$, and $0<\alpha<\alpha_{2r-1}$. The function (5) can be written in the form $Q_{r+1,\alpha}(x)=(\cos x-\cos\alpha)\sin x S_{r-1,\alpha}(x)/\sin(\alpha/2)$, where

$$S_{r-1,\alpha}(x)=\frac{\sin r\alpha\,\sin(r+1)x-\sin(r+1)\alpha\,\sin rx}{(\cos x-\cos\alpha)\sin x} \tag{12}$$

is a cosine polynomial of degree $r-1$. To study the zeros of the polynomial $S_{r-1,\alpha}$, we write it in the form $S_{r-1,\alpha}(x)=\dfrac{(f(x)-f(\alpha))\sin r\alpha\,\sin rx}{\sin x(\cos x-\cos\alpha)}$, where $f(x)=\dfrac{\sin(r+1)x}{\sin rx}$. When x runs over the intervals $(0,\pi/r)$, $((r-1)\pi/r,\pi)$, and $(k\pi/r,(k+1)\pi/r)$, $1\le k\le r-2$, then the values of f run continuously over the intervals $((r+1)/r,-\infty)$, $(+\infty,-(r+1)/r)$, and $(+\infty,-\infty)$, respectively. Thus, taking into account the definition (8) of α_{2r-1}, we see that for each α in the interval $(0,\alpha_{2r-1})$ the polynomial $S_{r-1,\alpha}$ has exactly $r-1$ zeros $x_2<x_3<\ldots<x_r$ in the interval (α,π). Moreover, these zeros are all simple since $S_{r-1,\alpha}$ has degree $r-1$. It is known [3, p. 403, formulae 30, 31, 33] that

$$\frac{1}{\pi}\int_0^\pi \frac{\sin mx\cos\nu x}{\sin x}\,dx=\begin{cases}1,\ m>\nu,\ m+\nu=2k-1;\\ 0,\ m>\nu,\ m+\nu=2k;\\ 0,\ m\le\nu.\end{cases} \tag{13}$$

It follows that for the polynomial (12) we have

$$\frac{1}{\pi}\int_0^\pi S_{r-1,\alpha}(x)\cos\nu x(1+\cos x)(\cos\alpha-\cos x)\,dx=\sin(r+1)\alpha-\sin r\alpha,\quad \nu=0,1,\ldots,r-1.$$

Consequently, for each cosine polynomial C_{r-1} of degree $r-1$ we have

$$\frac{1}{\pi}\int_0^\pi S_{r-1,\alpha}(x)C_{r-1}(x)(1+\cos x)(\cos\alpha-\cos x)\,dx=\big(\sin(r+1)\alpha-\sin r\alpha\big)C_{r-1}(0). \tag{14}$$

Thus, the polynomial $S_{r-1,\alpha}$ is orthogonal with the weight $(\cos x-\cos\alpha)(1-\cos x)(1+\cos x)$ to all cosine polynomials of degree $r-2$.

We will need the following known result (e.g., [1, pp. 162, 163]). Let the weight $\upsilon(x)$ and the points $a_1,\ldots,a_m$ in the interval $[0,\pi]$ be given. A quadrature formula of the form

$$\int_0^\pi C_{2\nu+m-1}(x)\upsilon(x)\,dx=\sum_{\ell=1}^m A_\ell C_{2\nu+m-1}(a_\ell)+\sum_{k=1}^\nu B_k C_{2\nu+m-1}(x_k)$$

which is exact for cosine polynomials of degree $2\nu+m-1$ exists if and only if there exists a cosine polynomial S_ν of degree ν which is orthogonal to all cosine polynomials of degree $\nu-1$ with the weight $\upsilon(x)(\cos x-\cos a_1)\ldots(\cos x-\cos a_m)$. The zeros of the polynomial S_ν coincide with the nodes $x_1,x_2,\ldots,x_\nu$; they should be all distinct and differ from the fixed nodes $a_1,\ldots,a_m$.

By this result, there exist numbers $\varepsilon_0,\ldots,\varepsilon_{r+1}$ such that for each cosine polynomial C_{2r} of degree $2r$ we have

$$\frac{1}{\pi}\int_0^\pi C_{2r}(x)\,dx=\sum_{k=0}^{r+1}\varepsilon_k C_{2r}(x_k), \tag{15}$$

where $x_2,x_3,\ldots,x_r$ are the zeros of the polynomial $S_{r-1,\alpha}$ in the interval (α,π), $x_0=0$, $x_1=\alpha$, $x_{r+1}=\pi$.

Note that, for $\nu=1,2,\ldots,r$, the zeros of the polynomial

$$S_{r-1,x_\nu}(x)=\frac{\sin rx_\nu\sin(r+1)x-\sin(r+1)x_\nu\sin rx}{(\cos x-\cos x_\nu)\sin x} \tag{16}$$

coincide with the zeros of the polynomial $(\cos x - \cos\alpha)S_{r-1,\alpha}(x)/(\cos x - \cos x_\nu)$. Thus,

$$S_{r-1,x_\nu}(x) = \mathcal{A}_\nu(\cos x - \cos\alpha)S_{r-1,\alpha}(x)/(\cos x - \cos x_\nu), \tag{17}$$

where $\mathcal{A}_\nu$ is a constant that does not depend on x.

It is not difficult to check that, for $\nu = 1, 2, \dots, r$, the polynomial (16) satisfies the equations

$$S_{r-1,x_\nu}(x) = \frac{D_r(x - x_\nu) - D_r(x + x_\nu)}{2\sin x} = 2\sum_{k=1}^{r} \frac{\sin kx_\nu \sin kx}{\sin x}, \tag{18}$$

where

$$D_r(x) = 1 + 2\sum_{k=1}^{r} \cos kx = \frac{\sin\frac{2r+1}{2}x}{\sin(x/2)}$$

is the Dirichlet kernel.

Using (18), we obtain

$$\frac{1}{\pi}\int_0^\pi S_{r-1,x_\nu}(x)(\sin x)^2\,dx = \sin x_\nu, \quad \nu = 1, 2, \dots, r. \tag{19}$$

Using (19), (17) and (18), one can calculate the following coefficients of the quadrature formula (15):

$$\varepsilon_\nu = 1\Big/\Big(2\sum_{k=1}^{r}(\sin kx_\nu)^2\Big) = \frac{\sin x_\nu}{r\sin x_\nu - \sin rx_\nu \cos(r+1)x_\nu}, \quad 1 \le \nu \le r. \tag{20}$$

By (14), we have

$$\frac{1}{\pi}\int_0^\pi S_{r-1,\alpha}(x)(1 + \cos x)(\cos\alpha - \cos x)\,dx = \sin(r+1)\alpha - \sin r\alpha.$$

Using (15) and (12), we obtain from here that

$$\varepsilon_0 = \frac{\sin r\alpha - \sin(r+1)\alpha}{2\big((r+1)\sin r\alpha - r\sin(r+1)\alpha\big)}. \tag{21}$$

By (13) and (12) we conclude that

$$\frac{1}{\pi}\int_0^\pi S_{r-1,\alpha}(x)(1 - \cos x)(\cos x - \cos\alpha)\,dx = (-1)^r\big(\sin r\alpha + \sin(r+1)\alpha\big). \tag{22}$$

Formulae (22), (15) and (12) imply

$$\varepsilon_{r+1} = \frac{\sin r\alpha + \sin(r+1)\alpha}{2\big((r+1)\sin r\alpha + r\sin(r+1)\alpha\big)}. \tag{23}$$

It is easy to check that

$$\Big(\frac{2\pi}{2r+1} - \alpha\Big)\varepsilon_r \ge 0 \tag{24}$$

for $0 < \alpha < \alpha_{2r-1}$. The statement of the lemma for $n = 2r - 1$, $r \ge 2$, now follows from (20), (21), (23) and (24).

Finally, let us consider the case when $n = 2r - 2$, $r \ge 2$, and $0 < \alpha < \pi/r$. Function (5) can be written in the form $Q_{(2r+1)/2,\alpha}(x) = \sin(x/2)(\cos x - \cos\alpha)\Theta_{r-1,\alpha}(x)/\sin(\alpha/2)$, where

$$\Theta_{r-1,\alpha}(x) = \frac{\sin\frac{2r-1}{2}\alpha\,\sin\frac{2r+1}{2}x - \sin\frac{2r+1}{2}\alpha\,\sin\frac{2r-1}{2}x}{(\cos x - \cos\alpha)\sin(x/2)} = \frac{\big(\varphi(x) - \varphi(\alpha)\big)\sin\frac{2r-1}{2}\alpha\,\sin\frac{2r-1}{2}x}{(\cos x - \cos\alpha)\sin(x/2)}; \tag{25}$$

here, $\varphi(x) = \left(\sin\frac{2r+1}{2}x\right)/\sin\frac{2r-1}{2}x$. When x runs over the intervals $(0, 2\pi/(2r-1))$, $(2(r-1)\pi/(2r-1), \pi)$ and $(2k\pi/(2r-1), 2(k+1)\pi/(2r-1))$, $1 \le k \le r-2$, then the values of the function φ run continuously over the intervals $((2r+1)/(2r-1), -\infty)$, $(+\infty, -1)$ and $(+\infty, -\infty)$, respectively. Thus, for $0 < \alpha < \pi/r$ the polynomial $\Theta_{r-1,\alpha}$ has exactly $r-1$ simple zeros $x_2 < x_3 < \ldots < x_r$ in the interval (α, π). With the help of (13) and (25), repeating the arguments used in the proof of formula (14), we see that

$$\frac{1}{\pi}\int_0^\pi \Theta_{r-1,\alpha}(x)C_{r-1}(x)(\cos\alpha - \cos x)\,dx = \Big(\sin\frac{2r+1}{2}\alpha - \sin\frac{2r-1}{2}\alpha\Big)C_{r-1}(0) \tag{26}$$

for all cosine polynomials C_{r-1} of degree $r-1$. Thus, the polynomial $\Theta_{r-1,\alpha}$ is orthogonal to all cosine polynomials of degree $r-2$ with the weight $(1-\cos x)(\cos x - \cos\alpha)$. It follows that there exist numbers $\delta_0, \delta_1, \ldots, \delta_r$ such that the quadrature formula

$$\frac{1}{\pi}\int_0^\pi C_{2r-1}(x)\,dx = \sum_{k=0}^{r}\delta_k C_{2r-1}(x_k), \tag{27}$$

where $x_2, x_3, \ldots, x_r$ are the zeros of the polynomial $\Theta_{r-1,\alpha}$ in the interval (α, π), $x_0 = 0$, $x_1 = \alpha$, is exact for all cosine polynomials C_{2r-1} of degree $2r-1$.

Note that, for $\nu = 1, 2, \ldots, r$, the polynomial

$$\Theta_{r-1,x_\nu}(x) = \frac{\sin\frac{2r-1}{2}x_\nu\,\sin\frac{2r+1}{2}x - \sin\frac{2r+1}{2}x_\nu\,\sin\frac{2r-1}{2}x}{(\cos x - \cos x_\nu)\sin(x/2)} \tag{28}$$

satisfies the equation

$$\Theta_{r-1,x_\nu}(x) = \mathcal{B}_\nu(\cos x - \cos\alpha)\Theta_{r-1,\alpha}(x)/(\cos x - \cos x_\nu), \quad 1 \le \nu \le r, \tag{29}$$

where $\mathcal{B}_\nu$ is a constant that does not depend on x.

Moreover, the polynomial (28) can be rewritten in the form

$$\Theta_{r-1,x_\nu}(x) = 2\sum_{k=1}^{r}\Big(\sin\frac{2k-1}{2}x_\nu\sin\frac{2k-1}{2}x\Big)\Big/\sin\frac{x}{2}. \tag{30}$$

This implies the equation

$$\frac{1}{\pi}\int_0^\pi \Theta_{r-1,x_\nu}(x)\Big(\sin\frac{x}{2}\Big)^2\,dx = \sin\frac{x_\nu}{2}, \quad 1 \le \nu \le r. \tag{31}$$

Formulae (31), (27), (29) and (30) yield

$$\delta_\nu = 1\Big/\Big(2\sum_{k=1}^{r}\Big(\sin\frac{2k-1}{2}x_\nu\Big)^2\Big) = \frac{2\sin x_\nu}{2r\sin x_\nu - \sin 2rx_\nu}, \quad 1 \le \nu \le r. \tag{32}$$

By (26) we obtain

$$\frac{1}{\pi}\int_0^\pi \Theta_{r-1,x_\nu}(x)(\cos\alpha - \cos x)\,dx = \sin\frac{2r+1}{2}\alpha - \sin\frac{2r-1}{2}\alpha. \tag{33}$$

Using (33), (27) and (25), we get

$$\delta_0 = \Big(\sin\frac{2r-1}{2}\alpha - \sin\frac{2r+1}{2}\alpha\Big)\Big/\Big((2r+1)\sin\frac{2r-1}{2}\alpha - (2r-1)\sin\frac{2r+1}{2}\alpha\Big). \tag{34}$$

The statement of the lemma for $n = 2r-2$, $r \ge 2$, now follows from (32) and (34). This completes the proof of the lemma. □

Lemma 2. *Let n be a nonnegative integer, $0 \le \alpha \le \alpha_n$ if n is even and $0 \le \alpha < \alpha_n$ if n is odd. For each polynomial $\tau_n \in T_n$ we have*

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\tau_n(x)(\cos\alpha-\cos x)\,dx &= \frac{\left(\sin\frac{n+3}{2}\alpha-\sin\frac{n+1}{2}\alpha\right)(1-\cos\alpha)}{(n+3)\sin\frac{n+1}{2}\alpha-(n+1)\sin\frac{n+3}{2}\alpha}\,\tau_n(0)\\ &+\sum_{k=1}^{[(n+1)/2]} g_{n+1}(x_k)(\cos\alpha-\cos x_k)\big(\tau_n(x_k)+\tau_n(-x_k)\big),\end{aligned} \tag{35}$$

where $x_1 < x_2 < \cdots < x_{[(n+1)/2]}$ are the zeros of the polynomial (7) in the interval $(\alpha,\pi]$, and the numbers $g_{n+1}(x_k)$, $k=1,2,\dots,[(n+1)/2]$, are defined by equations (11). Moreover, the coefficients $g_{n+1}(x_k)(\cos\alpha-\cos x_k)$, $k=1,2,\dots,[(n-1)/2]$, are nonnegative, as well as the number $\left(\frac{2\pi}{n+2}-\alpha\right)g_{n+1}(x_{[(n+1)/2]})(\cos\alpha-\cos x_{[(n+1)/2]})$.

P r o o f. For $0<\alpha<\alpha_n$, the statement is a straightforward consequence of Lemma 1. Let τ_n be an arbitrary polynomial of degree n, then the right-hand side of (35) and the coefficients of this quadrature formula tend uniformly to the claimed (bounded) values as $\alpha\to 0$, and the statement of the lemma follows for $\alpha=0$. The case of $\alpha=\alpha_n$ with even n can be proved in a similar way. As for the case of odd n, note that for an odd $n\ge 3$ we have $g_{n+1}(x_{[(n+1)/2]})(\cos\alpha-\cos x_{[(n+1)/2]}) = g(\pi)(\cos\alpha+1)\to-\infty$ as $\alpha\to\alpha_n$, while $g_{n+1}(x_{[(n-1)/2]})(\cos\alpha-\cos x_{[(n-1)/2]})\to+\infty$ as $\alpha\to\alpha_n$. □

P r o o f o f t h e t h e o r e m. The statement of the theorem follows from the fact that for each nonnegative polynomial τ_n and each number α in the interval $[0, 2\pi/(n+2)]$ we have, by Lemma 2, the inequality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\tau_n(x)(\cos\alpha-\cos x)\,dx \ge \frac{\left(\sin\frac{n+3}{2}\alpha-\sin\frac{n+1}{2}\alpha\right)(1-\cos\alpha)}{(n+3)\sin\frac{n+1}{2}\alpha-(n+1)\sin\frac{n+3}{2}\alpha}\,\tau_n(0).$$

This inequality turns into the equality for the polynomial $\tau_{n,\alpha}$. This proves the theorem. □

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REFERENCES

1. **Krylov V. I.** Approximate calculation of integrals. Fizmatgiz, Moscow. 1959. [Russian]
2. **Pólya, G., Szegő, G.** Problems and theorems in analysis. Springer, Berlin. Vol. 2. 1998.
3. **Prudnikov, A. P., Brychkov, Yu. A., Marichev, O. I.** Integrals and series. Nauka, Moscow. 1981. [Russian]
4. **Fejér L.** Über trigonometrische Polynome. Gesammelte Arbeit, Bd. 1, Akad. Kiado, Budapest. 1970. P. 842–872.