URAL MATHEMATICAL JOURNAL, Vol. 7, No. 2, 2021, pp. 121–135

DOI: 10.15826/umj.2021.2.009

ON HOP DOMINATION NUMBER OF SOME GENERALIZED GRAPH STRUCTURES

S. Shanmugavelan[†], **C. Natarajan**[††]

Srinivasa Ramanujan Centre,
SASTRA Deemed to be University,
Kumbakonam–612001, India

[†]shanmugavelan@src.sastra.edu, [††]natarajan_c@maths.sastra.edu

Abstract: A subset $H \subseteq V(G)$ of a graph G is a hop dominating set (HDS) if for every $v \in (V \setminus H)$ there is at least one vertex $u \in H$ such that $d(u,v)=2$. The minimum cardinality of a hop dominating set of G is called the hop domination number of G and is denoted by $\gamma_h(G)$. In this paper, we compute the hop domination number for triangular and quadrilateral snakes. Also, we analyse the hop domination number of graph families such as generalized thorn path, generalized ciliates graphs, glued path graphs and generalized theta graphs.

Keywords: Hop domination number, Snake graphs, Theta graphs, Generalized thorn path.

1. Introduction

Domination in graphs is fascinating topic in the field of graph theory. It is one of the most effective mathematical models for a variety of real world problems. A simple undirected finite graph G holds a vertex set $V(G)$ with vertices and an edge set $E(G)$ whose members are unordered pair of vertices called lines or edges of G. The *degree* of a vertex v, denoted by $d(v)$, is the number of edges that are incident with v and the *distance* $d(u,v)$ between any two distinct vertices u and v is the length of the shortest path connecting u and v in G. We use the symbol $[n]=\{1,2,\ldots,n\}$. For any other graph theory terminology not defined here, we follow [3].

In a graph G, a subset $D \subseteq V(G)$ is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a minimal dominating set of G is the domination number $\gamma(G)$. In the last three decades, several domination parameters have been established and they have been intensively investigated with applications in communication networks, facility location problems, game theory, mathematical chemistry, and so on. For a detailed study on domination concepts, one may refer [8–10].

Ayyasamy et al. [1] defined a new distance-based domination parameter called the hop domination number of a graph G. A subset $H \subseteq V(G)$ of a graph G is a hop dominating set (HDS) if for every vertex v not in H, there exists at least one vertex $u \in H$ such that $d(u,v)=2$. The minimum cardinality of a hop dominating set of G is called the hop domination number of G and is denoted by $\gamma_h(G)$. The hop degree of a vertex v in a graph denoted by $d_h(v)$ is the number of vertices at distance$=2$ from v. The hop graph $H(G)$ of a graph G is the graph having same vertex set and two vertices u,v are adjacent in $H(G)$ iff $d_G(u,v)=2$. Also, Ayyasamy et al. [2] obtained some bounds on hop domination number for trees and characterized trees attaining those bounds. Natarajan et al. [13] found characterization results for hop domination number equals other domination parameters like total domination number, connected domination number for several families of graphs. Many scholars have explored this parameter in the years thereafter, leading to novel versions such as connected hop domination, total perfect hop domination, Roman hop domination, Global hop domination, etc., [11, 12, 15–18, 20]. In 2018, Natarajan et al. [14] discussed hop domination number for some special families of graph like central graph, middle graph and total graph. Recently, Packiavathi et al. [6] obtained the hop domination number of a caterpillar graph $P_n(l_1, l_2, \dots, l_n)$ (a caterpillar is a graph obtained from the path by attaching leaves l_i to i^{th} vertex of the path P_n) and the domination number for some special families of snake graphs which occur as hop graph of $P_n(1, 1, \dots, 1)$ and $P_n(2, 2, \dots, 2)$. We refine their result on caterpillar graph and present an elegant result.

2. Main results

In this section, we study the hop domination number of snake graph families like triangular, alternate triangular, quadrilateral and alternate quadrilateral snakes. In addition, the hop domination number of some generalized structures like generalized theta graphs, generalized thorn paths and generalized ciliates graphs $GC(p, q, t)$ for $p = 3$ and $p = 4$ are determined.

Definition 1 [7]. *Let $l_1, l_2, \dots l_n$ be n positive integers. Then the thorn graph $G^t = G^t(l_1, l_2 \dots l_n)$ is obtained from a graph G by attaching l_i pendant vertices (thorns) to each vertex v_i of G, $i \in [n]$.*

In 2020, Getchial Pon Packiavathi et al. [6] obtained the following result on caterpillar graphs.

Theorem 1 [6]. $\gamma_h(P_n(1, 1, \dots 1)) = \gamma_h(P_n(2, 2, \dots 2)) = \begin{cases} 2r, & \text{if} \quad n = 2r; \\ 2r + 3, & \text{if} \quad n = 2r + 1. \end{cases}$

First, we observe that the result given in Theorem 1 is wrong. For example, $\gamma_h(P_4(1, 1, 1, 1)) = 2$ whereas from their computations it is 4. So, we refine the result by taking the more generalized version of caterpillar called thorn path P_n^t.

Theorem 2. *For $n > 1$,*

$$\gamma_h(P_n^t) = \begin{cases} \left\lfloor \dfrac{n-1}{2} \right\rfloor + 1, & \text{if} \quad n \equiv 0, 1, 3 \pmod 4; \\ \left\lceil \dfrac{n}{2} \right\rceil + 1, & \text{if} \quad n \equiv 2 \pmod 4. \end{cases}$$

P r o o f. Let $v_1, v_2, \cdots, v_n$ be the vertices of the central path P_n in P_n^t (see Fig. 1).

Case 1: $n \equiv 2 \pmod 4$. In this case, any γ_h-set is of the form

$$\{v_i \mid i \equiv 1 \pmod 4,\ 1 \le i \le (n-2)\} \cup \{v_j \mid j \equiv 2 \pmod 4,\ 2 \le j \le (n-1)\} \cup \{v_{n-1}\}.$$

Thus,

$$\gamma_h(P_n^t) \le \left\lceil \frac{n}{2} \right\rceil + 1$$

and it is easily seen that

$$\gamma_h(P_n^t) \ge \left\lceil \frac{n}{2} \right\rceil + 1.$$

Therefore,

$$\gamma_h(P_n^t) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Case 2: $n \equiv 0, 1, 3 \pmod 4$. In this case, any γ_h-set is of the form

$$\{v_i \mid i \equiv 2 \pmod 4,\ 2 \le i \le (n-2)\} \cup \{v_j \mid j \equiv 3 \pmod 4,\ 3 \le i \le (n-1)\} \cup \{v_{n-1}\}.$$

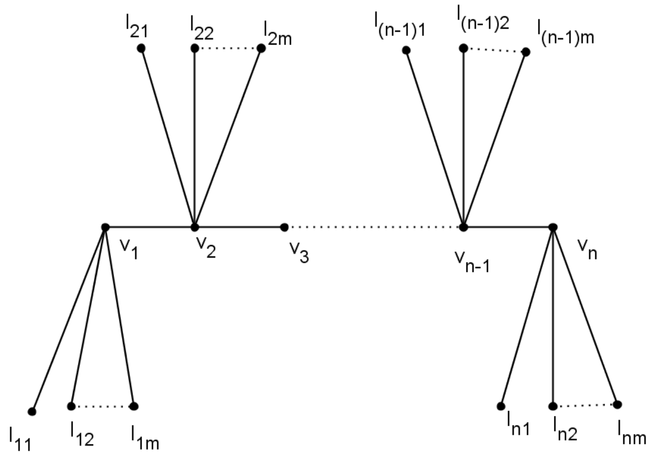

Figure 1. Thorny path P_n^t.

Thus,

$$\gamma_h(P_n^t) \leq \left\lceil \frac{n}{2} \right\rceil$$

and it is easily seen that

$$\gamma_h(P_n^t) \geq \left\lceil \frac{n}{2} \right\rceil.$$

Therefore,

$$\gamma_h(P_n^t) = \left\lceil \frac{n}{2} \right\rceil.$$

□

In [4], Derya Dogan et al. obtained some results for weak and strong domination in thorn graphs. Inspired by their results, we study our parameter namely, hop domination number for thorn rod given in [4], as well as for other generalized graph structures.

Lemma 1 [1]. $\gamma_h(P_n) = \begin{cases} 2r, & \textit{if} \quad n = 6r; \\ 2r+1, & \textit{if} \quad n = 6r+1; \\ 2r+2, & \textit{if} \quad n = 6r+s, \quad 2 \leq s \leq 5. \end{cases}$

Rewriting Lemma 1 in terms of congruence, we have

$$\gamma_h(P_n) = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor, & \text{if} \quad n \equiv 0 \pmod 6; \\ \left\lfloor \frac{n}{3} \right\rfloor + 1, & \text{if} \quad n \equiv 1,3,4,5 \pmod 6; \\ \left\lfloor \frac{n}{3} \right\rfloor + 2, & \text{if} \quad n \equiv 2 \pmod 6. \end{cases}$$

Definition 2 [4]. *A thorn rod is a graph $P_{n,t}$ which is obtained by taking a path on $n \geq 2$ vertices and attaching $(t-1)$ leaves, known as thorns, at each of the end of P_n.*

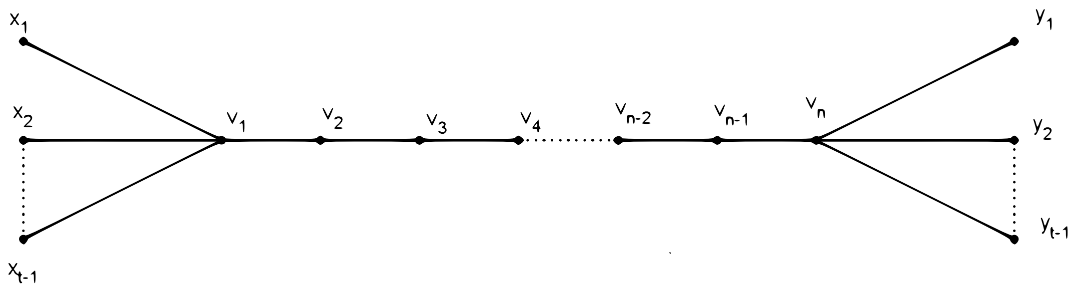

Figure 2. Thorn rod $P_{n,t}$.

Note that $P_{1,t}$ is a star graph $K_{1,t-1}$.

Theorem 3. $\gamma_h(P_{n,t}) = \begin{cases} \left\lfloor \dfrac{n-10}{3} \right\rfloor + 6, & \textit{if} \quad n \equiv 0 \pmod 6; \\ \left\lfloor \dfrac{n-10}{3} \right\rfloor + 5, & \textit{if} \quad n \equiv 1,2,3,5 \pmod 6; \\ \left\lfloor \dfrac{n-10}{3} \right\rfloor + 4, & \textit{if} \quad n \equiv 4 \pmod 6. \end{cases}$

P r o o f. Let us label the vertices of central path P_n as $v_1, v_2 \ldots v_n$. Let the leaves or thorns at the vertex v_1 be $x_1, x_2, \ldots x_{t-1}$ and the thorns at the vertex v_n be $y_1, y_2, \ldots y_{t-1}$.

From Fig. 2, it is clear that to hop dominate $2(t-1)$ leaves and their support vertices, any γ_h-set must include the vertices $v_2, v_3, v_{n-1}, v_{n-2}$.

Now, the subgraph induced by $P_n - \{v_1, v_2, v_3, v_4, v_5, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$, is clearly a path on $(n-10)$ vertices.

By Lemma 1,

$$\gamma_h(P_{n-10}) = \begin{cases} \left\lfloor \dfrac{n-10}{3} \right\rfloor, & \text{if} \quad n \equiv 4 \pmod 6; \\ \left\lfloor \dfrac{n-10}{3} \right\rfloor + 1, & \text{if} \quad n \equiv 0,1,3,5 \pmod 6; \\ \left\lfloor \dfrac{n-10}{3} \right\rfloor + 2, & \text{if} \quad n \equiv 2 \pmod 6. \end{cases}$$

and hence $\gamma_h(G) = 4 + \gamma_h(P_{n-10})$. Thus, the result follows. □

Definition 3. *A glued path $GP(n,t)$ is a graph obtained by gluing t copies of a path $P_n (n \geq 2)$ at a common vertex v such that v is the initial vertex in each copy of P_n.*

Theorem 4. $\gamma_h(GP(n,t)) = \begin{cases} 2\left\lceil \dfrac{n}{6} \right\rceil t, & \textit{if} \quad n \equiv 0,\ 5 \pmod 6; \\ 2\left\lfloor \dfrac{n}{6} \right\rfloor t + 1, & \textit{if} \quad n \equiv 1 \pmod 6; \\ 2\left\lceil \left\lceil \dfrac{n}{6} \right\rceil t - 1 \right\rceil + 1, & \textit{if} \quad n \equiv 4 \pmod 6; \\ 2\left\lfloor \dfrac{n}{6} \right\rfloor t + 2, & \textit{if} \quad n \equiv 2,\ 3 \pmod 6. \end{cases}$

P r o o f. Let us arrange the vertices of $GP(n,t)$ row-wise subject to the following conditions:

(i) Place the common vertex in the 1^{st} row R_1.

(ii) First vertex of each copy of the path P_n be placed in the 2^{nd} row R_2.

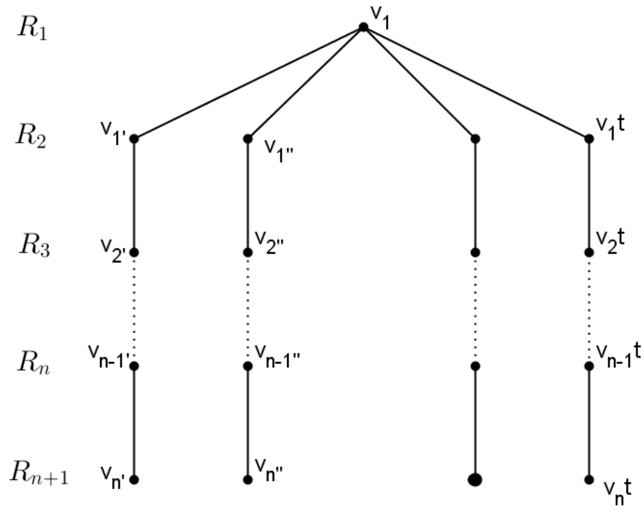

Figure 3. Glued path $GP(n,t)$.

(iii) In general, n^{th} vertex of each copy in the $(n+1)^{th}$ row R_{n+1}.

From Fig. 3, it is clear that, each row R_i has t vertices except the first row. That is, $V(R_i) = \{v_i, v_i^{''}, \ldots, v_i^{(t)}\}$, $2 \leq i \leq (n+1)$. To hop dominate all those leaves and support vertices, all the vertices in R_{n-1} and R_{n-2} must be selected from a γ_h-set of $GP(n,t)$. This choice will also hop dominate all of the vertices of R_{n-3} and R_{n-4}.

Case 1: $n \equiv 0 \pmod 6$. In this case, any γ_h-set contains $2\lceil n/6 \rceil t$ vertices from the following rows $\mathcal{R} = \{R_{(n-1)}, R_{(n-2)}, R_{(n-7)}, R_{(n-8)}, \ldots, R_4, R_3\}$. Thus,

$$\gamma_h(GP(n,t)) \leq 2\left\lceil \frac{n}{6} \right\rceil t.$$

It is easy to observe that any hop dominating set of $GP(n,t)$ contains at least $2\lceil n/6 \rceil t$ vertices. Therefore,

$$\gamma_h(GP(n,t)) = 2\left\lceil \frac{n}{6} \right\rceil t.$$

Case 2: $n \equiv 5 \pmod 6$. In this case, any γ_h-set includes $2\lceil n/6 \rceil t$ vertices from the rows $(\mathcal{R} \setminus R_3) \cup \{v_1\}$. Thus,

$$\gamma_h(GP(n,t)) \leq 2\left\lceil \frac{n}{6} \right\rceil t.$$

Also,

$$\gamma_h(GP(n,t)) \geq 2\left\lceil \frac{n}{6} \right\rceil t.$$

Therefore,

$$\gamma_h(GP(n,t)) = 2\left\lceil \frac{n}{6} \right\rceil t.$$

Case 3: $n \equiv 1 \pmod 6$. In this case, any γ_h-set includes vertices from the following rows $\mathcal{R}' = \{R_{(n-1)}, R_{(n-2)}, R_{(n-7)}, R_{(n-8)}, \ldots, R_5, R_4\} \cup \{v_1\}$. Thus,

$$\gamma_h(GP(n,t)) \leq 2\left\lfloor \frac{n}{6} \right\rfloor t + 1.$$

One can observe that

$$\gamma_h(GP(n,t)) \geq 2\left\lfloor \frac{n}{6} \right\rfloor t+1.$$

Therefore,

$$\gamma_h(GP(n,t)) = 2\left\lfloor \frac{n}{6} \right\rfloor t+1.$$

Similarly, the proof follows for other cases. □

The generalized thorn path can be defined as follows,

Definition 4. *The graph obtained by taking a path P_n and attaching t copies of P_r to every vertex of P_n is said to be a generalized thorn path and denoted by $G(n,r,t), n>1$.*

Theorem 5. $\gamma_h[G(n,r,t)] = \begin{cases} \gamma_h(P_n)+nt\left\lfloor \frac{r}{3} \right\rfloor, & \textit{if} \quad r \equiv 0 \pmod 6; \\ n+nt\left\lfloor \frac{r}{3} \right\rfloor, & \textit{if} \quad r \equiv 1,2,3 \pmod 6; \\ nt\left(\left\lfloor \frac{r}{3} \right\rfloor+1\right), & \textit{if} \quad r \equiv 4,5 \pmod 6. \end{cases}$

P r o o f. Let $S = \{v'_{ij} : 1 \leq i \leq r,\ 1 \leq j \leq t\}$ denote the vertices of the i^{th} copy of P_r as shown in Fig. 4.

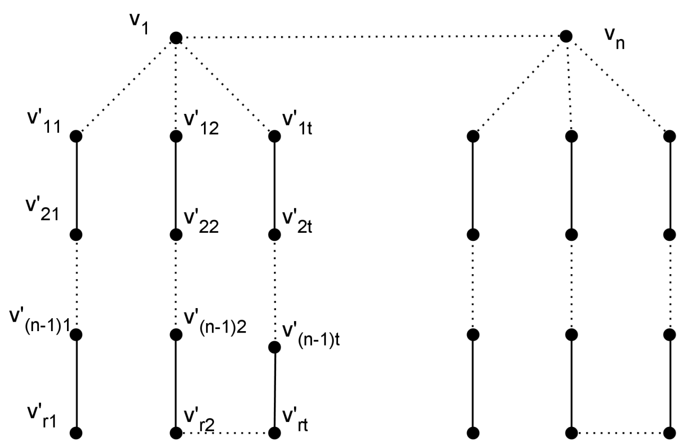

Figure 4. Generalized thorn path $G(n,r,t)$.

Case 1: $r \equiv 0 \pmod 6$. In this case, the set

$$H' = \{v'_{(n-2)j}, v'_{(n-1)j}, v'_{(n-8)j}, v'_{(n-9)j} \dots v'_{4j}, v'_{3j},\ 1 \leq j \leq t\}$$

hop dominates all vertices in each copy of P_r. In order to hop dominate the vertices of the path P_n, any γ_h-set of $G(n,r,t)$ should contains $\gamma_h(P_n)$ vertices. As a result,

$$\gamma_h[G(n,r,t)] \leq \gamma_h(P_n)+|H'| = \gamma_h(P_n)+nt\left\lfloor \frac{r}{3} \right\rfloor.$$

It is easily seen that

$$\gamma_h[G(n,r,t)] \geq \gamma_h(P_n)+|H'| = \gamma_h(P_n)+nt\left\lfloor \frac{r}{3} \right\rfloor.$$

Therefore,

$$\gamma_h[G(n,r,t)] = \gamma_h(P_n) + |H'| = \gamma_h(P_n) + nt\left\lfloor \frac{r}{3} \right\rfloor.$$

Case 2: $r \equiv 1,\ 2,\ 3\ (\mathrm{mod}\ 6)$. Here, any γ_h- set contains the set

$$H' = \{v'_{(n-2)j}, v'_{(n-3)j}, v'_{(n-8)j}, v'_{(n-9)j} \dots v'_{2j}, v'_{1j},\ 1 \le j \le t\}$$

and so $H' \cup V(P_n)$ forms a γ_h-set of cardinality $n + |H'|$.

Case 3: $r \equiv 4,\ 5\ (\mathrm{mod}\ 6)$. In this case, vertices in attached t copies are sufficient for a γ_h-set of $G(n,r,t)$. Therefore,

$$\gamma_h[G(n,r,t)] = nt\gamma_h(P_r) \le nt\Big(\left\lfloor \frac{r}{3} \right\rfloor + 1\Big).$$

Also, any minimal HDS of $G(n,r,t)$ requires at least

$$\gamma_h[G(n,r,t)] = nt\gamma_h(P_r) \ge nt\Big(\left\lfloor \frac{r}{3} \right\rfloor + 1\Big)$$

vertices. Hence,

$$\gamma_h[G(n,r,t)] = nt\gamma_h(P_r) = nt\Big(\left\lfloor \frac{r}{3} \right\rfloor + 1\Big).$$

□

Definition 5 [19]. *A generalized theta graph $\theta[nP(m)]$ is a graph obtained from n-internally disjoint paths, in which each path $P(m)$ contains m internal vertices and these paths share common end vertices u and v (see Fig. 5).*

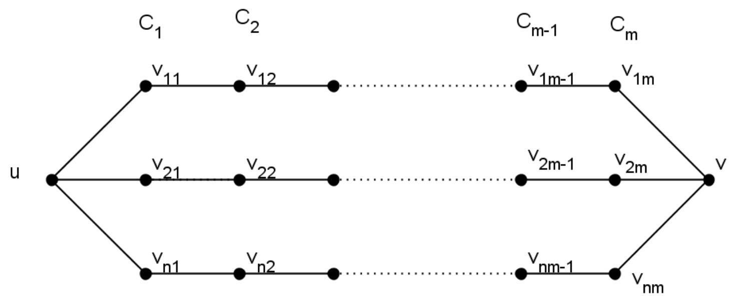

Figure 5. Generalized theta graph $\theta[nP(m)]$.

Theorem 6. $\gamma_h(\theta[nP(m)]) = \begin{cases} 4 + \left\lfloor \dfrac{m-6}{3} \right\rfloor, & \textit{if} \quad m \equiv 0\,(\mathrm{mod}\,6); \\ 4 + n\Big[\left\lfloor \dfrac{m-6}{3} \right\rfloor + 1\Big], & \textit{if} \quad m \equiv 1,3,4,5\,(\mathrm{mod}\,6); \\ 4 + n\Big[\left\lfloor \dfrac{m-6}{3} \right\rfloor + 2\Big], & \textit{if} \quad m \equiv 2\,(\mathrm{mod}\,6). \end{cases}$

P r o o f. Let us denote $\theta[nP(m)]$ by G for convenience. Clearly, $\{u,v\}$ should be included in any γ_h-set and any one vertex from column C_1 and C_m is enough to hop dominate the vertices in C_1 and C_m. The induced subgraph $\langle G - \{u,v\} \cup C_1 \cup C_2 \cup C_{m-1} \cup C_m \rangle$ is a collection of n-distinct paths P_{m-4}. As a consequence of Lemma 1, the result follows. □

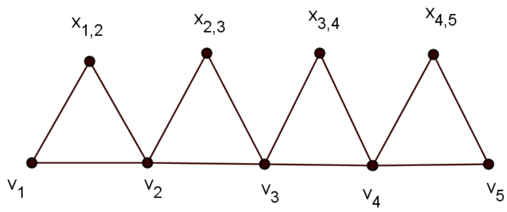

Figure 6. Triangular snake T_4.

Definition 6 [5]. *A triangular snake graph T_n is a graph obtained from the path P_n by replacing each edge by a cycle of length* 3*. For example, a triangular snake T_4 is shown in Fig. 6*

A double triangular snake DT_n consists of two triangular snakes that have a common path.

That is, a double triangular snake is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex x_i for $i = 1, 2, \ldots, n-1$ and to a new vertex y_i for $i = 1, 2, \ldots, n-1$. For example, a double triangular snake DT_6 is illustrated in Fig. 7.

A triple triangular snake TT_n is a graph in which three triangular snakes have a common path.

Similarly, a four triangular snake FT_n is a graph in which four triangles share a common path.

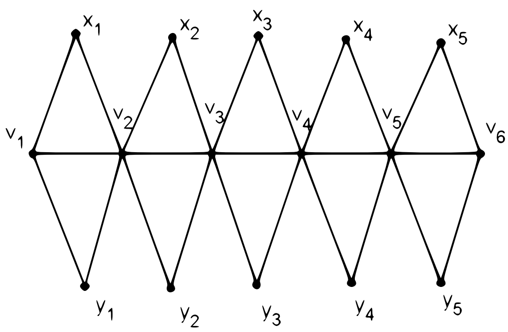

Figure 7. Double triangular snake DT_6.

Remark 1. $\gamma_h(T_n) \geq 3$.

Theorem 7.

$$\gamma_h(T_n) = \gamma_h(DT_n) = \gamma_h(TT_n) = \gamma_h(FT_n) = \begin{cases} 2 + \left\lfloor \dfrac{n-3}{2} \right\rfloor, & \text{if} \quad n \text{ is odd } (n \neq 3), \\ 2 + \left\lfloor \dfrac{n-2}{2} \right\rfloor, & \text{if} \quad n \text{ is even } (n \neq 6). \end{cases}$$

P r o o f. First, we observe that any γ_h-set of T_n will also be a γ_h-set for DT_n, TT_n and FT_n because they share a common path.

For $n = 3$ and 6, $\gamma_h(T_3) = \gamma_h(T_6) = 3$.

Let us label the vertices of the common path as $\{v_1, v_2 \ldots v_n\}$ and the remaining vertices of T_n be $S = \{x_{i,i+1}\}$, $1 \leq i \leq (n-1)$ as shown in Fig. 8.

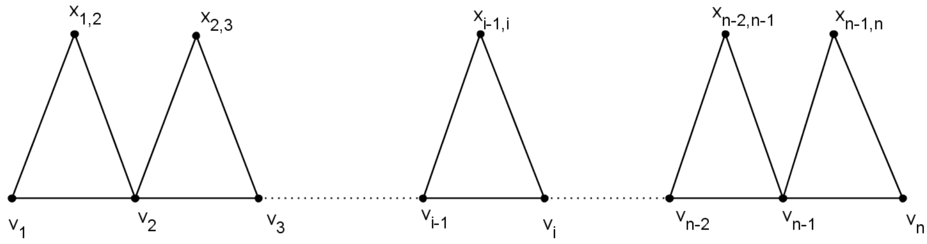

Figure 8. Triangular snake T_n.

Case 1: n is odd and $n \geq 5$. While finding any γ_h-set of T_n, the vertices v_2 and v_{n-2} are taken and the remaining vertices from the subset $S_k \subseteq S$ where

$$S_k = \{x_{i,i+1} | \, i \equiv 0 \pmod 2, \;\; 2 \leq i \leq (n-3)\}.$$

Clearly, $|S_k| = (n-3)/2$. Thus,

$$\gamma_h(T_n) \leq 2 + \left\lfloor \frac{n-3}{2} \right\rfloor.$$

It is easily seen that

$$\gamma_h(T_n) \geq 2 + \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Therefore,

$$\gamma_h(T_n) = 2 + \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Case 2: n is even and $n \neq 6$. Note that any common vertex say $x_{i,i+1}$ hop dominates the vertices v_{i-2} and v_i of P_n, $x_{i-2,i-1}$, $x_{i+1,i+2}$. Equivalently, the hop degree of any vertex is at most 4. Hence by choosing vertices from the set

$$S'_k = \{x_{i,i+1} | \, i \equiv 0 \pmod 2, \;\; 2 \leq i \leq (n-2)\} \subseteq S,$$

any γ_h-set can be obtained which includes the non-hop dominated vertices v_2 and v_{n-1} too. Thus,

$$\gamma_h(T_n) \leq 2 + \left\lfloor \frac{n-2}{2} \right\rfloor.$$

It is observed that

$$\gamma_h(T_n) \geq 2 + \left\lfloor \frac{n-2}{2} \right\rfloor.$$

Therefore,

$$\gamma_h(T_n) = 2 + \left\lfloor \frac{n-2}{2} \right\rfloor.$$

□

Definition 7 [5]. *An alternate triangular snake AT_n is a graph obtained from the path P_n, in which every alternate edge of a path is replaced by a cycle C_3. For example, an alternate double triangular snake is shown in Fig. 9.*

An alternate double triangular snake $AD(T_n)$ is obtained from two alternate triangular snakes that share a common path. For example, an alternate double triangular snake is illustrated in Fig. 10.

An alternate triple (four) triangular snakes $AT(T_n)(AF(T_n))$ consists of three (four) alternate triangular snakes that share a common path.

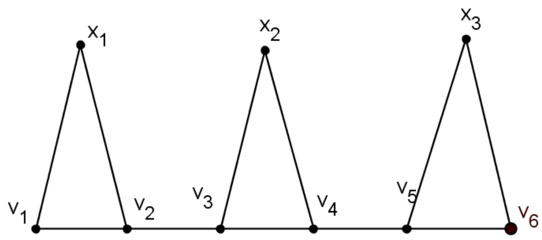

Figure 9. Alternate Triangular snake AT_6.

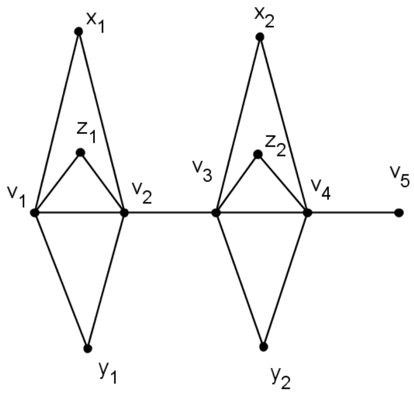

Figure 10. Alternate Triangular snake ADT_5.

Theorem 8.

$$\gamma_h(AT_n) = \gamma_h(AD(T_n)) = \gamma_h(AT(T_n)) = \gamma_h(AF(T_n)) = \begin{cases} \dfrac{n}{2}, & \textit{if} \quad n \equiv 0, 2 \pmod 4; \\ \dfrac{n+1}{2}, & \textit{if} \quad n \equiv 3 \pmod 4; \\ \dfrac{n-1}{2}, & \textit{if} \quad n \equiv 1 \pmod 4. \end{cases}$$

P r o o f. Let us follow the labeling of vertices as described in Theorem 7. Here, $d(v_i) = 3$, $i \neq 1, n$ and any vertex of path P_n hop dominates at most 3 vertices. In any γ_h-set, it is clear that central vertices of P_n alone appear consecutively (see Fig. 11–12).

Case 1: n is even.

Case 1.1: $n \equiv 0 \pmod 4$. Here, any γ_h-set is of the form

$$S = \big\{v_i|\ i \equiv 2 \pmod 4,\ \ 2 \leq i \leq (n-2)\} \cup \{v_j|\ j \equiv 3 \pmod 4,\ \ 3 \leq j \leq (n-1)\big\}.$$

Thus, $\gamma_h(AT_n) \leq n/2$. It is easily seen that, $\gamma_h(AT_n) \geq n/2$. Therefore, $\gamma_h(AT_n) = n/2$.

Case 1.2: $n \equiv 2 \pmod 4$ In this case, v_{n-2} must be chosen in any γ_h-set and the remaining vertices are chosen from $\{v_2, v_3, v_6, v_7, \ldots v_{n-4}, v_{n-3}\}$. Thus, $\gamma_h(AT_n) \leq (n-2)/2 + 1 = n/2$. It is easily seen that $\gamma_h(AT_n) \geq n/2$. Therefore, $\gamma_h(AT_n) = n/2$.

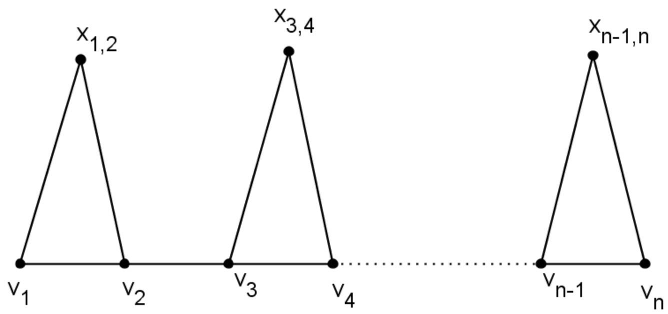

Figure 11. AT_n, when n is even.

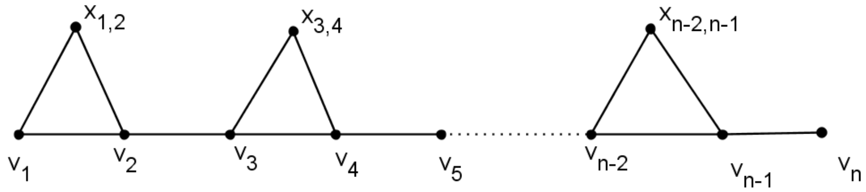

Figure 12. AT_n, when n is odd.

Case 2: n is odd.

Case 2.1: $n \equiv 1 \pmod 4$. In this case, any γ_h-set are chosen from $\{v_2, v_3, v_6, v_7, \ldots v_{n-3}, v_{n-2}\}$, with cardinality $(n-1)/2$. Thus, $\gamma_h(AT_n) \leq (n-1)/2$ and it is easy to verify that $\gamma_h(AT_n) \geq (n-1)/2$. Therefore, $\gamma_h(AT_n) = (n-1)/2$.

Similarly, the case for $n \equiv 3 \pmod 4$ follows. □

Definition 8. *A Quadrilateral snake Q_n is a graph obtained by replacing each edge of a path P_n by a cycle of length 4.*

An alternate quadrilateral snake AQ_n is obtained from the path P_n by replacing its alternate edges with C_4.

Proposition 1. (i) $\gamma_h(Q_n) = \begin{cases} \dfrac{n+2}{2}, & \text{if} \quad n \equiv 0,\, 2 \pmod 4; \\ \dfrac{n+3}{2}, & \text{if} \quad n \equiv 1 \pmod 4; \\ \dfrac{n+1}{2}, & \text{if} \quad n \equiv 3 \pmod 4. \end{cases}$

(ii) $\gamma_h(AQ_n) = \begin{cases} \dfrac{n+1}{2}, & \text{if} \quad n \equiv 1,\, 3 \pmod 4; \\ \dfrac{n+2}{2}, & \text{if} \quad n \equiv 0,\, 2 \pmod 4. \end{cases}$

Definition 9. *Ciliate is a graph $C(p, s)$ obtained from p disjoint copies of the path P_s by linking one end point of each such copy in the cycle C_p. For example, a Ciliate C(3, 3) is shown in Fig. 13.*

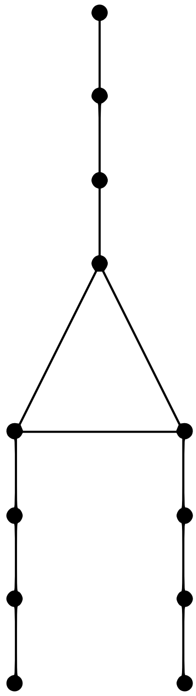

Figure 13. Ciliate $C(3,3)$.

Remark 2. $\gamma_h[C(p,q)] = p\,\gamma_h(P_q)$.

Definition 10. *A generalized ciliate $GC(p,s,t)$ is obtained by attaching t-copies of path P_s to each vertex of the cycle C_p.*

Proposition 2. $$\gamma_h[GC(3,s,t)] = \begin{cases} 2+3t\left\lfloor\dfrac{s}{3}\right\rfloor, & \textit{if} \quad s\equiv 0 \pmod 6; \\ 3+3t\left\lfloor\dfrac{s}{3}\right\rfloor, & \textit{if} \quad s\equiv 1,3 \pmod 6; \\ 3t\left\lfloor\dfrac{s}{3}\right\rfloor+1, & \textit{if} \quad s\equiv 4,5 \pmod 6; \\ 3+3t\left\lfloor\dfrac{s}{3}\right\rfloor, & \textit{if} \quad s\equiv 2 \pmod 6. \end{cases}$$

Theorem 9. $$\gamma_h[GC(4,s,t)] = \begin{cases} 2+4t\left\lfloor\dfrac{s}{3}\right\rfloor, & \textit{if} \quad s\equiv 0,1 \pmod 6; \\ 4+4t\left\lfloor\dfrac{s}{3}\right\rfloor, & \textit{if} \quad s\equiv 2 \pmod 6; \\ 4t\left\lfloor\dfrac{s}{3}\right\rfloor, & \textit{if} \quad s\equiv 3,4,5 \pmod 6. \end{cases}$$

P r o o f. Let us denote the vertices in the i^{th} copy of the path P_s as $\{v_1^i, v_2^i \dots v_s^i : 1 \le i \le t\}$. as shown in Fig. 14. Clearly, to hop dominate the leaves and its support vertices in every i^{th} copy of P_s, the vertices v_{s-2}^i and v_{s-3}^i $(1 \le i \le t)$ have to be chosen for any γ_h-set of $GC(4,s,t)$.

Case 1: $s \equiv 0, 1 \pmod 6$. To hop dominate v_1's and the vertices of the cycle, any γ_h-set includes u_2, u_3. The remaining vertices in each copy of P_s in GC(4,s,t) will induce a path, thus it is sufficient to add to $\{v_{s-2}^i, v_{s-3}^i, v_{s-8}^i, v_{s-9}^i \dots v_5^i, v_4^i\}$, $1 \le i \le s$ to γ_h-set of $GC(4,s,t)$. Thus,

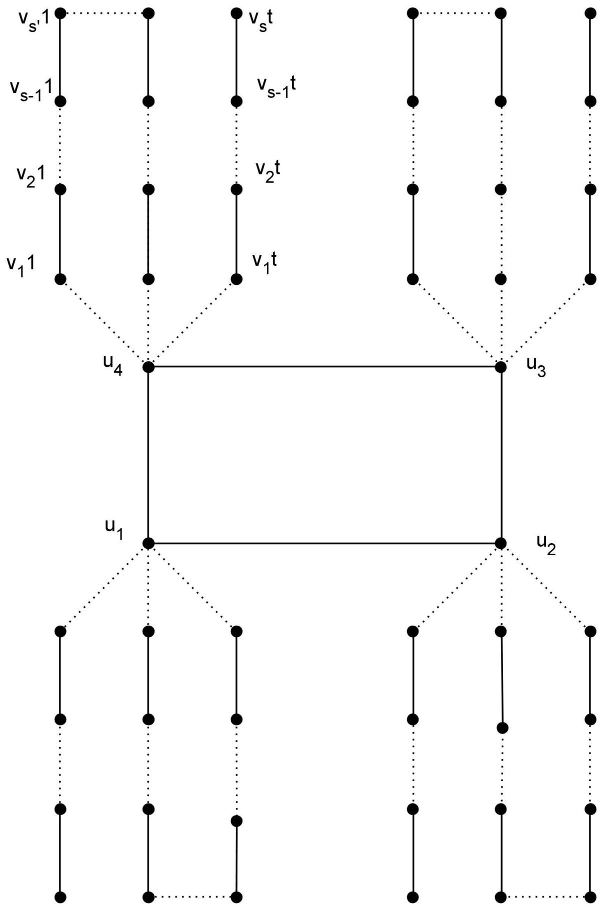

Figure 14. Generalized ciliate $GC(4, s, t)$.

$\gamma_h[GC(4, s, t)] \leq 2 + 4t\gamma_h(P_s) \leq 4t\lfloor s/3 \rfloor$ and it is easily seen that $\gamma_h[GC(4, s, t)] \geq 2 + 4t\gamma_h(P_s)$. Therefore,

$$\gamma_h[GC(4, s, t)] = 2 + 4t\gamma_h(P_s).$$

Case 2: $s \equiv 2 \pmod 6$. Any γ_h-set comprises u_1, u_2, u_3, u_4 to hop dominate v_1^i and v_2^i as well as the vertices of the cycle. Each copy's remaining vertices will induce a path on $(s-2)$ vertices. As a result, $\{v_{s-2}^i, v_{s-3}^i, v_{s-8}^i, v_{s-9}^i \ldots v_6^i, v_5^i\}$ are required to form a γ_h-set of $GC(4, s, t)$. Thus, $\gamma_h[GC(4, s, t)] \leq 4 + 4t\lfloor s/3 \rfloor$ and it is easy to show that $\gamma_h[GC(4, s, t)] \geq 4 + 4t\lfloor s/3 \rfloor$. Therefore,

$$\gamma_h[GC(4, s, t)] = 4 + 4t\lfloor s/3 \rfloor.$$

Case 3: $s \equiv 3,\ 4,\ 5 \pmod 6$. When $s \equiv 3, 5 \pmod 6$, $H = \{v_{s-2}^i, v_{s-3}^i, v_{s-8}^i, v_{s-9}^i \cdots v_3^i, v_2^i\}$ forms a γ_h-set of $GC(4, s, t)$, whereas for $s \equiv 4 \pmod 6$, $H = \{v_{s-2}^i, v_{s-3}^i, v_{s-8}^i, v_{s-9}^i \cdots v_2^i, v_1^i\}$ forms a γ_h-set. Thus, $\gamma_h[GC(4, s, t)] = 4t\gamma_h(P_s) \leq 4t(\lfloor s/3 \rfloor + 1)$. It is easily seen that $\gamma_h[GC(4, s, t)] \geq 4t(\lfloor s/3 \rfloor + 1)$. Therefore,

$$\gamma_h[GC(4, s, t)] = 4t(\lfloor s/3 \rfloor + 1).$$

$\square$

3. Conclusion

In this study, we computed hop domination number for some special families of graphs like triangular, quadrilateral, alternate triangular, alternate quadrilateral snake graphs and examined hop domination number for some generalized graph structures like generalized theta graph, glued path graph. In future, the result obtained for generalized ciliates $p = 3, 4$ may be extended to $p > 4$.

Acknowledgements

The authors thank the referees for their insightful comments. Also, the authors thank the Department of Science and Technology, Government of India for the financial support to the Department of Mathematics, SASTRA Deemed to be University under FIST Programme — Grant No.: SR/FST/MSI-107/2015(c).

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