

CARLEMAN’S FORMULA OF A SOLUTIONS OF THE POISSON EQUATION IN BOUNDED DOMAIN

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Abstract: We suggest an explicit continuation formula for a solution to the Cauchy problem for the Poisson equation in a domain from its values and values of its normal derivative on a part of the boundary. We construct the continuation formula of this problem based on the Carleman–Yarmuhamedov function method.

Keywords: Poisson equations, Ill-posed problem, Regular solution, Carleman–Yarmuhamedov function, Green’s formula, Carleman formula, Mittag–Leffler entire function.

1. Introduction

In this paper, we continue the research provided in [12]. We propose an explicit formula for the reconstruction of a solution of the Poisson equation in a bounded domain from its values and the values of its normal derivative on a part of the boundary, i.e., we give an explicit continuation formula for a solution to the Cauchy problem for the Poisson equation.

Let us introduce the following notation: \mathbb{R}^3 is a three-dimensional real Euclidean space,

$$\begin{aligned} x &= (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3) \in \mathbb{R}^3, \\ x' &= (x_1, x_2), \quad y' = (y_1, y_2) \in \mathbb{R}^2, \\ s &= \alpha^2 = |y' - x'|^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2, \\ r^2 &= s + (y_3 - x_3)^2 = |y - x|^2, \quad \tau = tg \frac{\pi}{2\rho}, \quad \rho > 1, \end{aligned}$$

$$G_\rho = \{y : |y'| < \tau y_3, y_3 > 0\}, \quad \partial G_\rho = \{y : |y'| = \tau y_3, y_3 > 0\}, \quad \overline{G}_\rho = G_\rho \cup \partial G_\rho,$$

ε , ε_1 , and ε_2 are sufficiently small positive constants,

$$G_\rho^\varepsilon = \{y : |y'| < \tau(y_3 - \varepsilon)\}, \quad \partial G_\rho^\varepsilon = \{y : |y'| = \tau(y_3 - \varepsilon)\}, \quad \overline{G}_\rho^\varepsilon = G_\rho^\varepsilon \cup \partial G_\rho^\varepsilon,$$

and Ω_ρ is a bounded simply connected domain whose boundary $\partial\Omega_\rho$ in R^3 consists of a part of the conic surface $T \equiv \partial G_\rho$ and a smooth surface S lying inside the cone \overline{G}_ρ . The case $\rho = 1$ is the limit case. In this case, G_1 is the half-space $y_3 > 0$, ∂G_1 is the hyperplane $y_3 = 0$, and Ω_1 is a bounded simply connected domain whose boundary consists of a compact connected part of the hyperplane $y_3 = 0$ and a smooth surface S in the half-space $y_3 \geq 0$, $\overline{\Omega}_\rho = \Omega_\rho \cup \partial\Omega_\rho$, and S_0 is the interior of S .

The Poisson equation or potential equation [15]

$$-\Delta U(x) \equiv -\sum_{i=1}^3 \frac{\partial^2 U}{\partial x_i^2} = f(x) \tag{1.1}$$

is a classical example of second-order elliptic partial differential equations and a mathematical model for some important physical phenomena. Let $H^\lambda(\Omega_\rho)$ be the set of real functions of the class

$C^{2,\lambda}(\Omega_\rho) \cap C^1(\overline{\Omega}_\rho)$ satisfying the Poisson equation. Let a function f be Hölder continuous with exponent $\lambda \in (0, 1)$, i.e., $f \in C^{s,\lambda}(\overline{\Omega}_\rho)$ and $s \in \mathbb{Z}_+$.

Problem 1. Assume that we know the Cauchy data for a solution to equation (1.1) on the surface S :

$$U(y) = f_1(y), \quad \frac{\partial U(y)}{\partial n} = f_2(y), \quad y \in S \quad (1.2)$$

where $n = (n_1, n_2, n_3)$ is the outward unit normal to the surface $\partial\Omega_\rho$ at a point y , and f_1 and f_2 are continuous functions. Given $f_1(y)$ and $f_2(y)$ on S , find $U(x)$, $x \in \Omega_\rho$.

Problem 2. Let f_1 and f_2 be given on S . Find conditions on f_1 and f_2 that are necessary and sufficient for the existence of a solution to system (1.1) satisfying (1.2) and from the class $H(\Omega_\rho)$.

It is well-known that the Cauchy problem (1.2) for the Poisson equation (1.1) is ill-posed [3, 5]. Hadamard [17] noted that a solution to Problem 1 is not stable. The possibility of introducing a positive parameter σ , depending on the accuracy of the initial data, was noticed by M.M. Lavrentev [23]. The uniqueness of the solution follows from the general theorem by Holmgren [6]. It has applications in many different areas such as plasma physic, electrocardiography, and corrosion non-destructive evaluation (e.g., [7, 9, 10, 13, 19]). Traditionally, regularization techniques, such as Tikhonov regularization [44] and the quasi-reversibility approach [22], were used to provide robust numerical schemes [18].

We suppose that a solution to the problem exists (in this event, it is unique) and is continuously differentiable in the closed domain, and the Cauchy data are given exactly. In this case, we establish an explicit continuation formula. This formula enables us to state a simple and convenient criterion for the solvability of the Cauchy problem.

The result established here is a multidimensional analog of theorems and Carleman-type formulas [4] by G.M. Goluzin, V.I. Krylov, V.A. Fok, and F.M. Kuni in the theory of holomorphic functions of one variable [14, 16].

The method for obtaining these results is based on an explicit form of the fundamental solution of the Poisson equation which depends on a positive parameter that vanishes together with its derivatives on a fixed cone and outside it, as the parameter tends to infinity, while the pole of the fundamental solution lies inside the cone. Following to M.M. Lavrent'ev, a fundamental solution with these properties is called a *Carleman function* for the cone [8, 23]. Having constructed a Carleman function explicitly, we write a continuation formula. The existence of a Carleman function follows from S.N. Mergelyan's approximation theorem [28]. However, this theorem shows no way for writing the Carleman function explicitly.

The Carleman function of the Cauchy problem for the Laplace equation and some close problems, in the case when $\partial\Omega_\rho \setminus S$ is a part of a conic surface, was constructed in [45]. Mergelyan [28] suggested a method to construct the Carleman function of the Cauchy problem for the Laplace equation in the case when S is a part, with a smooth boundary, of the boundary of a simply-connected domain. Based on [28] and approximative theorems, the Carleman matrix for elliptic systems was constructed in [41].

In [1], some theorems of existence of the Carleman matrix and a solvability criterion for a wider class of boundary value problems for elliptic systems were established. It was proved earlier in [1, 41] that, for every Cauchy problem for elliptic systems, the Carleman matrix exists if the Cauchy data are given on a boundary set of positive measure.

Following Tikhonov [21, 43], we call the family of functions $U_{\sigma\delta}(x)$ the regularized solution to the Cauchy problem for equation (1.1). The regularized solution determines the stability of the approximate method.

In the paper, based on results from [23, 45–48] on the Cauchy problem for the Laplace and Helmholtz equations, we construct the Carleman–Yarmukhamedov function in an explicit form. We use it to prove the Carleman formulas and a criterion for the solvability of the Cauchy problem.

In recent decades, interest in the classical ill-posed problems of mathematical physics has been preserved. This direction of investigation of the properties of solutions to the Cauchy problem for the Laplace equation was started in [2, 20, 23, 24, 42] and was further developed in [25–27, 30–40].

2. Construction of a Carleman–Yarmukhamedov function

According to [45], we define the Carleman–Yarmukhamedov function $\Phi(y, x)$ by the equality

$$-2\pi^2 K(0)\Phi(y, x) = \int_0^\infty \operatorname{Im} \left[\frac{K(w)}{w} \right] \frac{du}{\sqrt{s+u^2}}, \quad w = i\sqrt{s+u^2} + y_3 - x_3. \quad (2.1)$$

Here, $K(w)$ is an entire function of complex variable that takes real values for real w ($w = a+ib$, a and b are real numbers) such that $K(a) \neq \infty$, $|a| < \infty$, $K(0) \neq 0$, $\forall R > 0$, $\exists C_R > 0$

$$\sup_{|\operatorname{Re} w| < R, \operatorname{Im} w \leq -C_R} (|K(w)| + |\operatorname{Im} w| |K'(w)| + |\operatorname{Im} w|^2 |K''(w)|) < \infty.$$

For real w , since $K(w)$ is real, we have $\overline{K(\bar{w})} = K(w)$. Then (2.1) implies that $\forall R > 0$

$$\sup_{|\operatorname{Re} w| < R} \{|K(w)| + (1 + |\operatorname{Im} w|) |K'(w)| + (1 + |\operatorname{Im} w|^2) |K''(w)|\} < \infty. \quad (2.2)$$

Now we write (2.1) in the form

$$-2\pi^2 K(0)\Phi(y, x) = \int_0^\infty \left\{ \frac{(y_3 - x_3) \operatorname{Im} K(w)}{\sqrt{s+u^2}} - \operatorname{Re} K(w) \right\} \frac{du}{r^2 + u^2}, \quad (2.3)$$

where

$$\begin{aligned} \operatorname{Im} \left(\frac{K(w)}{w} \right) &= \frac{1}{2i} \left\{ \frac{K(w)}{w} - \frac{K(\bar{w})}{\bar{w}} \right\} = \frac{\bar{w}K(w) - wK(\bar{w})}{2i(r^2 + u^2)} \\ &= \frac{(y_3 - x_3) \operatorname{Im} K(w) - \sqrt{s+u^2} \operatorname{Re} K(w)}{r^2 + u^2}. \end{aligned} \quad (2.4)$$

From (2.2) and (2.3), it follows that, for $y \neq x$, the integral in (2.1) converges absolutely.

If $K(w) \equiv 1$, then the function $\Phi(y, x)$ is the classical fundamental solution to the Laplace equation, i.e.,

$$\Phi(y, x) \equiv \Phi_0(r) = 1/(4\pi r).$$

Theorem 1 [45]. *The function $\Phi(y, x)$ defined by (2.1) or (2.3)–(2.4) is representable in the form*

$$\Phi(y, x) = \Phi_0(r) + G(y, x), \quad (2.6)$$

where $\Phi_0(r) = 1/(4\pi r)$ and the function $G(y, x)$ is harmonic in the variable y in \mathbb{R}^3 , including $y = x$.

From Theorem 1 it follows that the function $\Phi(y, x)$ of the variable y is a fundamental solution of the Poisson equation. Therefore, for the function $U(y) \in H(\Omega_\rho)$ and for every point $x \in \Omega_\rho$, the Green's formula is valid [15]:

$$U(x) = \int_{\Omega_\rho} \Phi(y, x) f(y) dy - \int_{\partial\Omega_\rho} \left[U(y) \frac{\partial\Phi(y, x)}{\partial n} - \Phi(y, x) \frac{\partial U(y)}{\partial n} \right] dS_y, \quad (2.5)$$

where $f(x) \in C^\lambda(\Omega_\rho)$, $\lambda \in (0, 1)$, is bounded, i.e., the former integral on the right-hand side of (2.5) satisfies equation (1.1) in the domain.

3. The Mittag-Leffler entire function

The continuation formulas below are expressed explicitly in terms of the Mittag-Leffler entire function; therefore, we now present its basic properties without proof. These properties as well as detailed proofs can be found in [11, Chapter 3, §2], [47].

The Mittag-Leffler entire function is defined by the series

$$E_\rho(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(1 + n/\rho)}, \quad \rho > 0, \quad w \in C, \quad E_1(w) = e^w,$$

where Γ is the Euler gamma-function. Hereinafter, we suppose that $\rho > 1$. Let

$$\gamma = \gamma(1, \beta), \quad 0 < \beta < \frac{\pi}{\rho}, \quad \rho > 1,$$

be the contour in the complex w -plane that consists of the ray $\arg w = -\beta$, $|w| \geq 1$, the arc $-\beta \leq \arg w \leq \beta$ of the circle $|w| = 1$, and the ray $\arg w = \beta$, $|w| \geq 1$, which is passed so that $\arg w$ does not decrease. The contour γ splits the complex domain C into the two simply connected infinite domains Ω^- and Ω^+ lying to the left and to the right of γ , respectively. We suppose that

$$\frac{\pi}{2\rho} < \beta < \frac{\pi}{\rho}, \quad \rho > 1.$$

Under these conditions, the following integral representations are valid:

$$\begin{aligned} E_\rho(w) &= \rho e^{w^\rho} + \psi_\rho(w), \quad w \in \Omega^+, \\ E_\rho(w) &= \psi_\rho(w), \quad E'_\rho(w) = \psi'_\rho(w), \quad w \in \Omega^-, \end{aligned}$$

where

$$\psi_\rho(w) = \frac{\rho}{2\pi i} \int_{\gamma} \frac{e^{\zeta^\rho}}{\zeta - w} d\zeta, \quad \psi'_\rho(w) = \frac{\rho}{2\pi i} \int_{\gamma} \frac{e^{\zeta^\rho}}{(\zeta - w)^2} d\zeta. \quad (3.1)$$

Since $E_\rho(w)$ takes real vales for real w , we obtain

$$\begin{aligned} \operatorname{Re} \psi_\rho(w) &= \frac{\psi_\rho(w) + \psi_\rho(\bar{w})}{2} = \frac{\rho}{2\pi i} \int_{\gamma} \frac{e^{\zeta^\rho} (\zeta - \operatorname{Re} w)}{(\zeta - w)(\zeta - \bar{w})} d\zeta, \\ \operatorname{Im} \psi_\rho(w) &= \frac{\psi_\rho(w) - \psi_\rho(\bar{w})}{2i} = \frac{\rho \operatorname{Im} w}{2\pi i} \int_{\gamma} \frac{e^{\zeta^\rho}}{(\zeta - w)(\zeta - \bar{w})} d\zeta, \\ \operatorname{Im} \frac{\psi'_\rho(w)}{\operatorname{Im} w} &= \frac{\rho}{2\pi i} \int_{\gamma} \frac{2e^{\zeta^\rho} (\zeta - \operatorname{Re} w)}{(\zeta - w)^2 (\zeta - \bar{w})^2} d\zeta. \end{aligned}$$

Hereinafter, we take

$$\beta = \frac{\pi}{2\rho} + \frac{\varepsilon_2}{2}, \quad \rho > 1,$$

in the definition of the contour $\gamma(1, \beta)$. It is clear that, if

$$\frac{\pi}{2\rho} + \varepsilon_2 \leq |\arg w| \leq \pi, \quad (3.2)$$

then $w \in \Omega_\rho^-$ and $E_\rho(w) = \psi_\rho(w)$.

Define

$$T_{k,p}(w) = \frac{\rho}{2\pi i} \int_\gamma \frac{\zeta^p e^{\zeta^p}}{(\zeta - w)^k (\zeta - \bar{w})^k} d\zeta, \quad k = 1, 2, \dots, \quad p = 0, 1, \dots$$

The following inequalities are valid for $\pi/(2\rho) + \varepsilon_2 \leq |\arg w| \leq \pi$:

$$|E_\rho(w)| \leq \frac{C_1}{1 + |w|}, \quad |E'_\rho(w)| \leq \frac{C_2}{1 + |w|^2}, \quad (3.3)$$

$$|T_{k,p}(w)| \leq \frac{C_3}{1 + |w|^{2k}}, \quad k = 1, 2, \dots, \quad (3.4)$$

where C_1, C_2 , and C_3 are constants independent of w . Take in (2.1)

$$\beta = \frac{\pi}{2\rho} + \frac{\varepsilon_2}{2} < \frac{\pi}{\rho}, \quad \rho > 1.$$

Then $E_\rho(w) = \psi_\rho(w)$, where $\psi_\rho(w)$ is defined by (3.1). Moreover, note that $\cos \rho\beta < 0$ and the integral converges:

$$\int_\gamma |\zeta|^p e^{\cos \rho\beta |\zeta|^\rho} |d\zeta| < \infty, \quad p = 0, 1, \dots \quad (3.5)$$

4. Carleman formulas

Let the Mittag-Leffler entire function be the function $K(w)$ in (2.1):

$$K(w) = e^{aw^2} E_\rho(\sigma w),$$

where

$$\rho > 1, \quad w = i\sqrt{s + u^2} + y_3 - x_3, \quad K(0) = E_\rho(0) = 1, \quad a > 0 \quad \sigma \geq 0.$$

Denote by $\Phi_\sigma(y, x)$ the corresponding fundamental solution and by $\Phi_\sigma(y - x)$ its derivative with respect to the variable σ :

$$\Psi_\sigma(y - x) \equiv \frac{d\Phi_\sigma}{d\sigma}(y - x).$$

It follows from Theorem 1 that $\Psi_\sigma(y - x)$ satisfies the Poisson equation in \mathbb{R}^3 . Then

$$\begin{aligned} -2\pi^2 \Phi_\sigma(y - x) &= \int_0^\infty \operatorname{Im} \left[\frac{e^{aw^2} E_\rho(\sigma w)}{w} \right] \frac{du}{\sqrt{s + u^2}} \\ &= e^{a(y_3 - x_3)^2} \int_0^\infty \varphi_\sigma(y, x, u) \frac{e^{-as - au^2}}{u^2 + r^2} du, \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} \varphi_\sigma(y-x, u) &= \left[\frac{(y_3 - x_3)}{\sqrt{u^2 + s}} \operatorname{Im} E_\rho(\sigma w) - \operatorname{Re} E_\rho(\sigma w) \right] \cos(\nu \sqrt{s + u^2}) \\ &+ \left[\operatorname{Im} E_\rho(\sigma w) + \frac{(y_3 - x_3)}{\sqrt{s + u^2}} \operatorname{Re} E_\rho(\sigma w) \right] \sin(\nu \sqrt{s + u^2}), \quad \nu = 2a(y_3 - x_3), \\ \Psi_\sigma(y-x) &\equiv \frac{d\Phi_\sigma}{d\sigma}(y-x) = \int_0^\infty \operatorname{Im} \left[e^{aw^2} E'_\rho(\sigma w) \right] \frac{du}{\sqrt{s + u^2}}. \end{aligned} \quad (4.2)$$

Lemma 1 [47]. *Let M be a compact set in G_ρ , and let δ be the distance from M to ∂G_ρ . Then, for $\sigma \geq 0$, the following inequalities are valid for $x \in M$ and $y \in R^3 \setminus G_\rho$ ($|y'| \geq \tau y_3$):*

$$\begin{aligned} |\Phi_\sigma(y-x)| + \left| \frac{\partial}{\partial y_k} \Phi_\sigma(y-x) \right| + \left| \frac{\partial^i}{\partial x_j^i} \frac{\partial}{\partial y_k} \Phi_\sigma(y-x) \right| &\leq \frac{C_4(\rho, \delta)r}{1 + \sigma\delta}, \\ r \geq \delta > 0, \quad i, &= 0, 1, \quad k, j = 1, 2, 3. \end{aligned} \quad (4.3)$$

$$\begin{aligned} |\Psi_\sigma(y-x)| + \left| \frac{\partial}{\partial y_k} \Psi_\sigma(y-x) \right| + \left| \frac{\partial^i}{\partial x_j^i} \frac{\partial}{\partial y_k} \Psi_\sigma(y-x) \right| &\leq \frac{C_5(\rho, \delta)r}{1 + \sigma\delta}, \\ r \geq \delta > 0, \quad i &= 0, 1, \quad k, j = 1, 2, 3, \end{aligned} \quad (4.4)$$

where the constants C_4 and C_5 are independent of x, y , and σ .

Theorem 2. *Let f be bounded and locally Hölder continuous in Ω_ρ , $U(y) \in H^\lambda(\Omega_\rho)$, and*

$$U(y) = f_1(y), \quad \frac{\partial U}{\partial n}(y) = f_2(y), \quad y \in S,$$

where $f_1(y)$ and $f_2(y)$ are given functions of the class $C(S)$. Then the Carleman formulas

$$\begin{aligned} \frac{\partial^i U(x)}{\partial x_j^i} &= \lim_{\sigma \rightarrow \infty} \frac{\partial^i U_\sigma(x)}{\partial x_j^i} \\ &= \lim_{\sigma \rightarrow \infty} \left[\int_{\Omega_\rho} f(y) \frac{\partial^i \Phi_\sigma(y-x)}{\partial x_j^i} dy - \int_S \left\{ f_1(y) \frac{\partial^i}{\partial x_j^i} \frac{\partial \Phi_\sigma(y-x)}{\partial n} - f_2(y) \frac{\partial^i \Phi_\sigma(y-x)}{\partial x_j^i} \right\} dS_y \right] \end{aligned} \quad (4.5)$$

are valid for every $x \in \Omega_\rho$, where $i = 0, 1, j = 1, 2, 3$,

$$\frac{\partial^0 U_\sigma}{\partial x_j^0} = U_\sigma, \quad \frac{\partial^0 \Phi_\sigma}{\partial x_j^0} = \Phi_\sigma,$$

and the convergence in (4.5) is uniform on compact sets in Ω_ρ .

P r o o f. From Green's formula (2.5), for every $x \in \Omega_\rho$, we obtain

$$\frac{\partial^i U(x)}{\partial x_j^i} = \int_{\Omega_\rho} f(y) \frac{\partial^i \Phi_\sigma(y-x)}{\partial x_j^i} dy - \int_{\partial\Omega_\rho} \left[f_1(y) \frac{\partial^i}{\partial x_j^i} \frac{\partial \Phi_\sigma(y-x)}{\partial n} - f_2(y) \frac{\partial^i}{\partial x_j^i} \Phi_\sigma(y-x) \right] dS_y, \quad (4.6)$$

$\partial\Omega_\rho = S \cup (\partial\Omega_\rho \setminus S)$. According to [47], let us estimate

$$\Phi_\sigma, \quad \frac{\partial \Phi_\sigma}{\partial y_j}, \quad \frac{\partial^i}{\partial x_j^i} \frac{\partial \Phi_\sigma}{\partial y_j}.$$

Lemma 1 yields the assertion of Theorem 2. Indeed, if M is a compact set in Ω_ρ then $M \subset G_\rho$. Therefore, the inequalities in Lemma 1 for $\Phi_\sigma(y-x)$ and its derivatives remain also valid in the case where $x \in M \subset \Omega_\rho$ and $y \in \partial\Omega_\rho \setminus S \subset \partial G_\rho$ (in this case, δ is the distance from the compact set $M \subset \Omega_\rho$ to $\partial\Omega_\rho$). Now, let σ tend to infinity. The proof of Theorem 2 is complete. \square

We can write (4.5) in the following equivalent form:

$$\begin{aligned} \frac{\partial^i U(x)}{\partial x_j^i} &= \int_0^\infty \frac{\partial^i}{\partial x_j^i} J(\sigma, x) + \int_{\Omega_\rho} f(y) \frac{\partial^i \Phi_0(r)}{\partial x_j^i} dy \\ &- \int_S \left[f_1(y) \frac{\partial^i}{\partial x_j^i} \frac{\partial \Phi_0(r)}{\partial n} - f_2(y) \frac{\partial^i \Phi_0(r)}{\partial x_j^i} \right] dS_y, \quad x \in \Omega_\rho, \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} \frac{\partial^i}{\partial x_j^i} J(\sigma, x) &= \int_{\Omega_\rho} f(y) \frac{\partial^i \Psi_\sigma(y-x)}{\partial x_j^i} dy - \int_S \left[f_1(y) \frac{\partial^i}{\partial x_j^i} \frac{\partial \Psi_\sigma(y-x)}{\partial n} - f_2(y) \frac{\partial^i \Psi_\sigma(y-x)}{\partial x_j^i} \right] dS_y, \\ x \in \Omega_\rho, \quad i &= 0, 1, \quad j = 1, 2, 3, \quad \frac{\partial^0 U}{\partial x_j^0} = U, \quad \frac{\partial^0 \Phi_0}{\partial x_j^0} = \Phi_0, \quad \frac{\partial^0 \Psi_\sigma}{\partial x_j^0} = \Psi_\sigma, \quad \frac{\partial^0 J}{\partial x_j^0} = J. \end{aligned} \quad (4.8)$$

The functions $\Psi_\sigma(y-x)$ and $\Phi_0(r)$ are defined by equalities (4.2) and (4.1), respectively. The proof of (4.7) follows from the formulas

$$\lim_{\sigma \rightarrow \infty} \frac{\partial^i}{\partial x_j^i} P(\sigma, x) = \int_0^\infty \frac{\partial^i}{\partial x_j^i} \frac{\partial P(\sigma, x)}{\partial \sigma} + \frac{\partial^i}{\partial x_j^i} P(x)$$

and

$$\begin{aligned} \frac{\partial^i}{\partial x_j^i} \frac{\partial P(\sigma, x)}{\partial \sigma} &= \int_{\Omega_\rho} f(y) \frac{\partial^i \Psi_\sigma(y-x)}{\partial x_j^i} dy - \int_S \left[f_1(y) \frac{\partial^i}{\partial x_j^i} \frac{\partial \Psi_\sigma(y-x)}{\partial n} - f_2(y) \frac{\partial^i \Psi_\sigma(y-x)}{\partial x_j^i} \right] dS_y, \\ x \in \Omega_\rho, \quad i &= 0, 1, \quad j = 1, 2, 3; \end{aligned}$$

moreover, the differentiation under the integral sign is legal and

$$\frac{\partial^i}{\partial x_j^i} \frac{\partial P(\sigma, x)}{\partial \sigma} = \frac{\partial^i}{\partial x_j^i} J(\sigma, x).$$

Theorem 3. Let $S \subset C^2$, $f_1(y) \in C^1(S_0) \cap L(S)$, $f_2(y) \in C(S_0) \cap L(S)$, and let f be bounded and locally Hölder continuous in Ω_ρ . Then for the existence of a function $U(y) \in H^\lambda(\Omega_\rho) \cap C(S_0)$ such that

$$U(y) = f_1(y), \quad \frac{\partial U}{\partial n}(y) = f_2(y), \quad y \in S_0, \quad (4.9)$$

it is necessary and sufficient that the following improper integral converge (uniformly on compact sets in G_ρ) for each $x \in G_\rho$:

$$\left| \int_1^\infty J(\sigma, x) d\sigma \right| < \infty, \quad (4.10)$$

where $J(\sigma, x)$ is defined by (4.8). If (4.10) is satisfied, then harmonic continuation is performed by equivalent formulas (4.5) and (4.7).

P r o o f. *Necessity:* Let

$$U(y) \in H(\Omega_\rho) \cap C^1(\Omega_\rho \cup S_0) \cap L(S)$$

satisfy (4.10). Let M be a compact set in G_ρ , and let $\varepsilon > 0$ be such that $M \subset \overline{G_\rho^{2\varepsilon}} \subset \overline{G_\rho^\varepsilon} \subset G_\rho$. It is clear that the distance from M to $\partial G_\rho^\varepsilon$ is at least $\varepsilon\tau_1$ and the distance from $\partial G_\rho^{2\varepsilon}$ to $\partial G_\rho^\varepsilon$ is $\varepsilon\tau_1$. Now, let $y \in R^3 \setminus G_\rho^\varepsilon$ ($|y'| \leq \tau(y_3 - \varepsilon)$ and $y_3 > \varepsilon$) and $x \in M$ ($|x'| \leq \tau(x_3 - 2\varepsilon)$ and $x_3 > 2\varepsilon$). Then $\arg w = \arg(\sigma w) = \arg(i\tau\sqrt{u^2 + s} + \tau y_3 - \tau x_3)$ and

$$\begin{aligned} \tau w &= i\tau\sqrt{u^2 + s} + \tau y_3 - \tau x_3 = \sqrt{u^2 + s} \left(i\operatorname{tg} \frac{\pi}{2\rho} + \frac{\tau y_3 - \tau x_3}{\sqrt{u^2 + s}} \right), \quad u \geq 0, \quad \rho > 1, \\ \frac{\tau y_3 - \tau x_3}{\sqrt{u^2 + s}} &\leq \frac{|y'| - |x'| - \varepsilon\tau}{|y' - x'|} \leq 1 - \varepsilon_1, \quad y' \neq x', \quad \left| \arg(a \pm \operatorname{tg} \frac{\pi}{2\rho}) \right| \geq \frac{\pi}{2\rho}; \quad a \leq 1. \end{aligned}$$

Therefore, (2.5) is valid for $\arg w$; moreover, if $y' = x'$, then $\operatorname{Re} w < 0$, and this inequality also holds. Consequently, $\Phi_\sigma(y - x)$ and $\Psi_\sigma(y - x)$ satisfy estimates (3.2)–(3.5) from Lemma 1, where $\delta \geq \varepsilon\tau_1$. Define $S_\varepsilon = \overline{G_\rho^\varepsilon} \cap S$; in this case, the part $S_\varepsilon \subset S$ together with the part T_ε of the cone surface $\partial G_\rho^\varepsilon$ form a closed piecewise smooth surface $S_\varepsilon \cup T_\varepsilon$ (with the consistent direction of the outer normals) which is the boundary of a simply connected bounded domain. Represent the integral on the right-hand side of (4.8) as the sum of two integrals according to the representation $S = S_\varepsilon \cup (S \setminus S_\varepsilon)$. Since $\Psi_\sigma(y - x)$ is a regular solution of the Poisson equation, by Green's formula, the integral over the part S_ε is equal to the integral over T_ε ; moreover, $\Psi_\sigma(y - x)$ satisfies inequalities (4.7) and (4.9) for $y \in T_\varepsilon$ and $x \in M$, and the extended function $U(y)$ together with its gradient is bounded by a constant depending on ε . Therefore, the modulus of the integral over the part S_ε does not exceed the quantity

$$\frac{\operatorname{const}}{1 + \delta^2\sigma^2}, \quad \sigma \geq 0,$$

with a constant depending on $\rho, \varepsilon, \delta$, and the diameter of the domain Ω_ρ . Since $|y| \geq \tau(y_3 - \varepsilon)$, $y_3 \geq \varepsilon$, when $y \in S \setminus S_\varepsilon$ and $x \in K$ and $f_1(y), f_2(y) \in C(S_0) \cap L(S)$, these inequalities remain valid for the modulus of the integral over $S \setminus S_\varepsilon$ (of course, with other constants). Hence, we have (4.10).

Sufficiency: Under the assumptions of the theorem, define functions $U(x)$, $x \in G_\rho \setminus S_0$, by the right-hand side of (4.7). Consider the first term on the right-hand side of (4.7). Since $\Psi_\sigma(y)$ satisfies the Poisson equation in G_ρ for $\sigma \geq 0$, the function $J(\sigma, x)$ satisfies the Poisson equation with respect to x in G_ρ for $\sigma \geq 0$. Therefore, we conclude from (4.10) that the first term on the right-hand side of (4.7) satisfies the Poisson equation in G_ρ as the limit of the uniformly converging sequence of the solutions of the Poisson equations

$$U_n(x) = \int_0^n J(\sigma, x) d\sigma, \quad n = 1, 2, \dots$$

The second and third terms are the potential difference of the volume, single, and double layers and represent one solution of the Poisson equation in Ω_ρ and another in $\Omega'_\rho = G_\rho \setminus \overline{\Omega}_\rho$. Therefore, the right-hand side of (4.7) defines two different solutions of the Poisson equations $U^+(x)$ and $U^-(x)$ in Ω_ρ and Ω'_ρ . If x^1 and x^2 are two points on the normal at $x \in S_0$ symmetric with respect to x , then

$$\lim_{x^1 \rightarrow x} [U^+(x^1) - U^-(x^2)] = f_1(x), \quad \lim_{x^1 \rightarrow x} \left[\frac{\partial U^+}{\partial n}(x^1) - \frac{\partial U^-}{\partial n}(x^2) \right] = f_2(x), \quad x \in S_0;$$

moreover, the limit relations hold uniformly in x on each compact part S_0 . If $\max y_3 < x_3$, where $y \in S$ and $x \in G_\rho$, then $\operatorname{Re} w = y_3 - x_3 < 0$ and $\Phi_\sigma(y-x)$ and its derivatives satisfy inequalities (4.6) and (4.3). Now, from formula (4.5), which is equivalent to (4.7), we see that $U^-(x) = 0$ and $U^-(x) \equiv 0$, $x \in \Omega_\rho$, by the uniqueness theorem. It is clear that $U^-(x)$ extends smoothly to $\Omega'_\rho \cup S_0$. Then $U^+(x)$ extends smoothly as a function of the class $C^1(\Omega_\rho \cup S_0)$ (see [29]). Consequently,

$$U^+(x) = f_1(x), \quad \frac{\partial U^+}{\partial n}(x) = f_2(x), \quad x \in S_0.$$

Now, we set $U(x) = U^+(x)$, $x \in \Omega_\rho \cup S_0$. Theorem 3 is proved. \square

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