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SHILLA GRAPHS WITH $b = 5$ AND $b = 6$[1]

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Abstract: A Q-polynomial Shilla graph with $b = 5$ has intersection arrays $\{105t, 4(21t + 1), 16(t + 1); 1, 4(t + 1), 84t\}$, $t \in \{3, 4, 19\}$. The paper proves that distance-regular graphs with these intersection arrays do not exist. Moreover, feasible intersection arrays of Q-polynomial Shilla graphs with $b = 6$ are found.

Keywords: Shilla graph, Distance-regular graph, Q-polynomial graph.

1. Introduction

We consider undirected graphs without loops or multiple edges. For a vertex a of a graph Γ, denote by $\Gamma_i(a)$ the ith neighborhood of a, i.e., the subgraph induced by Γ on the set of all vertices at distance i from a. Define $[a] = \Gamma_1(a)$ and $a^\perp = \{a\} \cup [a]$.

Let Γ be a graph, and let $a, b \in \Gamma$. Denote by $\mu(a, b)$ (by $\lambda(a, b)$) the number of vertices in $[a] \cap [b]$ if a and b are at distance 2 (are adjacent) in Γ. Further, the induced $[a] \cap [b]$ subgraph is called *μ-subgraph* (*λ-subgraph*).

If vertices u and w are at distance i in Γ, then we denote by $b_i(u, w)$ (by $c_i(u, w)$) the number of vertices in the intersection of $\Gamma_{i+1}(u)$ (of $\Gamma_{i-1}(u)$, respectively) with $[w]$. A graph Γ of diameter d is called *distance-regular with intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if, for each $i = 0, \ldots, d$, the values $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of vertices u and w at distance i in Γ. Define $a_i = k - b_i - c_i$. Note that, for a distance regular graph, b_0 is the degree of the graph and a_1 is the degree of the local subgraph (the neighborhood of the vertex). Further, for vertices x and y at distance l in the graph Γ, denote by $p^l_{ij}(x, y)$ the number of vertices in the subgraph $\Gamma_i(x) \cap \Gamma_j(y)$. The numbers $p^l_{ij}(x, y)$ are called the intersection numbers of Γ (see [2]). In a distance-regular graph, they are independent of the choice of x and y.

A Shilla graph is a distance-regular graph Γ of diameter 3 with second eigenvalue θ_1 equal to $a = a_3$. In this case, a divides k and b is defined by $b = b(\Gamma) = k/a$. Morover, $a_1 = a - b$ and Γ has intersection array $\{ab, (a+1)(b-1), b_2; 1, c_2, a(b-1)\}$. Feasible intersection arrays of Shilla graphs are found in [6] for $b \in \{2, 3\}$.

Feasible intersection arrays of Shilla graphs are found in [1] for $b = 4$ (50 arrays) and for $b = 5$ (82 arrays). At present, a list of feasible intersection arrays of Shilla graphs for $b = 6$ is unknown. Moreover, the existence of Q-polynomial Shilla graphs with $b = 5$ also is unknown.

In this paper, we find feasible intersection arrays of Q-polynomial Shilla graphs with $b = 6$ and prove that Q-polynomial Shilla graphs with $b = 5$ do not exist.

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Theorem 1. *A Q-polynomial Shilla graph with $b = 6$ has intersection array*

(1) $\{42t, 5(7t+1), 3(t+3); 1, 3(t+3), 35t\}$, *where* $t \in \{7, 12, 17, 27, 57\}$;

(2) $\{372, 315, 75; 1, 15, 310\}$, $\{744, 625, 125; 1, 25, 620\}$ *or* $\{930, 780, 150; 1, 30, 775\}$;

(3) $\{312, 265, 48; 1, 24, 260\}$, $\{624, 525, 80; 1, 40, 520\}$, $\{1794, 1500, 200; 1, 100, 1495\}$ *or* $\{5694, 4750, 600; 1, 300, 4745\}$.

In view of Theorem 2 from [1], a Q-polynomial Shilla graph with $b = 5$ has intersection array $\{105t, 4(21t+1), 16(t+1); 1, 4(t+1), 84t\}$, $t \in \{3, 4, 19\}$.

Theorem 2. *Distance-regular graphs with intersection arrays* $\{315, 256, 64; 1, 16, 252\}$ *and* $\{1995, 1600, 320; 1, 80, 1596\}$ *do not exist.*

Theorem 3. *Distance-regular graphs with intersection array* $\{420, 340, 80; 1, 20, 336\}$ *do not exist.*

2. Proof of Theorem 1

In this section, Γ is a Q-polynomial Shilla graph with $b = 6$. Then $(a_2 - 5a - 6)^2 - 4(5b_2 - a_2)$ is the square of an integer. By [6, Lemma 8], we have

$$2a \leq c_2 b(b+1) + b^2 - b - 2;$$

therefore, $a \leq 21c_2 + 14$. It follows from the proof of Theorem 9 in [6] that either $k < b^3 - b = 6 \cdot 35$ or $v < k(2b^3 - b + 1) = 428k$. By [6, Corollary 17 and Theorem 20], the number $b_2 + c_2$ divides $b(b-1)b_2$ and

$$-34 = -b^2 + 2 \leq \theta_3 \leq -b^2(b+3)/(3b+1) \leq -18.$$

Theorem 2 from [7] implies the following lemma.

Lemma 1. *If $b_2 = c_2$, then Γ has an intersection arrays* $\{42t, 5(7t+1), 3(t+3); 1, 3(t+3), 35t\}$ *and* $t \in \{7, 12, 17, 27, 57\}$.

To the end of this section, assume that $b_2 \neq c_2$ and $k > \theta_1 > \theta_2 > \theta_3$ are eigenvalues of the graph Γ. Then

$$6(6b_2 + c_2)/(b_2 + c_2) = -\theta_3.$$

On the other hand, according to [6, Lemma 10], the number c_2 divides $(a+6)b_2$, $30a(a+1)$ and $(a+6)b_2 \geq (a+1)c_2$.

Lemma 2. *If $-34 \leq \theta_3 \leq -18$, then one of the following statements holds*:

(1) $\theta_3 = -31$ *and* Γ *has one of the intersection arrays* $\{372, 315, 75; 1, 15, 310\}$, $\{744, 625, 125; 1, 25, 620\}$, *and* $\{930, 780, 150; 1, 30, 775\}$;

(2) $\theta_3 = -26$ *and* Γ *has one of the intersection arrays* $\{312, 265, 48; 1, 24, 260\}$, $\{624, 525, 80; 1, 40, 520\}$, $\{1794, 1500, 200; 1, 100, 1495\}$, *and* $\{5694, 4750, 600; 1, 300, 4745\}$;

(3) $\theta_3 = -21$ *and* Γ *has one of the intersection arrays* $\{42t, 5(7t+1), 3(t+3); 1, 3(t+3), 35t\}$ *for* $t \in \{7, 12, 17, 27, 57\}$.

P r o o f. By [6, Lemma 10], c_2 divides $b(b-1)b_2 = 30b_2$ and, by [6, Corollary 17], the smallest nonprinciple eigenvalue θ_3 is equal to $b(bb_2 + c_2)/(b_2 + c_2)$. Therefore, $30(\theta_3 + 6)/(\theta_3 + 36)$ is an integer and $\theta_3 \in \{-34, -33, -32, -31, -30, -27, -26, -24, -21, -18\}$.

Let $\theta_3 = -34$. Then $3(6b_2 + c_2) = 17(b_2 + c_2)$ and $b_2 = 14c_2$. Further, θ_3 is a root of the equation $x^2 - (a_1 + a_2 - k)x + (b-1)b_2 - a_2 = 0$; therefore, $a = 425/28 \cdot c_2 - 34$. In this case, the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6/5 \cdot (2545c_2 - 5544)/c_2$, a contradiction with the fact that 5 does not divide $6 \cdot 5544$.

Let $\theta_3 = -33$. Then $2(6b_2 + c_2) = 11(b_2 + c_2)$ and $b_2 = 9c_2$. Further, $a = 275/27 \cdot c_2 - 33$ and the multiplicity of the first nonprincipal eigenvalue is equal to $m_1 = 6/5 \cdot (1645c_2 - 5184)/c_2$, a contradiction as above.

Let $\theta_3 = -32$. Then $3(6b_2 + c_2) = 16(b_2 + c_2)$ and $2b_2 = 13c_2$. Further, $a = 100/13 \cdot c_2 - 32$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6/5 \cdot (1195c_2 - 4836)/c_2$, a contradiction as above.

Let $\theta_3 = -31$. Then $6(6b_2 + c_2) = 31(b_2 + c_2)$ and $b_2 = 5c_2$. Further, $a = 31/5 \cdot c_2 - 31$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 30(37c_2 - 180)/c_2 = 1110 - 5400/c_2$. The number of vertices in the graph is $31/5 \cdot (222c_2^2 - 2005c_2 + 4500)/c_2$; hence, c_2 divides 900 and is a multiple of 5. By computer enumeration, we find that, only for $c_2 = 15, 25$ and 30, we have admissible intersection arrays $\{372, 315, 75; 1, 15, 310\}$, $\{744, 625, 125; 1, 25, 620\}$ and $\{930, 780, 150; 1, 30, 775\}$.

Let $\theta_3 = -30$. Then $(6b_2 + c_2) = 5(b_2 + c_2)$ and $b_2 = 4c_2$. Further, $a = 125/24 \cdot c_2 - 30$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6/5 \cdot (745c_2 - 4176)/c_2$, a contradiction as above.

Let $\theta_3 = -27$. Then $2(6b_2 + c_2) = 9(b_2 + c_2)$ and $3b_2 = 7c_2$. Further, $a = 25/7 \cdot c_2 - 25$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6/5 \cdot (445c_2 - 3276)/c_2$, a contradiction as above.

Let $\theta_3 = -26$. Then $3(6b_2 + c_2) = 13(b_2 + c_2)$ and $b_2 = 2c_2$. Further, $a = 13/4 \cdot c_2 - 26$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6(77c_2 - 600)/c_2 = 462 - 3600/c_2$. The number of vertices in the graph is $13/8 \cdot (231c_2^2 - 3340c_2 + 12000)/c_2$; hence, c_2 divides 1200 and is a multiple of 4. By computer enumeration, we find that only for $c_2 = 24, 40, 100$, and 300 we have admissible intersection arrays $\{312, 265, 48; 1, 24, 260\}$, $\{624, 525, 80; 1, 40, 520\}$, $\{1794, 1500, 200; 1, 100, 1495\}$, and $\{5694, 4750, 600; 1, 300, 4745\}$.

Let $\theta_3 = -21$. Then $2(6b_2 + c_2) = 7(b_2 + c_2)$ and $b_2 = c_2$. Further, $a = 7/3 \cdot c_2 - 21$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6(41c_2 - 360)/c_2 = 246 - 2160/c_2$. The number of vertices in the graph is $7/3 \cdot (82c_2^2 - 1335c_2 + 5400)/c_2$; hence, c_2 divides 1080 and is a multiple of 3. By computer enumeration, we find that, only for $c_2 = 18, 30, 45, 60, 90$, and 180, we have admissible intersection arrays $\{42t, 5(7t+1), 3(t+3); 1, 3(t+3), 35t\}$ for $t \in \{3, 7, 12, 17, 27, 57\}$. A graph with the array obtained for $t = 3$ does not exist by [5].

Let $\theta_3 = -18$. Then $6(6b_2 + c_2) = 19(b_2 + c_2)$, so $3b_2 = 2c_2$. Further, $a = 2512 \cdot c_2 - 18$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6/5 \cdot (145c_2 - 1224)/c_2$, a contradiction. The lemma is proved. □

Theorem 1 follows from Lemmas 1–2.

3. Triple intersection numbers

In the proof of Theorem 3, the triple intersection numbers [3] are used.

Let Γ be a distance-regular graph of diameter d. If u_1, u_2, u_3 are vertices of the graph Γ, then r_1, r_2, r_3 are non-negative integers not greater than d. Denote by $\left\{ {u_1u_2u_3 \atop r_1r_2r_3} \right\}$ the set of vertices $w \in \Gamma$ such that $d(w, u_i) = r_i$ and by $\left[{u_1u_2u_3 \atop r_1r_2r_3} \right]$ the number of vertices in $\left\{ {u_1u_2u_3 \atop r_1r_2r_3} \right\}$. The numbers $\left[{u_1u_2u_3 \atop r_1r_2r_3} \right]$ are called the triple intersection numbers. For a fixed triple of vertices u_1, u_2, u_3, instead of $\left[{u_1u_2u_3 \atop r_1r_2r_3} \right]$, we will write $[r_1r_2r_3]$. Unfortunately, there are no general formulas for the numbers $[r_1r_2r_3]$. However, [3] outlines a method for calculating some numbers $[r_1r_2r_3]$.

Let u, v, w be vertices of the graph Γ, $W = d(u, v)$, $U = d(v, w)$, and let $V = d(u, w)$. Since there is exactly one vertex $x = u$ such that $d(x, u) = 0$, then the number $[0jh]$ is 0 or 1. Hence $[0jh] = \delta_{jW}\delta_{hV}$. Similarly, $[i0h] = \delta_{iW}\delta_{hU}$ and $[ij0] = \delta_{iU}\delta_{jV}$.

Another set of equations can be obtained by fixing the distance between two vertices from $\{u, v, w\}$ and counting the number of vertices located at all possible distances from the third:

$$\begin{cases} \sum\limits_{l}^{d} [ljh] = p_{jh}^{U} - [0jh] \\ \sum\limits_{l}^{d} [ilh] = p_{ih}^{V} - [i0h] \\ \sum\limits_{l}^{d} [ijl] = p_{ij}^{W} - [ij0] \end{cases} \tag{3.1}$$

However, some triplets disappear. For $|i - j| > W$ or $i + j < W$, we have $p_{ij}^{W} = 0$; therefore, $[ijh] = 0$ for all $h \in \{0, ..., d\}$.

We set

$$S_{ijh}(u, v, w) = \sum_{r,s,t=0}^{d} Q_{ri}Q_{sj}Q_{th} \begin{bmatrix} uvw \\ rst \end{bmatrix}.$$

If the Krein parameter $q_{ij}^{h} = 0$, then $S_{ijh}(u, v, w) = 0$.

We fix vertices u, v, w of a distance-regular graph Γ of diameter 3 and set

$$\{ijh\} = \begin{Bmatrix} uvw \\ ijh \end{Bmatrix}, \quad [ijh] = \begin{bmatrix} uvw \\ ijh \end{bmatrix}, \quad [ijh]' = \begin{bmatrix} uwv \\ ihj \end{bmatrix}, \quad [ijh]^* = \begin{bmatrix} vuw \\ jih \end{bmatrix}, \quad [ijh]^\sim = \begin{bmatrix} wvu \\ hji \end{bmatrix}.$$

Calculating the numbers

$$[ijh]' = \begin{bmatrix} uwv \\ ihj \end{bmatrix}, \quad [ijh]^* = \begin{bmatrix} vuw \\ jih \end{bmatrix}, \quad [ijh]^\sim = \begin{bmatrix} wvu \\ hji \end{bmatrix}$$

(symmetrization of the triple intersection numbers) can give new relations that make it possible to prove the nonexistence of a graph.

4. Graphs with intersection arrays $\{315, 256, 64; 1, 16, 252\}$ and $\{1995, 1600, 320; 1, 80, 1596\}$

Let Γ be a distance-regular graph with intersection array $\{315, 256, 64; 1, 16, 252\}$. By [2, Theorem 4.4.3], the eigenvalues of the local subgraph of the graph Γ are contained in the interval $[-5, 59/5)$. Since the Terwilliger polynomial (see [4]) is $-4(5x - 59)(x + 5)(x + 1)(x - 43)$, then these eigenvalues lie in $[-5, -1] \cup (59/5.43]$. Hence, all nonprinciple eigenvalues are negative and the local subgraph is a union of isolated (a_1+1)-cliques, a contradiction with the fact that $a_1+1 = 49$ does not divide $k = 315$.

Thus, a distance-regular graph with intersection array $\{315, 256, 64; 1, 16, 252\}$ does not exist.

Let Γ be a distance-regular graph with intersection array $\{1995, 1600, 320; 1, 80, 1596\}$. Then Γ has $1 + 1995 + 39900 + 8000 = 49896$ vertices, spectrum $1995^1, 399^{495}, 15^{23275}, -21^{26125}$, and the dual matrix of eigenvalues

$$Q = \begin{pmatrix} 1 & 495 & 23275 & 26125 \\ 1 & 99 & 175 & -275 \\ 1 & 0 & -56 & 55 \\ 1 & -99/4 & 931/4 & -209 \end{pmatrix}.$$

The Terwilliger polynomial of the graph Γ is $-20(x+5)(x+1)(x-79)(x-299)$; hence, the eigenvalues of the local subgraph are contained in $[-5, -1] \cup \{79\} \cup \{394\}$.

Note that the multiplicity $m_1 = 495$ of the eigenvalue $\theta_1 = 399$ is less than k. By the corollary to Theorem 4.4.4 from [2] for $b = b_1/(\theta_1+1) = 4$, the graph $\Sigma = [u]$ has an eigenvalue $-1-b = -5$ of multiplicity at least $k - m_1 = 1500$.

Let the number of eigenvalues 79 of the graph Σ be equal to y. Then the sum of eigenvalues of the graph Σ is at most $-7500 - (494 - y) + 79y + 394$; therefore, $y \geq 95$. Now twice the number of edges in Σ is equal to

$$786030 = 1995 \cdot 394 = \sum_i m_i \theta_i^2$$

but not less than

$$25 \cdot 1500 + 399 + 95 \cdot 79^2 + 394^2 = 786030.$$

Hence, Σ has spectrum $394^1.79^{95}, -1^{399}, -5^{1500}$.

Now the number $t = k_\Sigma \lambda_\Sigma/2$ of triangles in Σ containing this vertex is equal to $\sum_i m_i \theta_i^3/(2v)$. Therefore,

$$t = \sum_i m_i \theta_i^3/(2v) = (394^3 + 79^3 \cdot 95 - 399 - 125 \cdot 1500)/3990 = 27021$$

and $\lambda_\Sigma = 54042/394$ is approximately equal to 137.16, a contradiction.

Thus, a distance-regular graph with intersection array $\{1995, 1600, 320; 1, 80, 1596\}$ does not exist.

Theorem 2 is proved.

5. Graph with array $\{420, 340, 80; 1, 20, 336\}$

Let Γ be a distance-regular graph with intersection array $\{420, 340, 80; 1, 20, 336\}$. Then Γ is a formally self-dual graph having $1 + 420 + 7140 + 1700 = 9261$ vertices, spectrum $420^1, 84^{420}, 0^{7140}, -21^{1700}$, and the dual matrix of eigenvalues

$$Q = \begin{pmatrix} 1 & 420 & 7140 & 1700 \\ 1 & 84 & 0 & -85 \\ 1 & 0 & -21 & 20 \\ 1 & -21 & 84 & -64 \end{pmatrix}.$$

The Terwilliger polynomial of the graph Γ is $-20(x+5)(x+1)(x-16)(x-59)$ and the eigenvalues of the local subgraph are contained in $[-5, -1] \cup \{16\} \cup \{79\}$. If the nonprinciple eigenvalues of a local subgraph are negative, then this subgraph is a union of isolated (a_1+1)-cliques, a contradiction with the fact that $a_1+1 = 80$ does not divide $k = 420$. Hence, the local subgraph has eigenvalue 6.

Lemma 3. *Intersection numbers of a graph* Γ *satisfy the equalities*

(1) $p^1_{11} = 79$, $p^1_{21} = 340$, $p^1_{32} = 1360$, $p^1_{22} = 5440$, $p^1_{33} = 340$,

(2) $p^2_{11} = 20$, $p^2_{12} = 320$, $p^2_{13} = 80$, $p^2_{22} = 5519$, $p^2_{23} = 1300$, $p^2_{33} = 320$;

(3) $p^3_{12} = 336$, $p^3_{13} = 84$, $p^3_{22} = 5460$, $p^3_{23} = 1344$, $p^3_{33} = 271$.

P r o o f. Direct calculations. □

Let u, v, and w be vertices of a graph Γ, $[rst] = \left[\begin{smallmatrix} uvw \\ rst \end{smallmatrix}\right]$, $\Omega = \Gamma_3(u)$, and let $\Sigma = \Omega_2$. Then Σ is a regular graph of degree 1344 on 1700 vertices.

Lemma 4. *Let* $d(u, v) = d(u, w) = 3$ *and* $d(v, w) = 1$. *Then the following equalities hold*:

(1) $[122] = 2r_6/5 - 136$, $[123] = [132] = -2r_6/5 + 472$, $[133] = 2r_6/5 - 388$;

(2) $[211] = r_6/10 - 38$, $[212] = [221] = -r_6/10 + 374$, $[222] = -14r_6/10 + 5576$, $[223] = [232] = 3r_6/2 - 490$, $[233] = -3r_6/2 + 1834$;

(3) $[311] = -r_6/10 + 117$, $[312]=[321]=r_6/10 - 34$, $[322] = r_6$, $[323] = [332] = -11r_6/10 + 1378$, $[333] = 11r_6/10 - 1107$,

where $r_6 \in \{1010, 1020, \ldots, 1170\}$.

P r o o f. A simplification of formulas (3.1) taking into account the equalities $S_{113}(u, v, w) = S_{131}(u, v, w) = S_{311}(u, v, w) = 0$. □

By Lemma 4, we have $1010 \leq [322] = r_6 \leq 1170$.

Lemma 5. *Let* $d(u, v) = d(u, w) = d(v, w) = 3$. *Then the following equalities hold*:

(1) $[122] = -r_{17} + 336$, $[123] = [132] = r_{17}$, $[133] = -r_{17} + 84$;

(2) $[213] = [231] = r_{17}$, $[212] = [221] = -r_{17} + 336$, $[222] = 39r_{17}/4 + 3444$, $[223] = [232] = -35r_{17}/4 + 1680$, $[233] = 31r_{17}/4 - 336$;

(3) $[313] = [331] = -r_{17} + 84$, $[312] = [321] = r_{17}$, $[322] = -35r_{17}/4 + 1680$, $[323] = [332] = 31r_{17}/4 - 336$, $[333] = -27r_{17}/4 + 522$,

where $r_{17} \in \{44, 48, \ldots, 76\}$.

P r o o f. A simplification of formulas (3.1) taking into account the equalities $S_{113}(u, v, w) = S_{131}(u, v, w) = S_{311}(u, v, w) = 0$. □

By Lemma 5, we have $1015 \leq [322] = -35r_{17}/4 + 1680 \leq 1295$.
The number d of edges between $\Sigma(w)$ and $\Sigma - (\{w\} \cup \Lambda(w))$ satisfies the inequalities

$$359905 = 84 \cdot 1010 + 271 \cdot 1015 \leq d \leq 84 \cdot 1170 + 271 \cdot 1295 = 449225,$$
$$267.786 \leq 1343 - \lambda \leq 334.245,$$
$$1008.755 \leq \lambda \leq 1075.214,$$

where λ is the mean value of the parameter $\lambda(\Sigma)$.

Lemma 6. *Let $d(u,v) = d(u,w) = 3$ and $d(v,w) = 2$. Then the following equalities hold*:

(1) $[122] = (-64r_{15} + 4r_{16} + 7364)/27$, $[123] = [132] = (64r_{15} - 4r_{16} + 1708)/27$, $[133] = (-64r_{15} + 4r_{16} + 560)/27$;

(2) $[211] = -r_{15}+20$, $[212] = [221] = (71r_{15}+4r_{16}+6392)/27$, $[222] = (-17r_{15}-13r_{16}+38311)/9$, $[223] = [232] = (-20r_{15} + 35r_{16} + 26095)/27$, $[233] = (64r_{15} - 31r_{16} + 8053)/27$;

(3) $[311] = r_{15}$, $[312] = [321] = (-71r_{15} - 4r_{16} + 2248)/27$, $[313] = (44r_{15} + 4r_{16} + 20)/27$, $[322] = (115r_{15}+35r_{16}+26716)/27$, $[323] = [332] = (-44r_{15}-31r_{16}+7297)/27$, $[333] = r_{16}$,

where $-10r_{15} + 4r_{16} + 20$ *is a multiple of* 27, $r_{15} \in \{0, 1, \dots, 20\}$, *and* $r_{16} \in \{0, 1, \dots, 235\}$.

P r o o f. A simplification of formulas (3.1) taking into account the equalities $S_{113}(u,v,w) = S_{131}(u,v,w) = S_{311}(u,v,w) = 0$. □

By Lemma 6, we have

$$998 \le [322] = (115r_{15} + 35r_{16} + 26716)/27 \le 1294.$$

Let us count the number h of pairs of vertices y and z at distance 3 in the graph Ω, where

$$y \in \begin{Bmatrix} uv \\ 31 \end{Bmatrix}, \quad z \in \begin{Bmatrix} uv \\ 32 \end{Bmatrix}.$$

On the one hand, by Lemma 4, we have $[323] = -11r_6/10+1378$, where $r_6 \in \{1010, 1020, ..., 1170}$, therefore

$$7644 = 8491 \le h \le 84267 = 22428.$$

On the other hand, by Lemma 6, we have $[313] = (44r_{15} + 4r_{16} + 20)/27$, where $r_{15} \in \{0, 1, ..., 20\}$, $r_{16} \in \{0, 1, ..., 235\}$, therefore

$$7644 \le \sum_i (44r^i_{15} + 4r^i_{16}) + 995.55 \le 22428,$$

$$6648.44 \le \sum_i (44r^i_{15} + 4r^i_{16}) \le 21432.45,$$

$$4.946 \le \sum_i (11r^i_{15} + r^i_{16})/1344 \le 15.947.$$

If $r_{15} = 0$, then $r_{16} + 5$ is a multiple of 27 and $r_{16} = 22.49, \dots$.
If $r_{15} = 1$, then $2r_{16} + 5$ is a multiple of 27 and $r_{16} = 11.38, \dots$.
In any case,

$$\sum_i (11r^i_{15} + r^i_{16})/1344 \ge 22,$$

a contradiction.

Theorem 3 is proved. □

Conclusion

The following are the main steps in creating a theory of Shilla graphs:
(1) finding a list of feasible intersection arrays of Shilla graphs with $b = 6$;
(2) classification of Q-polynomial Shilla graphs with $b_2 = c_2$.

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