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# INDUCED $nK_2$ DECOMPOSITION OF INFINITE SQUARE GRIDS AND INFINITE HEXAGONAL GRIDS

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**Abstract:** The induced $nK_2$ decomposition of infinite square grids and hexagonal grids are described here. We use the multi-level distance edge labeling as an effective technique in the decomposition of square grids. If the edges are adjacent, then their color difference is at least 2 and if they are separated by exactly a single edge, then their colors must be distinct. Only non-negative integers are used for labeling. The proposed partitioning technique per the edge labels to get the induced $nK_2$ decomposition of the ladder graph is the square grid and the hexagonal grid.

**Keywords:** Distance labelling, Channel assignment, $L(h,k)$-colouring, Rectangular grid, Hexagonal grid.

## 1. Introduction

Decomposition of graphs has been an intriguing area of study in Graph Theory. Many of the decomposition problems could be addressed by even a beginner in graph theory. However, it needs a crafty and involved work to achieve certain types of decompositions of graphs. Given a graph with vertices and edges, the task in decomposition is to find subgraphs with a particular property. The disjoint union of these subgraphs is the given graph itself. For a perfect decomposition, there should not be any edges left over apart from the decomposed subgraphs. In an optimal decomposition if there are some edges left over, the collection of such edges is known as leave. In case, with the compromise of some overlapping edges, if we can find subgraphs whose union is the given graph, the set of repeated (overlapping) edges is known as padding.

The problem of decomposition of graphs dates back to some real-life problems. The famous Kirkman's schoolgirl problem and the 9-prisoner's problem are some of them. For many years, decomposition was identified as $G$-design. This has its origin from the design of experiments. If we have $n$ samples to be compared optimally, we can use decomposition as tool. Steiner triple system also needs decomposition techniques.

In this paper, we go for some techniques of a particular type of decomposition known as induced decomposition. We consider only simple connected graphs. We use the definitions and notations mostly from [2] unless otherwise defined here. Two adjacent vertices are referred to as neighbors of each other. The set of neighbors of a vertex $v$ is called the open neighborhood of $v$ and is denoted by $N_G(v)$ or $N(v)$. The set $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of $v$. Similarly, we can consider the set of neighbors of an edge as well.

An edge-induced subgraph is a subset of the edges of a graph $G$ together with any vertices that are their end vertices. As seen in [1], if $G$ is a connected graph, and $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$

are two edges of $G$, then the *distance between edges or edge distance* of $e_1$ and $e_2$ is defined as

$$ed(e_1, e_2) = \min\{d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2)\}.$$

If $ed(e_1, e_2) = 0$, then these edges are called *neighbor edges* or adjacent edges. For the induced decomposition, we use the colouring technique known as the $L'(2,1)$–edge coloring of a graph $G$ which is defined as in [7] and used in [4]. For non-negative integers $i$ and $j$, an $L'(i,j)$–edge coloring of a graph $G$ is an assignment of non-negative integers to the edges $e_1$ and $e_2$ of $G$ such that $|c(e_1) - c(e_2)| \geq i$ if $ed(e_1, e_2) = 0$ and $|c(e_1) - c(e_2)| \geq j$ if $ed(e_1, e_2) = 1$. No condition is placed on colors assigned to the edges $e_1$ and $e_2$ if $ed(e_1, e_2) \geq 2$.

In this paper we study the case where $i = 2$ and $j = 1$. For an $L'(i,j)$–edge coloring $c$ of a graph $G$, the $c$-span of $G$ is the maximum value of $|c(e_1) - c(e_2)|$ over all pairs of edges $e_1$ and $e_2$ of $E(G)$. It is denoted by $\lambda'_{i,j}(c)$. That is,

$$\lambda'_{i,j}(c) = \max\big\{|c(e_1) - c(e_2)| : e_1, e_2 \in E(G)\big\}.$$

In particular, for $i = 2$ and $j = 1$, from [7], we have the c-span of $G$ with respect to the $L'(2,1)$–edge coloring as

$$\lambda'_{2,1}(c) = \max\{|c(e_1) - c(e_2)| : e_1, e_2 \in E(G)\}.$$

We use Stiebitz et al. [6] for the terminologies of $\chi'$–critical graph and $\chi'$–critical edge.

## 2. Rectangular grids

We use the rectangular grid graph (RGG) concept from [3, 5]. Given a RGG, we apply $L'(2,1)$–edge coloring technique and thereby obtain the $L'(2,1)$–edge coloring number of infinite rectangular grids. For convenience we denote RGG as $G_{m,n}$, an $m \times n$ rectangular grid graph. A particular case is the *Ladder graph* denoted as $P_2 \times P_m$. The $L'(2,1)$–edge coloring number of ladder graph is obtained as follows.

**Theorem 1** [4, Theorem 4].

$$\lambda'(P_2 \times P_m) = \begin{cases} 4 & \textit{if} \quad m = 2, \\ 6 & \textit{if} \quad m = 3, \\ 7 & \textit{if} \quad m \geq 4. \end{cases}$$

**Theorem 2** [3, Theorem 2]. *The $L'(2,1)$–edge coloring number of the rectangular grid $G_{3,4}$ is 8. That is, $\lambda'(G_{3,4}) = 8$.*

By the following theorem, we obtain the smallest positive integer or the smallest maximum color used among the different $L'(2,1)$–edge coloring of the infinite rectangular grids.

**Theorem 3** [3, Theorem 3]. *The $L'(2,1)$–edge coloring number of $G_{m,n}$ is at most 9. That is, $\lambda'(G_{m,n}) \leq 9$, for any positive integers $m$ and $n$.*

See Fig. 1 for a sample coloring [3].

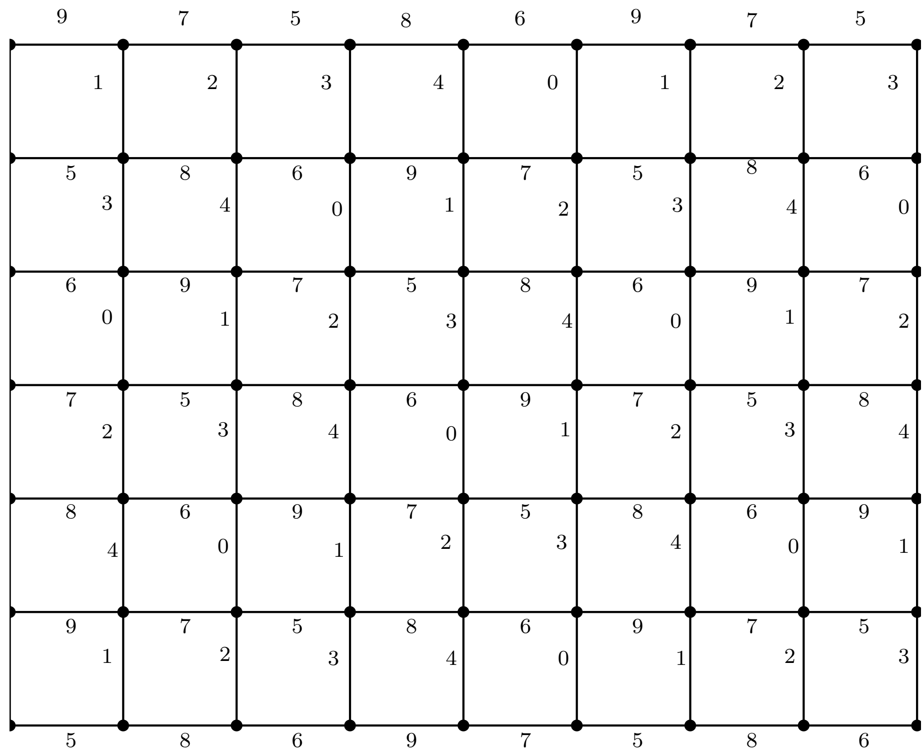



Figure 1. An optimal $L'(2,1)$–labeling of a fragment of rectangular grid.

### 2.1. The $nK_2$ decomposition of the Ladder graph $P_2 \times P_m$

In this section, we find the $nK_2$ decomposition of the ladder graph. By the symbol $\mathcal{P}_n(K_2)(G)$ we denote the graph $G$ that has the property $\mathcal{P}_n(K_2)$, i.e., induced $nK_2$. We use the edge coloring as a tool used in [4], see Fig. 2.

**Theorem 4.** *A ladder graph $P_2 \times P_m$ can have $\mathcal{P}_n(K_2)$ if and only if*

(i) $n|q$ *is such that* $2 \le n \le \left\lfloor \frac{q}{6} \right\rfloor$, *where $q$ is the size of the ladder graph, and*

(ii) $\operatorname{diam}(P_2 \times P_m) \ge 6$.

P r o o f. We see that the edge partition number of the ladder graph is

$$\pi_\nu{}'(P_2 \times P_m) = d(u) + d(w) = 6,$$

where $u$ is the vertex of maximum degree lying on the cycle $C_4$ and $w$ is such that $d(w)$ is maximum; $w \in N(u)$. Hence, the bound for $n$ in $\mathcal{P}_n(K_2)$ becomes

$$2 \le n \le \left\lfloor \frac{q}{6} \right\rfloor,$$

where $q$ is the size of the ladder graph and $q = 3m - 2$.

By the diameter condition, ladder graph has $\mathcal{P}_n(K_2)$ only if the diameter is at least six. Hence, consider the ladder graph $P_2 \times P_6$ which is of diameter 6 and size, $q = 16$. The bound for $n$ in $\mathcal{P}_n(K_2)$ is calculated as

$$2 \le n \le \left\lfloor \frac{16}{6} \right\rfloor = 2.$$

That is, there is every possibility that the graph $P_2 \times P_6$ can have $\mathcal{P}_2(K_2)$. We now verify the existence of $\mathcal{P}_2(K_2)$ in $P_2 \times P_6$ by the $L'(2,1)$–edge coloring technique given in [4].

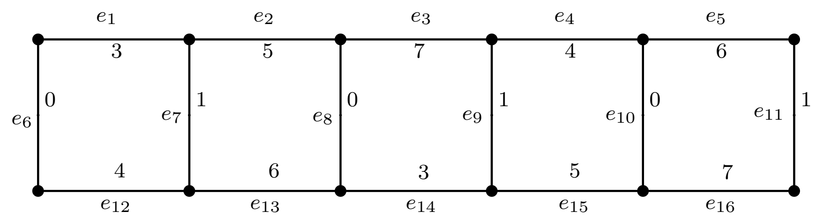



Figure 2. Optimal $L'(2,1)$–coloring of $P_2 \times P_6$.

Partition the edge set $E(P_2 \times P_6)$ into independent sets $E_1,\ E_2,\ \ldots$ such that the set $E_j$ consists of edges which receives color $j$. Note that the edges of $P_2 \times P_6$ are labeled $e_1,\ e_2,\ \ldots$ consecutively and selected under each $E_j$ such that the suffixes of the edge labels are in ascending order.

Consider the following partitioning of $E(P_2 \times P_6)$

$$\begin{array}{lll} E_0 = \{e_6,\ e_8,\ e_{10}\}, & E_1 = \{e_7,\ e_9,\ e_{11}\}, & E_2 = \varnothing, \\ E_3 = \{e_1,\ e_{14}\}, & E_4 = \{e_4,\ e_{12}\}, & E_5 = \{e_2,\ e_{15}\}, \\ E_6 = \{e_5,\ e_{13}\}, & E_7 = \{e_3,\ e_{16}\}. & \end{array}$$

As we aim at $2K_2$ decomposition, select two edges from $E_0$ and $E_1$ respectively such that the remaining two edges having different colors are at edge distance at least two

$$\begin{array}{lllll} E_0{}' = \{e_6,\ e_{10}\}, & E_1{}' = \{e_7,\ e_9\}, & E_2{}' = \varnothing, & E_3{}' = E_3, & E_4{}' = E_4, \\ E_5{}' = E_5, & E_6{}' = E_6, & E_7{}' = E_7, & E_8{}' = \{e_8,\ e_{11}\}. & \end{array}$$

As $E_2{}'$ is empty, we see that the distinct $E_j{}'$ for $j \neq 2$, forms the eight subsets with respect to $\mathcal{P}_2(K_2)(P_2 \times P_6)$ (see Fig. 3).

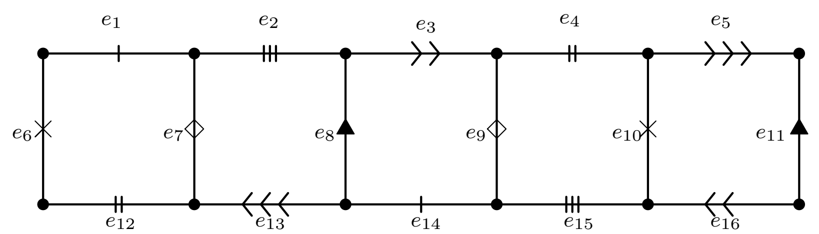



Figure 3. $\mathcal{P}_2(K_2)(P_2 \times P_6)$.

Note that the edges designed in the similar manner come under the same subset of decomposition.

As the size of $P_2 \times P_7$ is 19, it cannot have $\mathcal{P}_2(K_2)$ for any $n$. So consider the ladder graph $P_2 \times P_8$, whose size is 22. As seen earlier, the bound for $n$ in $\mathcal{P}_n(K_2)$ is calculated as,

$$2 \leq n \leq \left\lfloor \frac{22}{6} \right\rfloor,$$

which implies that $n$ takes up values 2 and 3. However, we see that the ladder graph $P_2 \times P_8$ is bound to have only $\mathcal{P}_2(K_2)$. We now verify the existence of $\mathcal{P}_2(K_2)$ in $P_2 \times P_8$ by the $L'(2,1)$–coloring technique and partitioning of the edge set of $P_2 \times P_8$ as follows

$$\begin{array}{ll} E_0 = \{e_8,\ e_{10},\ e_{12},\ e_{14}\}, & E_1 = \{e_9,\ e_{11},\ e_{13},\ e_{15}\}, \\ E_2 = \varnothing, & E_3 = \{e_1,\ e_6,\ e_{18}\}, \\ E_4 = \{e_4,\ e_{16},\ e_{21}\}, & E_5 = \{e_2,\ e_7,\ e_{19}\}, \\ E_6 = \{e_5,\ e_{17},\ e_{22}\}, & E_7 = \{e_3,\ e_{20}\}. \end{array}$$

As we aim at $2K_2$ decomposition, rearrange it to form a new partition as done earlier

$$\begin{array}{lll} {E_0}' = \{e_8,\ e_{10}\}, & {E_1}' = \{e_9,\ e_{11}\}, & {E_2}' = E_2 = \phi, \\ {E_3}' = \{e_1,\ e_6\}, & {E_4}' = \{e_4,\ e_{16}\}, & {E_5}' = \{e_2,\ e_7\}, \\ {E_6}' = \{e_5,\ e_{17}\}, & {E_7}' = \{e_3,\ e_{20}\}, & {E_8}' = \{e_{12}, e_{14}\}, \\ {E_9}' = \{e_{13},\ e_{15}\}, & {E_{10}}' = \{e_{18},\ e_{21}\}, & {E_{11}}' = \{e_{19},\ e_{22}\}. \end{array}$$

They form the eleven subsets with respect to $\mathcal{P}_2(K_2)(P_2 \times P_8)$, see Fig. 4 and Fig. 5. The optimal coloring, labeling and induced $2K_2$ decomposition of the ladder graph $P_2 \times P_8$ are given below.

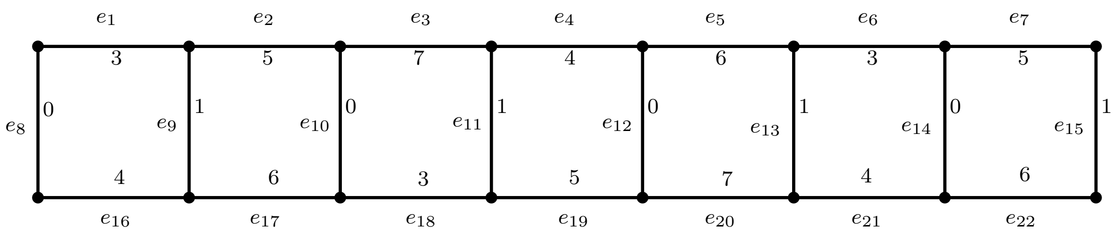



Figure 4. Optimal $L'(2,1)$–coloring in $(P_2 \times P_8)$.

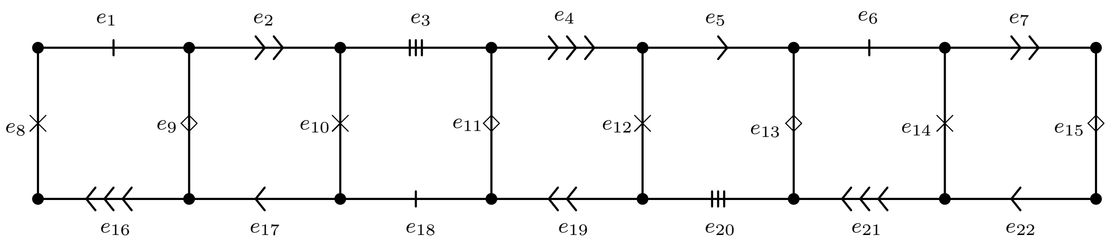



Figure 5. $\mathcal{P}_2(K_2)(P_2 \times P_8)$.

In this manner, we can have $\mathcal{P}_n(K_2)$ for any ladder graph of diameter at least six. Hence, we conclude that a ladder graph $P_2 \times P_m$ can have $\mathcal{P}_n(K_2)$ if and only if

(i) $n|q$ such that $2 \leq n \leq \left\lfloor \dfrac{q}{6} \right\rfloor$; where $q$ is the size of the ladder graph and

(ii) $\operatorname{diam}(P_2 \times P_m) \geq 6$.

$\square$

### 2.2. Rectangular grid and $\mathcal{P}_n(K_2)$

We now give a process of the induced $nK_2$ decomposition of rectangular grid.

We first find $\mathcal{P}_n(K_2)(P_3 \times P_m)$ or the induced $nK_2$ decomposition in the rectangular grid graph, $(P_3 \times P_m)$ using the optimal edge coloring discussed earlier as a tool. We see that the edge partition number of the rectangular grid graph is $\pi_\nu{}'(P_n \times P_m) = d(u) + d(w) = 8$, where $u$ is the vertex of maximum degree which lying on the cycle $C_4$ and $w$ is such that $d(w)$ is maximum; $w \in N(u)$. Hence, the bound for $n$ in $\mathcal{P}_n(K_2)$ becomes

$$2 \leq n \leq \left\lfloor \frac{q}{8} \right\rfloor,$$

where $q$ is the size of the rectangular grid graph. While considering the grid graph $(P_3 \times P_6)$, whose size is 27, we see that the bound for $n$ in $\mathcal{P}_n(K_2)$ is

$$2 \leq n \leq \left\lfloor \frac{27}{8} \right\rfloor = 3$$

and as $n$ must divide $q$, we have that $n = 3$. That is, $(P_3 \times P_6)$ can have $\mathcal{P}_3(K_2)$ and the verification of its existence is done by the optimal $L'(2,1)$–edge coloring technique as follows.

Partition the edge set $E(P_3 \times P_6)$ into independent sets $E_1,\ E_2,\ \ldots$ such that the set $E_j$ consists of edges with color $j$. Note that the edges of $P_3 \times P_6$ are labeled $e_1,\ e_2,\ \ldots$ consecutively and selected under each $E_j$ such that the suffixes of the edge labels are in ascending order, as done in ladder graph. Consider the following partitioning of $E(P_3 \times P_6)$ (see Fig. 7)

$$\begin{array}{lll} E_0 = \{e_6,\ e_{11},\ e_{20}\}, & E_1 = \{e_7,\ e_{21}\}, & E_2 = \{e_8,\ e_{17},\ e_{22}\}, \\ E_3 = \{e_9,\ e_{18}\}, & E_4 = \{e_{10},\ e_{19}\}, & E_5 = \{e_3,\ e_{12},\ e_{26}\}, \\ E_6 = \{e_5,\ e_{14},\ e_{23}\}, & E_7 = \{e_2,\ e_{16},\ e_{25}\}, & E_8 = \{e_4,\ e_{13},\ e_{27}\}, \\ E_9 = \{e_1,\ e_{15},\ e_{24}\}. & & \end{array}$$

As we aim at induced $3K_2$ decomposition, we will diffuse one of the subsets to have exactly three edges under each set. We also interchange the edge $e_{22}$ in $E_2$ for the same. Hence, the new partition can be considered as follows. Consider the following partitioning of $E(P_3 \times P_6)$ (see Fig. 7)

$$\begin{array}{lll} E_1{}' = \{e_7,\ e_{22},\ e_{20}\} & E_2{}' = \{e_8,\ e_{17},\ e_{21}\} & E_3{}' = \{e_9,\ e_{11},\ e_{18}\} \\ E_4{}' = \{e_6,\ e_{10},\ e_{19}\} & E_5{}' = E_5 = \{e_3,\ e_{12},\ e_{26}\} & E_6{}' = E_6 = \{e_5,\ e_{14},\ e_{23}\} \\ E_7{}' = E_7 = \{e_2,\ e_{16},\ e_{25}\} & E_8{}' = E_8 = \{e_4,\ e_{13},\ e_{27}\} & E_9{}' = E_9 = \{e_1,\ e_{15},\ e_{24}\}. \end{array}$$

This is the required $\mathcal{P}_3(K_2)$ of the grid graph $P_3 \times P_6$ (see Fig. 6 and Fig. 7).

We partition the edge set $E(P_3 \times P_{11})$ according to their edge labels as follows

$$\begin{array}{ll} E_0 = \{e_{11},\ e_{16},\ e_{21},\ e_{35},\ e_{40}\}, & E_1 = \{e_{12},\ e_{17},\ e_{36},\ e_{41}\}, \\ E_2 = \{e_{13},\ e_{18},\ e_{32},\ e_{37},\ e_{42}\}, & E_3 = \{e_{14},\ e_{19},\ e_{33},\ e_{38}\}, \\ E_4 = \{e_{15},\ e_{20},\ e_{34},\ e_{39}\}, & E_5 = \{e_3,\ e_8,\ e_{22},\ e_{27},\ e_{46},\ e_{51}\}, \\ E_6 = \{e_5,\ e_{10},\ e_{24},\ e_{29},\ e_{43},\ e_{48}\}, & E_7 = \{e_2,\ e_7,\ e_{26},\ e_{31},\ e_{45},\ e_{50}\}, \\ E_8 = \{e_4,\ e_9,\ e_{23},\ e_{28},\ e_{47},\ e_{52}\}, & E_9 = \{e_1,\ e_6,\ e_{25},\ e_{30},\ e_{44},\ e_{49}\}. \end{array}$$

In a similar manner, $P_4 \times P_6$ will have only $\mathcal{P}_2(K_2)$ as its size is 38 and due to the divisibility criteria. We can find $\mathcal{P}_n(K_2)$ for any grid $P_m \times P_6$ using the similar conditions as that of a ladder graph mentioned in the earlier section. Now consider the rectangular grid graph, $P_3 \times P_{11}$, whose size is 52 and the bound for $n$ in $\mathcal{P}_2(K_2)$ is obtained to be

$$2 \leq n \leq \left\lfloor \frac{q}{8} \right\rfloor.$$

By applying the divisibility conditions, we see that $n$ takes up the values 2 and 4. That is, the graph $P_3 \times P_{11}$ can have only $\mathcal{P}_2(K_2)$ and $\mathcal{P}_4(K_2)$ (see Fig. 8).

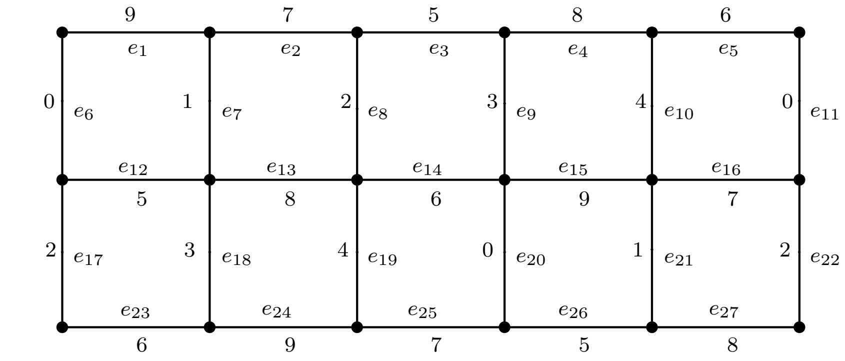



Figure 6. Optimal $L'(2,1)$–edge coloring of $P_3 \times P_6$.

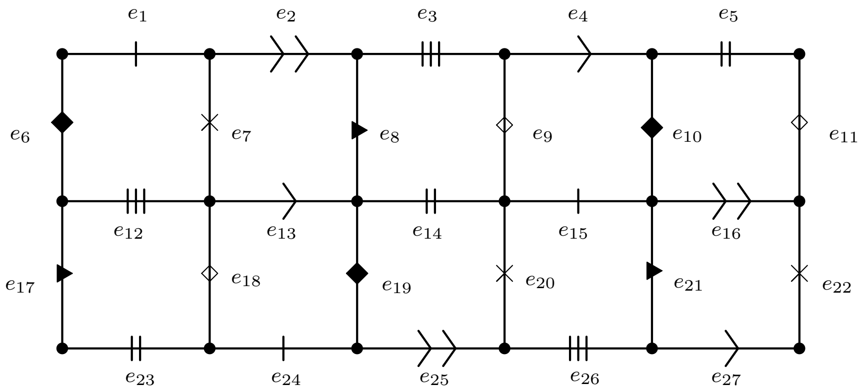



Figure 7. $\mathcal{P}_3(K_2)(P_3 \times P_6)$.

As we aim at $2K_2$ decomposition, further partitioning is required to have induced $2K_2$ in each subset. As the sets $E_0$ and $E_2$ have five edges, we eliminate one from each to form a new subset with two edges at distance at least two. Also eliminate two edges from each subset, with four and six elements, to form new subsets with only two edges in each. The resulting 26 subsets containing two elements each are the required $\mathcal{P}_2(K_2)$ in $E(P_3 \times P_{11})$ (see Fig. 9)

$$\begin{array}{lll}
E_0{}' = \{e_{11},\ e_{16}\}, & E_1{}' = \{e_{12},\ e_{17}\}, & E_2{}' = \{e_{13},\ e_{18}\}, \\
E_3{}' = \{e_{14},\ e_{19}\}, & E_4{}' = \{e_{15},\ e_{20}\}, & E_5{}' = \{e_3,\ e_8\}, \\
E_6{}' = \{e_5,\ e_{10}\}, & E_7{}' = \{e_2,\ e_7\}, & E_8{}' = \{e_4,\ e_9\}, \\
E_9{}' = \{e_1,\ e_6\}, & E_{10}{}' = \{e_{40},\ e_{42}\}, & E_{11}{}' = \{e_{21},\ e_{35}\}, \\
E_{12}{}' = \{e_{36},\ e_{41}\}, & E_{13}{}' = \{e_{32},\ e_{37}\}, & E_{14}{}' = \{e_{33},\ e_{38}\}, \\
E_{15}{}' = \{e_{34},\ e_{39}\}, & E_{16}{}' = \{e_{22},\ e_{27}\}, & E_{17}{}' = \{e_{46},\ e_{51}\}, \\
E_{18}{}' = \{e_{24},\ e_{29}\}, & E_{19}{}' = \{e_{43},\ e_{48}\}, & E_{20}{}' = \{e_{26},\ e_{31}\}, \\
E_{21}{}' = \{e_{45},\ e_{50}\}, & E_{22}{}' = \{e_{23},\ e_{28}\}, & E_{23}{}' = \{e_{47},\ e_{52}\}, \\
E_{24}{}' = \{e_{25},\ e_{30}\}, & E_{25}{}' = \{e_{44},\ e_{49}\}. &
\end{array}$$

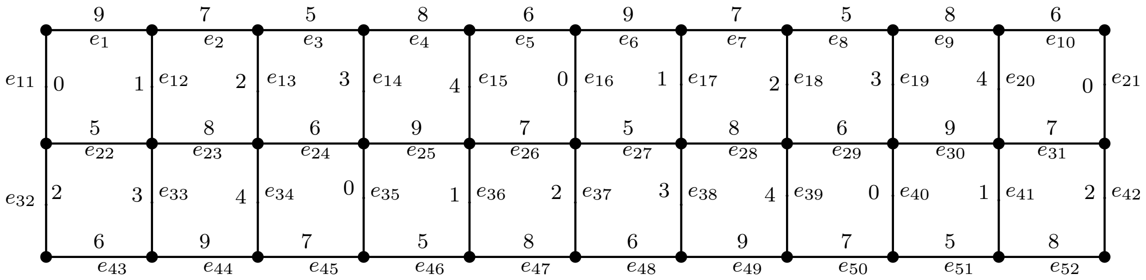



Figure 8. Optimal $L'(2,1)$–labeling of $P_3 \times P_{11}$.

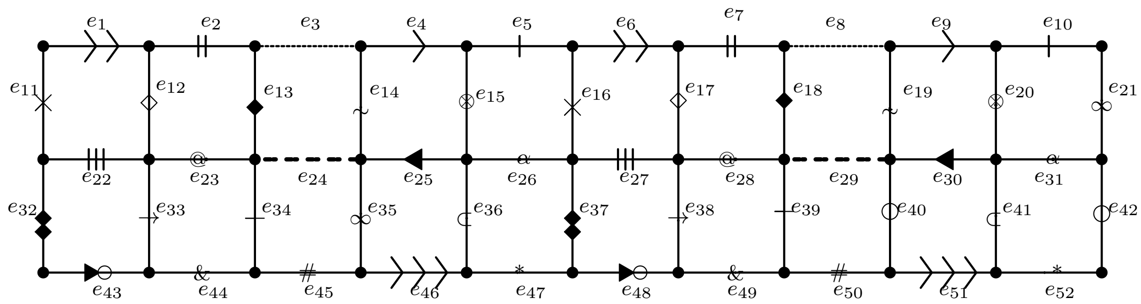



Figure 9. $\mathcal{P}_2(K_2)(P_3 \times P_{11})$.

In a similar manner we can have $\mathcal{P}_4(K_2)(P_3 \times P_{11})$ with the following partition (see Fig. 10)

$$
\begin{array}{ll}
E_0{}'' = \{e_{11},\ e_{16},\ e_{21},\ e_{35}\}, & E_1{}'' = \{e_{12},\ e_{17},\ e_{36},\ e_{41}\}, \\
E_2{}'' = \{e_{13},\ e_{18},\ e_{32},\ e_{37}\}, & E_3{}'' = \{e_{14},\ e_{19},\ e_{33},\ e_{38}\}, \\
E_4{}'' = \{e_{15},\ e_{20},\ e_{34},\ e_{39}\}, & E_5{}'' = \{e_{22},\ e_{27},\ e_{46},\ e_{51}\}, \\
E_6{}'' = \{e_5,\ e_{10},\ e_{24},\ e_{29}\}, & E_7{}'' = \{e_{26},\ e_{31},\ e_{45},\ e_{50}\}, \\
E_8{}'' = \{e_4,\ e_9,\ e_{23},\ e_{28}\}, & E_9{}'' = \{e_1,\ e_6,\ e_{25},\ e_{30}\}, \\
E_{10}{}'' = \{e_{40},\ e_{42},\ e_{43},\ e_{48}\}, & E_{11}{}'' = \{e_3,\ e_8,\ e_{44},\ e_{49}\}, \\
E_{12}{}'' = \{e_2,\ e_7,\ e_{47},\ e_{52}\}. &
\end{array}
$$

Similarly, we can form $\mathcal{P}_n(K_2)$ for any rectangular grid $P_m \times P_r$, where $r = 5x + 1$ for $x \geq 1$ by following the conditions mentioned under the ladder graph and the above pattern. The same partitioning technique can be applied to study the existence of $\mathcal{P}_n(K_2)$ in hexagonal grid as well.

## 3. Hexagonal Grids and $\mathcal{P}_n(K_2)$

The hexagonal grid or honeycomb topology has wide range of application in Network-on-chip (NoC) which is an effective architecture in chip designing. It is evident that the hexagonal grid or

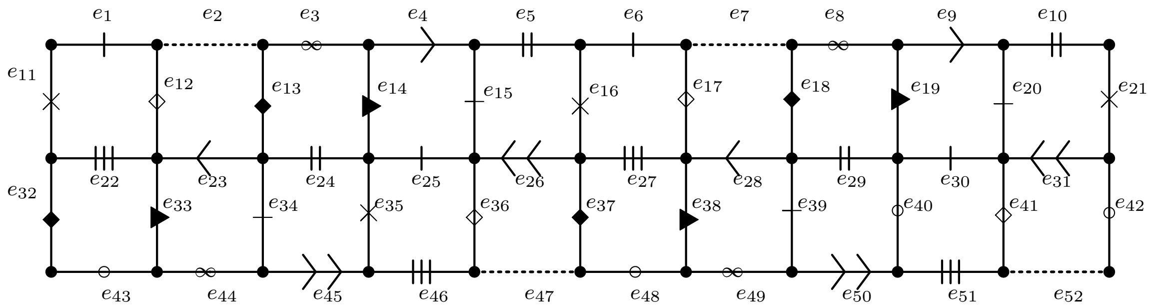
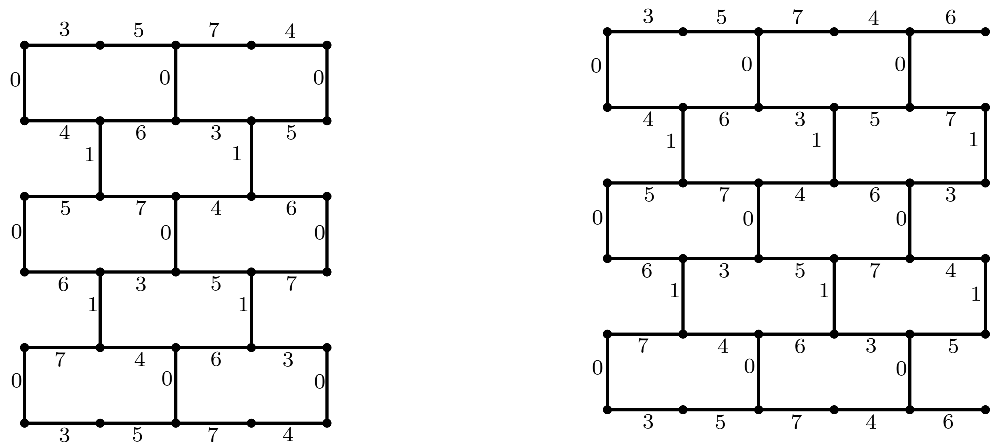



Figure 10. $\mathcal{P}_4(K_2)(P_3 \times P_{11})$.



(a) $H_{6,5}$.

(b) $H_{6,6}$.

Figure 11. Hexagonal grids.

honeycomb structure denoted by $H_{m,n}$ is a spanning subgraph of a rectangular grid with $V(H_{m,n}) = V(G_{m,n})$ and $E(H_{m,n}) \subset E(G_{m,n})$ implying that the value of $\Delta$ decreases and hence $\lambda'(H_{m,n}) < 9$. However, it is proved in [3] that $\lambda'(H_{m,n}) = 7$.

Consider the hexagonal grid $H_{6,5}$ whose size is 37 (Fig. 11 (a)). Then by the condition of $\mathcal{P}_n(K_2)$ we have that there exists no $\mathcal{P}_n(K_2)$ as the size of this grid is prime. Consider the hexagonal grid $H_{6,6}$ whose size is 45. Clearly, nine copies of $\mathcal{P}_5(K_2)$ and five copies of $\mathcal{P}_9(K_2)$ exist in $H_{6,6}$ (Fig. 11 (b)). As $\lambda'(H_{6,6})$ is 7, the edges of $H_{6,6}$, can be partitioned into independent sets $E_0, E_1, E_3, E_4, E_5, E_6, E_7$ such that the set $E_j$ consists of edges which receive color $j$.
Here, $|E_0| = 9,\ |E_1| = 6,\ |E_3| = 6,\ |E_4| = 6,\ |E_5| = 6,\ |E_6| = 6,\ |E_7| = 6$.

For $\mathcal{P}_5(K_2)$, as we aim at induced $5K_2$ in each subset, further partitioning is required as the cardinality of each subset is greater than 5. Let the new partitioning be $|E_0|' = 5, |E_1|' = 5$, $|E_3|' = 5,\ |E_4|' = 5,\ |E_5|' = 5,\ |E_6|' = 5, |E_7|' = 5$. Remaining four edges from $|E_0|$ and one edge each from other subsets of the first partition can be put under two newly formed subsets of cardinality 5. This results in nine copies of $\mathcal{P}_5(K_2)$ as required. For $\mathcal{P}_9(K_2)$, the twelve edges of $|E_6| = 6, |E_7|$ can be distributed equally among the other independent sets of cardinality 6, which

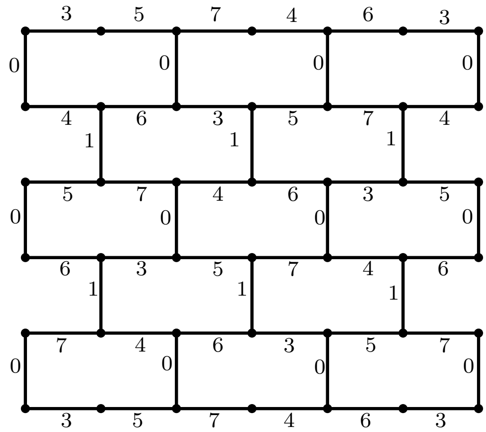



Figure 12. Hexagonal grid $H_{6,7}$.

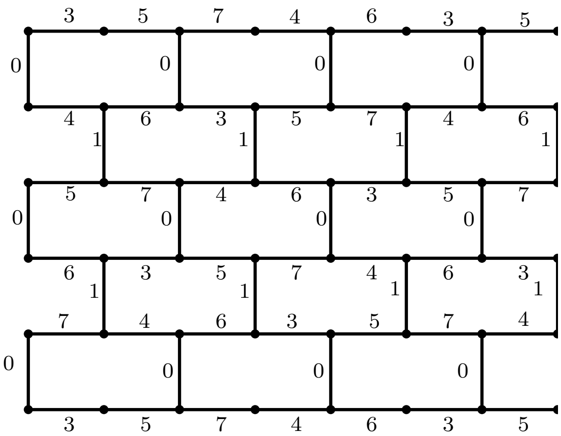


Figure 13. Hexagonal grid $H_{6,8}$.

gives $\mathcal{P}_9(K_2)$ of $H_{6,6}$.

Similarly, $H_{6,7}$ (Fig. 12) is of size 54 and by the condition of $\mathcal{P}_n(K_2)$ we have that there exist twenty seven copies of $\mathcal{P}_2(K_2)$, eighteen copies of $\mathcal{P}_3(K_2)$, nine copies of $\mathcal{P}_6(K_2)$, six copies of $\mathcal{P}_9(K_2)$ and three copies of $\mathcal{P}_{18}(K_2)$. However, as the size of $H_{6,8}$ (Fig. 13) is 62, it has thirty one copies of $\mathcal{P}_2(K_2)$ only.

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