URAL MATHEMATICAL JOURNAL, Vol. 8, No. 1, 2022, pp. 13–22
DOI: 10.15826/umj.2022.1.002

ON $A^{\mathcal{I}^{\mathcal{K}}}$–SUMMABILITY

Chiranjib Choudhury[†], **Shyamal Debnath**[††]

Tripura University (A Central University),
Suryamaninagar-799022, Agartala, India

[†]chiranjibchoudhury123@gmail.com, chiranjib.mathematics@tripurauniv.in
[††]shyamalnitamath@gmail.com

Abstract: In this paper, we introduce and investigate the concept of $A^{\mathcal{I}^{\mathcal{K}}}$-summability as an extension of $A^{\mathcal{I}^*}$-summability which was recently (2021) introduced by O.H.H. Edely, where $A=(a_{nk})_{n,k=1}^{\infty}$ is a non-negative regular matrix and $\mathcal{I}$ and $\mathcal{K}$ represent two non-trivial admissible ideals in $\mathbb{N}$. We study some of its fundamental properties as well as a few inclusion relationships with some other known summability methods. We prove that $A^{\mathcal{K}}$-summability always implies $A^{\mathcal{I}^{\mathcal{K}}}$-summability whereas $A^{\mathcal{I}}$-summability not necessarily implies $A^{\mathcal{I}^{\mathcal{K}}}$-summability. Finally, we give a condition namely $AP(\mathcal{I},\mathcal{K})$ (which is a natural generalization of the condition AP) under which $A^{\mathcal{I}}$-summability implies $A^{\mathcal{I}^{\mathcal{K}}}$-summability.

Keywords: Ideal, Filter, $\mathcal{I}$-convergence, $\mathcal{I}^{\mathcal{K}}$-convergence, $A^{\mathcal{I}}$-summa-bility, $A^{\mathcal{I}^{\mathcal{K}}}$-summability.

1. Introduction

In 2000, Kostrkyo and Salat [12] introduced the notion of ideal convergence. They studied several fundamental properties of $\mathcal{I}$ and $\mathcal{I}^*$-convergence and showed that their idea was the extended version of so many known convergence methods. Based on $\mathcal{I}$-convergence several generalizations were made by researchers and several analytical and topological properties have been investigated (see [1, 9, 11, 15–19, 21, 22] where many more references can be found) and this area becomes one of the most focused areas of research.

In 2011, M. Macaj and M. Sleziak [13] generalized the idea of $\mathcal{I}^*$-convergence to $\mathcal{I}^{\mathcal{K}}$-convergence by involving two ideals $\mathcal{I}$ and $\mathcal{K}$. In the case of $\mathcal{I}^{\mathcal{K}}$-convergence, the convergence along the large set is taken with regard to another ideal $\mathcal{K}$ instead of considering ordinary convergence. So from that point of view the concept of $\mathcal{I}^{\mathcal{K}}$-convergence being an extension of $\mathcal{I}^*$-convergence shows a strong analogy for further investigation. Recent developments in the direction of $\mathcal{I}^{\mathcal{K}}$-convergence from topological aspects can be found from the works of Das et al. [4, 5], Banerjee and Paul [2, 3] and many others.

If $x=(x_k)$ be a real-valued sequence and $A=(a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix, then Ax is the sequence having n^{th} term $A_n(x)=\sum_{k=1}^{\infty}a_{nk}x_k$. A sequence $x=(x_k)$ is said to be A-summable to L, if $\lim\limits_{n\to\infty}A_n(x)=L$. A matrix $A=(a_{nk})_{n,k=1}^{\infty}$ is said to be regular if it maps a convergent sequence into a convergent sequence keeping the same limit i.e., $A\in(c,c)_{reg}$ if $A\in(c,c)$ and $\lim\limits_{n\to\infty}A_n(x)=\lim\limits_{k\to\infty}x_k$. Here c, (c,c), and $(c,c)_{reg}$ denote the collection of all real-valued convergent sequences, collection of all matrices which maps an element of c to an element of c, and the collection of all regular matrices which maps an element of c to an element of c, respectively. The necessary and sufficient Silverman–Toeplitz conditions for an infinite matrix $A=(a_{nk})_{n,k=1}^{\infty}$ to be regular are as follows:

(i) $\sup\limits_{n}\sum_{k=1}^{\infty}|a_{nk}|<\infty$;

(ii) For any $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} a_{nk} = 0$;

(iii) $\lim\limits_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

In 2008, Edely and Mursaleen [7] generalized the notion of A-summability to statistical A-summability by using the concept of natural density. Recently, Edely [6] further extended the notion of statistical A-summability to $A^{\mathcal{I}}$-summability, where $\mathcal{I}$ represents an ideal in $\mathbb{N}$. In this paper we intend to introduce the notion of $A^{\mathcal{I}^{\mathcal{K}}}$-summability which is a natural generalization of $A^{\mathcal{I}^*}$-summability. For more details regarding summability theory, one may refer to [8, 10, 14, 20].

Throughout the paper, we will use (y_n) to denote the image $(A_n(x))$ of the sequence $x = (x_k)$ under the transformation of the non-negative regular infinite matrix A.

2. Definitions and preliminaries

Definition 1. *A collection $\mathcal{I}$ containing subsets of a nonempty set X is called an ideal in X if and only if* (i) $\emptyset \in \mathcal{I}$, (ii) $P, Q \in \mathcal{I}$ *implies* $P \cup Q \in \mathcal{I}$ *(Additive), and* (iii) $P \in \mathcal{I}, Q \subset P$ *implies* $Q \in \mathcal{I}$ *(Hereditary).*

If for any $x \in X$ $\{\{x\}\} \subset \mathcal{I}$ then it is said that $\mathcal{I}$ satisfies the admissibility property or simply is called admissible. Also $\mathcal{I}$ is called non-trivial if $X \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$.

Some standard examples of ideal are given below:

(i) The set $\mathcal{I}_f$ consisting of all subsets of $\mathbb{N}$ having finite cardinality is an admissible ideal in $\mathbb{N}$.

(ii) The set $\mathcal{I}_d$ of all subsets of natural numbers having natural density 0 is an ideal in $\mathbb{N}$ which is also admissible.

(iii) The set $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\}$ is an ideal in $\mathbb{N}$ which also has the so called admissibility property.

(iv) Suppose $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} D_p$, where $D_p \subset \mathbb{N}$ for any $p \in \mathbb{N}$ and for $i \neq j$, $D_i \cap D_j = \emptyset$. Then, the set $\mathcal{I}$ of all subsets of $\mathbb{N}$ which intersects finitely many D_p's forms an ideal in $\mathbb{N}$.

More important examples can be found in [9] and [11].

Definition 2. *A collection $\mathcal{F}$ containing subsets of a nonempty set X is called a filter in X if and only if (i) $\emptyset \notin \mathcal{F}$ (ii) $M, N \in \mathcal{F}$ implies $M \cap N \in \mathcal{F}$ and (iii) $M \in \mathcal{F}, N \supset M$ implies $N \in \mathcal{F}$.*

If $\mathcal{I}$ is a proper non-trivial ideal in X, then the collection $\mathcal{F}(\mathcal{I}) = \{M \subset X : \exists\, P \in \mathcal{I}\, such\, that\, M = X \setminus P\}$ forms a filter in X. It is known as the filter associated with the ideal $\mathcal{I}$.

Definition 3 [12]. *Let $\mathcal{I}$ be an ideal in $\mathbb{N}$ which satisfies the admissibility property. A real-valued sequence $x = (x_k)$ is called $\mathcal{I}$-convergent to l if for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ is contained in $\mathcal{I}$. The number l is called the $\mathcal{I}$-limit of the sequence $x = (x_k)$. Symbolically, $\mathcal{I} - \lim x = l$.*

Definition 4 [12]. *Let $\mathcal{I}$ be an ideal in $\mathbb{N}$ which satisfies the admissibility property. A sequence $x = (x_k)$ is called $\mathcal{I}^*$-convergent to l, if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\}$ in the associated filter $\mathcal{F}(\mathcal{I})$, for which $\lim\limits_{k} x_{m_k} = l$ holds.*

Definition 5 [13]. *Let $\mathcal{I}, \mathcal{K}$ denote two ideals in $\mathbb{N}$. A sequence $x = (x_k)$ is called $\mathcal{I}^{\mathcal{K}}$-convergent to l if the associated filter $\mathcal{F}(\mathcal{I})$ contains a set M such that the sequence $y = (y_k)$ defined by*

$$y_k = \begin{cases} x_k, & k \in M, \\ l, & k \notin M \end{cases}$$

is $\mathcal{K}$-convergent to l.

If we consider $\mathcal{K} = \mathcal{I}_f$ then $\mathcal{I}^{\mathcal{K}}$-convergence concept coincides with $\mathcal{I}^*$-convergence [12].

Definition 6 [13]. *Let $\mathcal{K}$ be an ideal in $\mathbb{N}$. Then, $P \subset_{\mathcal{K}} Q$ denotes the property $P \setminus Q \in \mathcal{K}$. Also $P \subset_{\mathcal{K}} Q$ and $Q \subset_{\mathcal{K}} P$ together implies $P \sim_{\mathcal{K}} Q$. Thus $P \sim_{\mathcal{K}} Q$ if and only if $P \triangle Q \in \mathcal{K}$. A set P is said to be $\mathcal{K}$-pseudointersection of a system $\{P_i : i \in \mathbb{N}\}$ if for every $i \in \mathbb{N}$ $P \subset_{\mathcal{K}} P_i$ holds.*

Definition 7 [13]. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals on $\mathbb{N}$. Then $\mathcal{I}$ is said to have the additive property with respect to $\mathcal{K}$ or the condition $AP(\mathcal{I}, \mathcal{K})$ holds if every sequence $(F_n)_{n\in\mathbb{N}}$ of sets from $\mathcal{F}(\mathcal{I})$ has $\mathcal{K}$-pseudointersection in $\mathcal{F}(\mathcal{I})$.*

Definition 8 [6]. *A real-valued sequence $x = (x_k)$ is said to be $A^{\mathcal{I}}$-summable to a real number L, if the transformed sequence $(A_n(x))$ is $\mathcal{I}$-convergent to L. Symbolically, it is written as $A^{\mathcal{I}} - \lim x_k = L$.*

Definition 9 [6]. *A real-valued sequence $x = (x_k)$ is said to be $A^{\mathcal{I}^*}$-summable to a real number L, if there exists a set $M = \{m_1 < m_2 < ... < m_i < ...\} \in \mathcal{F}(\mathcal{I})$ such that*

$$\lim_{i\to\infty} \sum_k a_{m_i k} x_k = \lim_{i\to\infty} y_{m_i} = L.$$

3. Main results

Throughout the section, for a sequence $x = (x_k)$ we will use $y = (y_n)$ to denote the transformed sequence $(A_n(x))$ where $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$.

Definition 10. *Let $A = (a_{nk})_{n,k=1}^{\infty}$ be a non-negative regular matrix and suppose $\mathcal{I}, \mathcal{K}$ be two admissible ideals in $\mathbb{N}$. A real-valued sequence $x = (x_k)$ is said to be $A^{\mathcal{I}^{\mathcal{K}}}$-summable to $L \in \mathbb{R}$, if there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $z = (z_k)$ defined by*

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}$$

is $\mathcal{K}$-convergent to L, where the sequence $y = (y_n)$ is defined as

$$y_n = A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

In this case we write, $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$.

Example 1. Consider the decomposition of $\mathbb{N}$ given by

$$\mathbb{N} = \bigcup_{i=1}^{\infty} D_i, \quad D_i = \{2^{i-1}(2s-1) : \ s = 1, 2, 3, ...\}.$$

Let $\mathcal{I}$ denotes the ideal consisting of all subsets of $\mathbb{N}$ which intersects finitely many of D_i's. Consider the sequence $x = (x_k)$ defined by $x_k = 1/i$ if $k \in D_i$ and the infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ as

$$a_{nk} = \begin{cases} 1, & k = n+2, \\ 0, & otherwise. \end{cases}$$

Then, the sequence is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to 0 for $\mathcal{K} = \mathcal{I}$.

Justification: Clearly,

$$y_n = \sum_{k=1}^{\infty} a_{nk} x_k = \frac{1}{i}, \quad n+2 \in D_i.$$

Let $M = \mathbb{N} \setminus D_1$. Then, $M \in \mathcal{F}(\mathcal{I})$ and it is easy to verify that the sequence $z = (z_k)$ defined by

$$z_k = \begin{cases} y_k, & k \in M, \\ 0, & k \notin M \end{cases}$$

is $\mathcal{I}$-convergent to 0. Hence, $A^{\mathcal{I}^{\mathcal{I}}} - \lim x_k = 0$.

Theorem 1. *Let $A^{\mathcal{I}^*} - \lim x_k = L$ then $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$.*

P r o o f. Let $A^{\mathcal{I}^*} - \lim x_k = L$. Then, there exists a set

$$M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{I})$$

such that $\lim_i y_{m_i} = L$. This implies that the sequence $z = (z_k)$ defined as

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}$$

is usual convergent to L. Now by Theorem 2.1 of [11], we can say that for any ideal $\mathcal{K}$, the sequence $z = (z_k)$ is $\mathcal{K}$-convergent to L. Hence, $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$. □

Theorem 2. *Let $A^{\mathcal{K}} - \lim x_k = L$ then $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$.*

P r o o f. Since $A^{\mathcal{K}} - \lim x = L$, so for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : |y_k - L| \geq \varepsilon\} \in \mathcal{K}. \tag{3.1}$$

Choose $M = \mathbb{N}$ from $\mathcal{F}(\mathcal{I})$. Consider the sequence $z = (z_k)$ defined by $z_k = y_k, \ k \in M$. Then, using (3.1), we get for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : |z_k - L| \geq \varepsilon\} \in \mathcal{K}$$

i.e. $z = (z_k)$ is $\mathcal{K}$-convergent to L. Hence $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$. □

Remark 1. Converse of the above theorem is not necessarily true.

Example 2. Consider the ideals

$$\mathcal{I}_c = \{B \subseteq \mathbb{N} : \sum_{b \in B} b^{-1} < \infty\}, \quad \mathcal{I}_d = \{B \subseteq \mathbb{N} : d(B) = 0\}$$

and the infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ defined by

$$a_{nk} = \begin{cases} 1, & k = n, \\ 0, & otherwise. \end{cases}$$

Let $x = (x_k)$ be the sequence defined as

$$x_k = \begin{cases} 1, & k \text{ is prime}, \\ 0, & k \text{ is not prime}. \end{cases}$$

Then, there exists set M of all non prime numbers $\in \mathcal{F}(\mathcal{I}_{\mathrm{d}})$ such that the sequence $z = (z_k)$ defined as

$$z_k = \begin{cases} y_k, & k \in M, \\ 0, & k \notin M \end{cases}$$

is $\mathcal{I}_c$-converegnt to 0. Hence, $A^{\mathcal{I}_d{}^{\mathcal{I}_c}} - \lim x_k = 0$. But we claim that $A^{\mathcal{I}_c} - \lim x_k \neq 0$. Because if $A^{\mathcal{I}_c} - \lim x_k = 0$, then for any particular ε with $0 < \varepsilon < 1$, we have the set

$$\{k \in \mathbb{N} : |y_k - 0| \geq \varepsilon\} = \text{set of all prime numbers} \in \mathcal{I}_c,$$

it is a contradiction.

The next theorem gives the condition under which $A^{\mathcal{I}^{\mathcal{K}}}$-summability implies $A^{\mathcal{K}}$-summability.

Theorem 3. *Let $\mathcal{I}$ and $\mathcal{K}$ be two admissible ideals in $\mathbb{N}$. If $\mathcal{I} \subseteq \mathcal{K}$ then $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$ implies $A^{\mathcal{K}} - \lim x_k = L$.*

P r o o f. Let $\mathcal{I} \subseteq \mathcal{K}$. Then, $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$ gives the assurance of the existence of a set $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $z = (z_k)$ defined as

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}$$

is $\mathcal{K}$-convergent to L and subsequently, we have

$$\forall \varepsilon > 0, \quad \{k \in M : |y_k - L| \geq \varepsilon\} \in \mathcal{K}. \tag{3.2}$$

Now as the inclusion

$$\{k \in \mathbb{N} : |y_k - L| \geq \varepsilon\} \subseteq \{k \in M : |y_k - L| \geq \varepsilon\} \cup (\mathbb{N} \setminus M)$$

holds and by our assumption, $\mathbb{N} \setminus M \in \mathcal{I} \subseteq \mathcal{K}$, from (3.2) we have

$$\{k \in \mathbb{N} : |y_k - L| \geq \varepsilon\} \in \mathcal{K}.$$

Hence, $A^{\mathcal{K}} - \lim x_k = L$. □

Theorem 4. *If every subsequence of $x = (x_k)$ is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L, then x is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L.*

P r o o f. If possible let us assume the contrary. Then, for every $M \in \mathcal{F}(\mathcal{I})$, the sequence $z = (z_k)$ defined as

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}$$

is not $\mathcal{K}$-convergent to L. In other words, for any $M \in \mathcal{F}(\mathcal{I})$, there exists an $\varepsilon_M > 0$ such that

$$B = M \cap \{k \in \mathbb{N} : |y_k - L| \geq \varepsilon_M\} \notin \mathcal{K}.$$

Since $\mathcal{K}$ is admissible, so B is infinite. Let $B = \{b_1 < b_2 < ... < b_k < ...\}$. Construct a subsequence $w = (w_k)$ defined as $w_k = y_{b_k}$ for $k \in \mathbb{N}$. Then, $A^{\mathcal{I}^{\mathcal{K}}} - \lim w_k \neq L$, we get a contradiction to the hypothesis. □

Theorem 5. *Let $x = (x_k)$ be a sequence such that $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$. Then, every subsequence of x is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L if and only if both $\mathcal{I}$ and $\mathcal{K}$ does not contain infinite sets.*

P r o o f. There are two possible cases.

Case I. Let $\mathcal{K}$ contain an infinite set. Suppose C be an infinite set and $C \in \mathcal{K}$. Then, $\mathbb{N} \setminus C \in \mathcal{F}(\mathcal{K})$ and $\mathbb{N} \setminus C$ is also infinite. Let $\varepsilon > 0$ be arbitrary. Choose $L_1 \in \mathbb{R}$ such that $L_1 \neq L$. Consider the infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$, defined as

$$a_{nk} = \begin{cases} 1, & k = n, \\ 0, & otherwise, \end{cases}$$

and the sequence $x = (x_k)$ such that

$$x_k = \begin{cases} L_1, & k \in C, \\ L, & k \in \mathbb{N} \setminus C. \end{cases}$$

Then,

$$\{k \in \mathbb{N} : |y_k - L| \geq \varepsilon\} \subseteq C \in \mathcal{K}.$$

This means that x is $A^{\mathcal{K}}$-summable to L. Therefore by Theorem 2, x is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L. But clearly the subsequence $(x_k)_{k \in C}$ of x is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L_1 and not to L.

Case II. Let $\mathcal{K}$ does not contain an infinite set. Then $\mathcal{K} = \mathcal{I}_f$ and $A^{\mathcal{I}^{\mathcal{K}}}$-summability concept coincides with $A^{\mathcal{I}^*}$-summability.

Subcase I: if $\mathcal{I}$ contains an infinite set. Let B be any infinite set such that $B \in \mathcal{I}$. Then, $\mathbb{N} \setminus B \in \mathcal{F}(\mathcal{I})$ and $\mathbb{N} \setminus B$ is also infinite. Define a sequence $x = (x_k)$ as

$$x_k = \begin{cases} \xi, & k \in B, \\ L, & k \in \mathbb{N} \setminus B, \end{cases}$$

where $\xi(\neq L) \in \mathbb{R}$. Clearly x is $A^{\mathcal{I}^*}$-summable to L for the infinite matrix considered in Case I. But clearly the subsequence $(x_k)_{k \in B}$ of x is not $A^{\mathcal{I}^*}$-summable to L.

Subcase II: if $\mathcal{I}$ does not contain an infinite set. In this subcase, we have $\mathcal{I} = \mathcal{K} = \mathcal{I}_f$ and therefore $A^{\mathcal{I}^{\mathcal{K}}}$-summability concept coincides with ordinary summability ([10]) so any subsequence of x is ordinary summable to L. □

Remark 2. If a sequence is $A^{\mathcal{I}^{\mathcal{K}}}$-summable then it may not be $A^{\mathcal{I}}$-summable.

Example 3. Let us consider the ideal $\mathcal{I}$ which is defined in Example 1 and the ideal

$$\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a \in A} a^{-1} < \infty\}.$$

Let $M = \{k \in \mathbb{N} : k = 2^p$ for some non-negative integer p$\}$. Then, for the regular matrix $A = (a_{nk})_{n,k=1}^{\infty}$ defined as

$$a_{nk} = \begin{cases} 1, & k = n, \\ 0, & otherwise, \end{cases}$$

the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & k \in M, \\ 0, & k \notin M \end{cases}$$

is $A^{\mathcal{I}^{\mathcal{I}_c}}$-summable to 0 but x is not $A^{\mathcal{I}}$-summable to 0.

Theorem 6. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$. Let $x = (x_k)$ be any real-valued sequence. Then, $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$ implies $A^{\mathcal{I}} - \lim x_k = L$ if and only if $\mathcal{K} \subseteq \mathcal{I}$.*

P r o o f. Let $\mathcal{K} \subseteq \mathcal{I}$ and suppose $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$. Then, the result follows directly from the following inclusion

$$\{k \in \mathbb{N} : |y_k - L| \geq \varepsilon\} \subseteq \{k \in M : |y_k - L| \geq \varepsilon\} \cup (\mathbb{N} \setminus M).$$

For the converse part, we assume the contrary. Then, there exists a set say $C \in \mathcal{K} \setminus \mathcal{I}$. Let L_1 and L_2 be two real numbers such that $L_1 \neq L_2$. Define a sequence $x = (x_k)$ as

$$x_k = \begin{cases} L_1, & k \in C, \\ L_2, & k \in \mathbb{N} \setminus C \end{cases}$$

and the regular matrix $A = (a_{nk})_{n,k=1}^{\infty}$ as

$$a_{nk} = \begin{cases} 1, & k = n, \\ 0, & otherwise. \end{cases}$$

Then, for any $\varepsilon > 0$ we have,

$$\{k \in \mathbb{N} : |y_k - L_2| \geq \varepsilon\} \subseteq C \in \mathcal{K}$$

which means that x is $A^{\mathcal{K}}$-summable to L_2. Therefore by Theorem 2, x is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L_2. By hypothesis x is $A^{\mathcal{I}}$-summable to L_2. Therefore for $\varepsilon = |L_1 - L_2|$,

$$\{k \in \mathbb{N} : |y_k - L_2| \geq |L_1 - L_2|\} = C \in \mathcal{I},$$

it is a contradiction. Hence we must have $\mathcal{K} \subseteq \mathcal{I}$. □

Remark 3. If a sequence is $A^{\mathcal{I}}$-summable then it may not be $A^{\mathcal{I}^{\mathcal{K}}}$-summable. Consider the ideal $\mathcal{I}$ and the sequence $x = (x_k)$ defined in Example 1. Then, proceeding as Example 1 of [6], we can prove that $A^{\mathcal{I}^{\mathcal{I}_f}} - \lim x_k \neq 0$ although $A^{\mathcal{I}} - \lim x_k = 0$.

Theorem 7. *Let $\mathcal{I}$ and $\mathcal{K}$ be two admissible ideals of $\mathbb{N}$ such that the condition $AP(\mathcal{I}, \mathcal{K})$ holds. Then, for a sequence $x = (x_k)$, $A^{\mathcal{I}}$-summability implies $A^{\mathcal{I}^{\mathcal{K}}}$-summability to the same limit.*

P r o o f. Let $A^{\mathcal{I}} - \lim x_k = L$. Choose a sequence of rationales $(\varepsilon_i)_{i\in\mathbb{N}}$. Then, for every i,

$$M_i = \{k \in \mathbb{N} : |y_k - L| < \varepsilon_i\} \in \mathcal{F}(\mathcal{I}).$$

Thus by Definition 7, there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that for any $i \in \mathbb{N}$, $M \setminus M_i \in \mathcal{K}$. Consider the sequence $z = (z_k)_{k\in\mathbb{N}}$ defined by

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M. \end{cases}$$

To complete the proof, it is sufficient to show that the sequence $z = (z_k)$ is $\mathcal{K}$-convergent to L. Now,

$$\begin{aligned} \{k \in \mathbb{N} : |z_k - L| < \varepsilon_i\} &= \{k \in M : |z_k - L| < \varepsilon_i\} \cup \{k \in \mathbb{N} \setminus M : |z_k - L| < \varepsilon_i\} \\ &= (\mathbb{N} \setminus M) \cup \{k \in M : |z_k - L| < \varepsilon_i\} \\ &= (\mathbb{N} \setminus M) \cup (M_i \cap M) \\ &= \mathbb{N} \setminus (M \setminus M_i). \end{aligned}$$

Now as $M \setminus M_i \in \mathcal{K}$, so $\mathbb{N} \setminus (M \setminus M_i) \in \mathcal{F}(\mathcal{K})$ and consequently we have

$$\{k \in \mathbb{N} : |z_k - L| < \varepsilon_i\} \in \mathcal{F}(\mathcal{K})$$

i.e. $\mathcal{K} - \lim z_k = L$. Hence, $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$. This completes the proof. □

Theorem 8. *Let $\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2, \mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$ be admissible ideals in $\mathbb{N}$ satisfying $\mathcal{I}_1 \subseteq \mathcal{I}_2$ and $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Then,*

(i) $A^{\mathcal{I}_1^{\mathcal{K}}} - \lim x_k = L$ *implies* $A^{\mathcal{I}_2^{\mathcal{K}}} - \lim x_k = L$;

(ii) $A^{\mathcal{I}^{\mathcal{K}_1}} - \lim x_k = L$ *implies* $A^{\mathcal{I}^{\mathcal{K}_2}} - \lim x_k = L$.

P r o o f. (i) Suppose $A^{\mathcal{I}_1^{\mathcal{K}}} - \lim x_k = L$. Then, by Definition 10, there exists $M \in \mathcal{F}(\mathcal{I}_1)$ such that the sequence $z = (z_k)$ defined as

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}$$

is $\mathcal{K}$-convergent to L. Now since $M \in \mathcal{F}(\mathcal{I}_1)$, we have $\mathbb{N} \setminus M \in \mathcal{I}_1$ and therefore by hypothesis $\mathbb{N} \setminus M \in \mathcal{I}_2$, which again implies $M \in \mathcal{F}(\mathcal{I}_2)$. Hence we must have that $A^{\mathcal{I}_2^{\mathcal{K}}} - \lim x_k = L$.

(ii) Suppose $A^{\mathcal{I}^{\mathcal{K}_1}} - \lim x_k = L$. Then, by Definition 10, there exists $M \in \mathcal{F}(\mathcal{I}_1)$ such that the sequence $z = (z_k)$ defined as,

$$z_k = \begin{cases} y_k, & k \in M, \\ l, & k \notin M \end{cases}$$

satisfies the following property $\forall \varepsilon > 0$,

$$\{k \in \mathbb{N} : |z_k - l| \geq \varepsilon\} \in \mathcal{K}_1.$$

Now by hypothesis the inclusion $\mathcal{K}_1 \subseteq \mathcal{K}_2$ holds, so we must have for $\forall \varepsilon > 0$,

$$\{k \in \mathbb{N} : |z_k - l| \geq \varepsilon\} \in \mathcal{K}_2.$$

Hence $A^{\mathcal{I}^{\mathcal{K}_2}} - \lim x_k = L$. □

4. Conclusion

Summability plays an important role in mathematics, particularly in mathematical analysis. In this paper, we introduce and investigate a few properties of $A^{\mathcal{I}^{\mathcal{K}}}$-summability. We generate a few examples and counterexamples in order to study some inclusion relationships with some known methods of summability. But the main focus was to link $A^{\mathcal{I}}$ and $A^{\mathcal{I}^*}$-summability with $A^{\mathcal{I}^{\mathcal{K}}}$-summability. We prove that the condition $AP(\mathcal{I},\mathcal{K})$ plays a crucial role in this regard. In the future, this idea can be utilized by the researchers to develop some other forms of summability.

Acknowledgements

The authors thank the anonymous referees for their constructive suggestions to improve the quality of the paper. The first author is grateful to the University Grants Commission, India for their fellowships funding under the UGC-JRF scheme (F. No. 16-6(DEC. 2018)/2019(NET/CSIR)) during the preparation of this paper.

REFERENCES

1. Altinok M., Küçükaslan M. Ideal limit superior-inferior. *Gazi Univ. J. Sci.*, 2017. Vol. 30, No. 1. P. 401–411.
2. Banerjee A. K., Paul M. *Weak and Weak* I^K-convergence in Normed Spaces.* 2018. 10 p. arXiv:1811.06707 [math.GN]
3. Banerjee A. K., Paul M. *A Note on I^K and I^{K^*}-convergence in Topological Spaces.* 2018. 10 p. arXiv:1811.06707v1 [math.GN]
4. Das P., Sleziak M., Toma V. I^K-Cauchy functions. *Topology Appl.*, 2014. Vol. 173, P. 9–27. DOI: 10.1016/j.topol.2014.05.008
5. Das P., Sengupta S., Supina J. I^K-convergence of sequence of functions. *Math. Slovaca.*, 2019. Vol. 69, No. 5. P. 1137–1148. DOI: 10.1515/ms-2017-0296
6. Edely O. H. H. On some properties of A^I-summability and A^{I^*}-summability. *Azerb. J. Math.*, 2021. Vol. 11, No. 1. P. 189–200.
7. Edely O. H. H., Mursaleen M. On statistical A-summability. *Math. Comput. Model*, 2009. Vol. 49, No. 3. P. 672–680. DOI: 10.1016/j.mcm.2008.05.053
8. Freedman A. R., Sember J. J. Densities and summability. *Pacific J. Math.*, 1981. Vol. 95, No. 2. P. 293–305.
9. Gogola J., Mačaj M., Visnyai T. On $\mathcal{I}_c^{(q)}$-convergence. *Ann. Math. Inform.*, 2011. Vol. 38, P. 27–36.
10. Jarrah A. M., Malkowsky E. Ordinary, absolute and strong summability and matrix transformations. *Filomat*, 2003. No. 17. P. 59–78. DOI: 10.2298/FIL0317059J
11. Kostyrko P., Mačaj M., Šalát T., Sleziak M. $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points. *Math. Slovaca*, 2005. Vol. 55, No. 4. P. 443–464.
12. Kostyrko P., Šalát T., Wilczyński W. I-convergence. *Real Anal. Exch.*, 2000–2001. Vol. 26, No. 2. P. 669–686.
13. Mačaj M., Sleziak M. $\mathcal{I}^K$-convergence. *Real Anal. Exch.*, 2010–2011. Vol. 36, No. 1. P. 177–194.
14. Mursaleen M. On some new invariant matrix methods of summability. *Q. J. Math.*, 1983. Vol. 34, No. 1. P. 77–86. DOI: 10.1093/qmath/34.1.77
15. Mursaleen M., Alotaibi A. On $\mathcal{I}$-convergence in random 2-normed spaces. *Math. Slovaca*, 2011. Vol. 61, No. 6. P. 933–940. DOI: 10.2478/s12175-011-0059-5
16. Mursaleen M., Mohiuddine S. A. On ideal convergence in probabilistic normed spaces. *Math. Slovaca*, 2012. Vol. 62, No. 1. P. 49–62. DOI: 10.2478/s12175-011-0071-9
17. Nabiev A., Pehlivan S., Gürdal M. On $\mathcal{I}$-Cauchy sequences. *Taiwanese J. Math.*, 2007. Vol. 11, No. 2. P. 569–576.
18. Šalát T., Tripathy B. C., Ziman M. On some properties of $\mathcal{I}$-convergence. *Tatra Mt. Math. Publ.*, 2004. Vol. 28, No. 2. P. 274–286.

19. Savaş E. Generalized asymptotically $\mathcal{I}$-lacunary statistical equivalent of order α for sequences of sets. *Filomat*, 2017. Vol. 31, No. 6. P. 1507–1514. DOI: 10.2298/FIL1706507S

20. Savaş E. General inclusion relations for absolute summability. *Math. Inequal. Appl.*, 2005. Vol. 8, No. 3. P. 521–527.

21. Savaş E., Gürdal M. Ideal convergent function sequences in random 2-normed spaces. *Filomat*, 2016. Vol. 30, No. 3. P. 557–567. DOI: 10.2298/FIL1603557S

22. Tripathy B. C., Hazarika B. Paranorm I-convergent sequence spaces. *Math. Slovaca*, 2009. Vol. 59, No. 4. P. 485–494. DOI: 10.2478/s12175-009-0141-4