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TERNARY ∗-BANDS ARE GLOBALLY DETERMINED

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Abstract: A non-empty set S together with the ternary operation denoted by juxtaposition is said to be ternary semigroup if it satisfies the associativity property $ab(cde) = a(bcd)e = (abc)de$ for all $a, b, c, d, e \in S$. The global set of a ternary semigroup S is the set of all non empty subsets of S and it is denoted by $P(S)$. If S is a ternary semigroup then $P(S)$ is also a ternary semigroup with a naturally defined ternary multiplication. A natural question arises: "Do all properties of S remain the same in $P(S)$?" The global determinism problem is a part of this question. A class K of ternary semigroups is said to be globally determined if for any two ternary semigroups S_1 and S_2 of K, $P(S_1) \cong P(S_2)$ implies that $S_1 \cong S_2$. So it is interesting to find the class of ternary semigroups which are globally determined. Here we will study the global determinism of ternary ∗-band.

Keywords: Rectangular ternary band, Involution ternary semigroup, Involution ternary band, Ternary ∗-band, Ternary projection.

1. Introduction

In our previous paper [7] we have discussed the global determinism of ternary groups and finite left zero ternary semigroups. Here we will discuss some properties of a rectangular ternary band and of a proper rectangular ternary band and also discuss the global determinism problem of ternary ∗-band.

Let us briefly present the literature on the problem of global determinism. In 1960 B.M. Shane formulated the importance of studying the problem of global determinism. In 1967, T. Tamura and J. Shafer [11] proved that groups are globally determined. In 1984, T. Tamura [10] proved that rectangular groups are globally determined. In 1984, M. Gould and J.A. Iskra [4] also studied some globally determined classes of semigroups. M. Gould, J.A. Iskra, C. Tsinakis [5, 6] also studied the global determinism problem of semigroup theory. In 1984, Y. Kobayashi [9] proved that semilattices are globally determined. At present, the problem of global determinism is a well-known research problem. M. Vinčić [13] established in 2001, that ∗-bands are globally determined. In 2014, A. Gan, X. Zhao and Y. Shao [1] proved that clifford semigroups are globally determined. In 2015, A. Gan, X. Zhao and M. Ren [3] studied the global determinism of semigroups having regular globals. A. Gan, X. Zhao and Y. Shao [2] also discussed the globals of idempotent semigroups in 2016 and in 2017, B. Yu, X. Zhao, A. Gan [12] proved that idempotent semigroups are globally determined.

So the problem of global determinism is important and relevant in the ternary theory of semigroups. Here we will prove that ternary ∗-bands are globally determined.

2. Preliminaries

First we provide the basic definitions and results which are used in the rest of the paper.

Definition 1. *A ternary semigroup S is said to be left (resp. right) zero ternary semigroup if for $a, b, c \in S$, $abc = a$ (resp. $abc = c$).*

Definition 2. *A ternary semigroup S is said to be a ternary band if every element of S is idempotent, i.e. $a^3 = a$ for all $a \in S$.*

Definition 3. *A ternary semigroup S is said to be rectangular ternary band if $aba = a$ for all $a, b \in S$.*

Although the definition of rectangular ternary band and rectangular band in binary are similar, but all the rectangular ternary bands are not rectangular bands in binary. The following example illustrates this fact.

Example 1. Let $M_2(\mathbb{R})$ is the set of all 2×2 matrices over $\mathbb{R}$. This is a ternary semigroup w.r.t. the natural ternary matrix multiplication.
(i) $\left\{ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \right\} \in M_2(\mathbb{R})$. This is a rectangular ternary band w.r.t. natural ternary matrix multiplication.
(ii) $\left\{ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \right\} \in M_2(\mathbb{R})$. This is a rectangular ternary band w.r.t. natural ternary matrix multiplication.

Lemma 1. *A ternary semigroup S is rectangular ternary band if and only if $ababa = a$ and $abcde = ace$ for all $a, b, c, d, e \in S$.*

P r o o f. Let S be a rectangular ternary band. Then $aba = a$ for all a, $b \in S$. Therefore,

$$ababa = (aba)ba = aba = a \quad \text{for all} \quad a, b \in S.$$

Now

$$\begin{aligned} abcde &= a(b(adc)b)(cde) = (aba)(d(cbc)d)e = a(dcd)e = (adc)(d(ace)d)e \\ &= (a(dcd)a)c(ede) = (ada)ce = ace. \end{aligned}$$

Conversely, suppose that $ababa = a$ and $abcde = ace$. Then

$$aba = (ababa)ba = a(bab)aba = aaa = a^3 = a.$$

Therefore, S is the rectangular ternary band. □

Lemma 2. *A ternary semigroup S is a rectangular ternary band if and only if it can be expressed as a cartesian product of left zero and right zero ternary semigroups.*

P r o o f. Let S be a rectangular ternary band and u be a fixed element of S. Define two sets L, R such that

$$L = \{xuu : \ x \in S\}, \quad R = \{uux : \ x \in S\}.$$

Since

$$(xuu)(yuu)(zuu) = x(uuy)u(uzu)u = xuu$$

for all $xuu, yuu, zuu \in L$, we have, L is left zero ternary semigroup.
Similarly,

$$(uux)(uuy)(uuz) = u(uxu)u(yuu)z = uuz$$

for all uux, uuy, $uuz \in R$ implies that R is right zero ternary semigroup.

Define a mapping $\phi : S \longrightarrow L \times R$ such that $\phi(x) = (xuu, uux)$ for all $x \in S$. Here the ternary operation on $L \times R$ is as follows:

$$(a, b)(c, d)(e, f) = (ace, bdf) = (a, f) \quad \text{for all} \quad (a, b), \ \ (c, d), \ \ (e, f) \in L \times R.$$

Let $\phi(x) = \phi(y)$. This implies that $xuu = yuu$, $uux = uuy$. Now

$$x = xux = xuuux = (xuu)ux = (yuu)ux = yu(uux) = yu(uuy) = yuuuy = yuy = y.$$

Therefore, ϕ is one-to-one mapping.

$$\phi(xuz) = (xuzuu, uuxuz) = ((xuu)(uuu)(zuu), (uux)(uuu)(uuz)) = (xuu, uuz).$$

Therefore, ϕ is an onto mapping.

$$\begin{aligned}\phi(x)\phi(y)\phi(z) &= (xuu, uux)(yuu, uuy)(zuu, uuz)\\ &= (xuuyuuzuu, uuxuuyuuz) = (xuu, uuz) = \phi(xyz).\end{aligned}$$

Thus ϕ is a ternary homomorphism. Hence ϕ is an isomorphism and $S \cong L \times R$.

Conversely, suppose that S is isomorphic to $L \times R$, where L, R are left zero, right zero ternary semigroups respectively. Now $(a, c)(e, g)(a, c) = (aea, cgc) = (a, c)$. Therefore, $L \times R$ is a rectangular ternary band. Since S is isomorphic to a rectangular ternary band, S is also rectangular ternary band. □

Let $S \cong L \times R$ where L be the left zero, R be the right zero ternary semigroup and μ be an isomorphism from S to $L \times R$. The ternary operation on $L \times R$ is defined as

$$(a, b)(c, d)(e, f) = (ace, bdf) = (a, f) \quad \text{for all} \quad (a, b), (c, d), (e, f) \in L \times R.$$

There are some notions defined as follows:
Let $A \in P(S)$, where $P(S)$ is the global of S.

$\pi_L(A) = \{i \in L : \ \exists \ k \in R \text{ such that } (i, k) \in \mu(A)\}$.
$\pi_R(A) = \{k \in R : \ \exists \ i \in L \text{ such that } (i, k) \in \mu(A)\}$.

If $S = L \times R$ then for any $A \in P(S)$, we have

$\pi_L(A) = \{i \in L : \ \exists \ k \in R \text{ such that } (i, k) \in (A)\}$.
$\pi_R(A) = \{k \in R : \ \exists \ i \in L \text{ such that } (i, k) \in (A)\}$.

Definition 4. *A rectangular ternary band S is said to be a proper rectangular ternary band if it is not left zero, right zero and lateral zero ternary semigroup. By the notation TRB_2 we mean the proper rectangular ternary band.*

Definition 5. *(i) A ternary semigroup S is said to be an involution ternary semigroup if it is equipped with a unary operation $*$ such that $(xyz)^* = z^*y^*x^*$ and $(x^*)^* = x$.*

(ii) An idempotent involution ternary semigroup is said to be an involution ternary band.

Definition 6. *(i) Let S be an involution ternary semigroup. If for each $x \in S$, $xx^*x = x$, $x^*xx^* = x^*$ and $x^2y = xy^2$ for all $x, y \in S$, then S is said to be a ternary $*$-semigroup.*

(ii) A ternary semigroup S is said to be ternary $*$-band if S is an idempotent ternary $*$-semigroup.

Definition 7. *Let S be an involution ternary semigroup. An element $x \in S$ is said to be a projection of S if it is idempotent and is a fixed point of involution, i.e. $x^3 = x$ and $x^* = x$.*

Definition 8. *Let S be an involution ternary semigroup. Then $X \subseteq S$ is said to be an involution ternary subsemigroup if X is a subsemigroup of S and $X^* \subseteq X$, where $X^* = \{x^* : x \in X\}$.*

Two important notations of this paper are as follows:
The set of all subsemigroups of a ternary $*$-band S is denoted by $\mathcal{S}(S)$ and

$$Ch(S) = \{X \in \mathcal{S}(S) : \; X = Y^3 \Longrightarrow X = Y \text{ for all } Y \in P(S)\}.$$

Remark 1. Every bijection between two left (resp. right) zero ternary semigroups is the isomorphism between them.

We have already discussed in [7], that finite left (resp. right) zero ternary semigroups are globally determined. In this paper, we generalize this result for arbitrary left (resp. right) zero ternary semigroups.

Here we assume the generalized continuum hypothesis which states that if cardinality of an infinite set lies between that of an infinite set A and that of the power set $P(A)$ of A, then it has the same cardinality as either A or $P(A)$.

Lemma 3. *If $P(S_1) \cong P(S_2)$ then $|S_1| = |S_2|$ where $|S_1|$ and $|S_2|$ denote cardinality of S_1, S_2 respectively.*

P r o o f. To prove the result we consider the following three cases.
Case 1. Suppose that S_1, S_2 both are finite sets. Let $|S_1| = m$ and $|S_2| = n$.
Since $P(S_1) \cong P(S_2)$, so $|P(S_1)| = |P(S_2)|$. Again $|P(S_1)| = 2^m$ and $|P(S_2)| = 2^n$. Therefore, $2^m = 2^n$. This implies that $m = n$.

Case 2. Suppose that S_1 is a finite set and S_2 is an infinite set and $|S_1| = m$. Then $|P(S_1)| = 2^m$, i.e. a finite number but $|P(S_2)|$ is not a finite number. Therefore, $|P(S_1)| \neq |P(S_2)|$. Hence Case 2 is not true.

Case 3. Let us assume that both S_1, S_2 are infinite sets. Then the following three situations may arise.
(i) If S_1 and S_2 both are countable then $|S_1| = |S_2| = \aleph_0$[1]. So there is nothing to prove.
(ii) Suppose S_1 is a countable set and S_2 is an uncountable set. Then $|S_1| = \aleph_0 \Longrightarrow |P(S_1)| = 2^{\aleph_0}$ and $|S_2| \geq 2^{\aleph_0}$. Therefore, $|P(S_2)| > 2^{\aleph_0} = |P(S_1)|$. But this is not possible.
(iii) Suppose S_1 and S_2 both are uncountable. If possible, let $|S_1| \neq |S_2|$. Then $|S_1| = \mathbf{c}$ and $|S_2| = \mathbf{c_1}$ where $\mathbf{c}, \mathbf{c_1} \geq 2^{\aleph_0}$. Therefore, $|P(S_1)| = 2^{\mathbf{c}}$ and $|P(S_2)| = 2^{\mathbf{c_1}}$. Since $\mathbf{c} \neq \mathbf{c_1}$ thus $2^{\mathbf{c}} \neq 2^{\mathbf{c_1}}$. This contradicts our assumption. Therefore, $|S_1| = |S_2|$. □

Theorem 1. *Left (resp. right) zero ternary semigroups are globally determined.*

P r o o f. Let S_1 and S_2 be two left zero ternary semigroups and $\phi : P(S_1) \longrightarrow P(S_2)$ is an isomorphism, i.e. $P(S_1) \cong P(S_2)$. This implies that $|P(S_1)| = |P(S_2)|$. Hence by Lemma 3, $|S_1| = |S_2|$. Thus there is a bijection from S_1 to S_2. Since S_1 and S_2 are left zero ternary semigroups, by Remark 1, it follows that the bijection is an isomorphism. So it is clear that $S_1 \cong S_2$. Hence the class of all left zero ternary semigroups is globally determined.
Similarly, we can show that right zero ternary semigroups are globally determined. □

[1] $\aleph$ is the cardinality of the set of all natural number.

3. Main result

A rectangular ternary band is said to be a proper rectangular ternary band if it is not a left zero, right zero or lateral zero ternary semigroup. In this section, we provide some results of proper rectangular ternary bands and show that the proper rectangular ternary band satisfies the strong isomorphism property. By strong isomorphism property we mean that any isomorphism ϕ from $P(S)$ to $P(S_1)$ is also an isomorphism from S to S_1. Here we also discuss that ternary $*$-bands are globally determined. Unless otherwise stated, in this section, we assume that S is a proper rectangular ternary band.

Lemma 4. *Let S and S_1 be two ternary semigroups such that $P(S) \cong P(S_1)$ and ψ is an isomorphism from $P(S)$ to $P(S_1)$. Then the restriction $\psi \mid_{Ch(S)}$ is a bijection from $Ch(S)$ to $Ch(S_1)$.*

P r o o f. Let $A \in Ch(S)$. This implies that $A \in \mathcal{S}(S)$. Therefore, $\psi(A) \in \mathcal{S}(S_1)$.

Let $\psi(A) = A_1$. If possible, there exists $B_1 \in P(S_1)$ such that $B_1^3 = A_1$. Since ψ is an isomorphism there exists $B \in P(S)$ such that $\psi(B) = B_1$. Therefore, $A_1 = B_1^3 = (\psi(B))^3 = \psi(B^3)$.

Hence $\psi(A) = \psi(B^3)$. This implies that $A = B^3$. Since $A \in Ch(S)$, we have $A = B$. Therefore, $\psi(A) = \psi(B)$. This implies that $A_1 = B_1$. Hence $A_1 \in Ch(S_1)$. Therefore, $\psi \mid_{Ch(S)}$ is a bijection from $Ch(S)$ to $Ch(S_1)$. □

Lemma 5. *Let S be a proper rectangular ternary band such that $S = L \times R$. Then $A \in Ch(S)$ if and only if $|\pi_L(A)| = 1$ or $|\pi_R(A)| = 1$.*

P r o o f. The proof is similar to the binary result of [2]. □

Theorem 2. *Rectangular ternary bands are globally determined.*

P r o o f. Proof of the theorem immediately follows from the binary result of [10]. □

Theorem 3. *Proper rectangular ternary band satisfies the strong isomorphism property.*

P r o o f. The proof is similar to the binary result of [2]. □

A restricted class of a involution ternary semigroup is ternary $*$-band. Unless otherwise stated, in the rest of this section, B denotes an involution ternary band and $\mathcal{S}(B)$ denotes the set of all involution ternary subsemigroups of B.

Lemma 6. *For any involution ternary band B, $\mathcal{S}(B)$ coincides with the set of all projections of $P(B)$. Therefore, if B_1 and B_2 be two involution ternary bands, then every isomorphism $\psi : P(B_1) \longrightarrow P(B_2)$ induces a bijection from $\mathcal{S}(B_1)$ to $\mathcal{S}(B_2)$.*

P r o o f. Since B is a involution ternary band, for any subset X of B, we have $X \subset X^3$ and $(X^*)^* = X$.

Now let $X \in \mathcal{S}(B)$. This implies $X^3 \subseteq X$, $X^* \subseteq X$ and $X \subseteq B$. Thus $X^3 = X$ and $(X^*)^* \subseteq X^*$. Hence $X^3 = X$ and $X \subseteq X^*$.

Therefore, $X^3 = X$ and $X^* = X$. So X is a projection of $P(B)$.

Conversely, if X is a projection of $P(B)$ then $X^3 = X$ and $X = X^*$. Therefore, $X^3 \subseteq X$ and $X^* \subseteq X$. Thus $X \in \mathcal{S}(B)$.

Hence $\mathcal{S}(B)$ is the set of all projections of $P(B)$.

Let $\psi : P(B_1) \longrightarrow P(B_2)$ be an isomorphism, where B_1, B_2 be two involution ternary bands. We have to show that $X \in \mathcal{S}(B_1)$ implies that $\psi(X) \in \mathcal{S}(B_2)$. Let us extend ψ on $(P(B_1))^*$ as $\psi : (P(B_1))^* \longrightarrow (P(B_2))^*$ such that $\psi(X^*) = (\psi(X))^*$.

If $X = X^*$ then $\psi(X) = \psi(X^*) = (\psi(X))^*$. Hence $X \in \mathcal{S}(B_1)$ implies that $\psi(X) \in \mathcal{S}(B_2)$. Thus $\psi(\mathcal{S}(B_1)) \subseteq \mathcal{S}(B_2)$. Similarly, $\psi^{-1}(\mathcal{S}(B_2)) \subseteq \mathcal{S}(B_1)$. This implies that

$$\psi(\psi^{-1}(\mathcal{S}(B_2))) \subseteq \psi(\mathcal{S}(B_1)) \Longrightarrow \mathcal{S}(B_2) \subseteq \psi(\mathcal{S}(B_1)).$$

Therefore, $\mathcal{S}(B_2) = \psi(\mathcal{S}(B_1))$. Thus ψ induces a bijection between $\mathcal{S}(B_1)$ and $\mathcal{S}(B_2)$. □

Lemma 7. *Let B_1, B_2 be two involution ternary bands. Any isomorphism from $P(B_1)$ to $P(B_2)$ induces a bijection from $Ch(B_1)$ to $Ch(B_2)$.*

P r o o f. If we are able to show that for any isomorphism $\psi : P(B_1) \longrightarrow P(B_2)$, $\psi(Ch(B_1)) = Ch(B_2)$ then $\psi : Ch(B_1) \longrightarrow Ch(B_2)$ becomes onto mapping. Again since ψ is an isomorphism from $P(B_1)$ to $P(B_2)$ and $Ch(B_1) \subseteq P(B_1)$ so $\psi : Ch(B_1) \longrightarrow Ch(B_2)$ is one-to-one mapping hence a bijection.

Let $X \in Ch(B_1)$. If possible there exists $Y' \in P(B_2)$ such that $Y'^3 = \psi(X)$. Then there exists $Y \in P(B_1)$ such that $\psi(Y) = Y'$. Therefore,

$$\psi(X) = Y'^3 = (\psi(Y))^3 = \psi(Y^3).$$

This implies that $X = Y^3$ because $X \in Ch(B_1)$. Hence $X = Y$. Thus $\psi(X) = \psi(Y)$. Therefore, $\psi(X) = Y'$. So $X \in Ch(B_1)$ implies that $\psi(X) \in Ch(B_2)$. Hence $\psi(Ch(B_1)) \subseteq Ch(B_2)$.

Since ψ is an isomorphism, ψ^{-1} is also an isomorphism. Hence $Y \in Ch(B_2)$ implies that $\psi^{-1}(Y) \in Ch(B_1)$. Thus

$$\psi^{-1}(Ch(B_2)) \subseteq Ch(B_1) \Longrightarrow \psi(\psi^{-1}(Ch(B_2))) \subseteq \psi(Ch(B_1)) \Longrightarrow Ch(B_2) \subseteq \psi(Ch(B_1)).$$

Hence $\psi(Ch(B_1)) = Ch(B_2)$.

Therefore, ψ is a bijection from $Ch(B_1)$ to $Ch(B_2)$. □

Let us a define partial ordering and a chain on a ternary band as follows.

Definition 9. *Let B be a ternary band. A partial order $\leq$ on a ternary band B can be defined as $a \leq b$ if and only if*

$$a = a^2 b = ab^2 = b^2 a = ba^2.$$

Definition 10. *Let A be a non empty subset of a ternary band B. Then A is said to be a chain of B if for all $a, b \in A$ either $a \leq b$ or $b \leq a$.*

Lemma 8. *Let B be a ternary $*$-band. Then $X \in Ch(B)$ if and only if X is a chain of projections.*

P r o o f. Let B be a ternary $*$-band. Then

$$x^3 = x, \quad (x^*)^* = x, \quad x^*xx^* = x^*, \quad xx^*x = x \quad \text{and} \quad x^2y = xy^2 \quad \text{for all} \quad x, y \in B.$$

Suppose $X \in Ch(B)$ and $x, y \in X$. If possible let $x^2y \notin \{x, y\}$. Construct $Y = X \setminus \{x^2y\}$. Now $x^2y \in Y^3$ and $Y \subseteq X$. Since B is ternary $*$-band and $Y \subseteq X \subseteq B$, we have $Y \subseteq Y^3$. Thus it follows that $Y^3 = X$. Again

$$Y \subseteq X \Longrightarrow Y^3 \subseteq X^3 = X.$$

Now

$$Y = X \setminus \{x^2y\} \subseteq Y^3 \subseteq X.$$

Since $x^2y \in Y^3$, it follows that $Y^3 \neq Y$. Hence $Y^3 = X$. By definition of $Ch(B)$ we find that $Y = X$. This contradicts our assumption that $x^2y \notin \{x, y\}$. Therefore, $x^2y \in \{x, y\}$. Similarly, $xy^2, yx^2, y^2x, xyx, yxy$ all are in $\{x, y\}$. Hence X is a chain.

Now our aim is to show that $x \in X$ implies that x is a projection. Since $X \in Ch(B)$, $X^3 = X$. Therefore, $x^* \in X$ for all $x \in X$. Now $x^2x^* \in \{x, x^*\}$.

Suppose $x^2x^* = x$. This implies that

$$(x^2x^*)^* = x^* \Longrightarrow xx^*x^* = x^* \Longrightarrow x^2x^* = x^*.$$

Therefore, $x = x^*$. Hence x is the projection.

Again if

$$x^2x^* = x^* \Longrightarrow xx^*x^* = x \Longrightarrow x^2x^* = x.$$

Hence $x^* = x$. Therefore, x is the projection.

This implies that X is a chain of projections.

Conversely, suppose that $X \in P(B_1)$ is a chain of projections. Suppose there exists $Y \in P(B_1)$ such that $Y^3 = X$. It is clear that $Y \subseteq Y^3$. Therefore, $Y \subseteq X$. Subset of a chain must be a chain. Hence $Y^3 \subseteq Y$. This implies $X \subseteq Y \subseteq X$. Thus $X = Y$. Therefore, $X \in Ch(B)$. □

Let B be a ternary $*$-band. Let define a partial ordering " $\leq$ " on $P(B)$ as follows:

$X \leq Y$ if and only if $X = X^2Y = YX^2$ for all $X, Y \in P(B)$.

$X \twoheadrightarrow Y$ if and only if $X < Y$ in $\mathcal{S}(B)$ and there does not exist any $Z \in \mathcal{S}(B)$ such that $X < Z < Y$.

Again $X \longrightarrow Y$ if and only if $X < Y$ and there does not exist any $Z \in Ch(B)$ such that $X < Z < Y$.

It is clear that

$$X \twoheadrightarrow Y \Longrightarrow X \longrightarrow Y.$$

Remark 2. If B is a ternary band then B is also a ternary semigroup. So ideal of a ternary band is the same as the ideal of a ternary semigroup.

Lemma 9. *Let B be a ternary band. Thus there exists some ternary semilattice S such that there is a homomorphism $\sigma : B \longrightarrow S$ such that $\sigma(B) = S$.*

P r o o f. Let B be a ternary band. Define

$$I_a = \{xay : x, y \in B\}$$

for any $a \in B$. Then I_a is an ideal of B, generated by a.

Let us define a relation ρ on B such that $a\rho b$ if and only if $I_a = I_b$. There is no doubt that ρ is an equivalence relation on B. Now let $I_{a_1} = I_{b_1}$, $I_{a_2} = I_{b_2}$, $I_{a_3} = I_{b_3}$. Therefore,

$$b_1 = x_1a_1y_1, \quad b_2 = x_2a_2y_2, \quad b_3 = x_3a_3y_3.$$

Hence

$$\begin{aligned} b_1b_2b_3 &= x_1a_1y_1x_2a_2y_2x_3a_3y_3 = x_1(a_1y_1x_2{a_2}^2)a_2y_2x_3a_3y_3 \\ &= x_1(a_1y_1x_2{a_2}^2)(a_1y_1x_2{a_2}^2)(a_1y_1x_2{a_2}^2)a_2y_2x_3a_3y_3 = x_1X(a_2a_1y_1x_2a_2y_2x_3)a_3y_3 \\ &= x_1X(a_2a_1Y{a_3}^2)a_3y_3 = x_1X(a_2a_1Y{a_3}^2)(a_2a_1Y{a_3}^2)(a_2a_1Y{a_3}^2)a_3y_3 \\ &= (x_1Xa_2a_1Y{a_3}^2a_2a_1Ya_3)(a_3a_2a_1)(Ya_3y_3) = X_1(a_3a_2a_1)Y_1, \end{aligned}$$

where

$$X = a_1y_1x_2{a_2}^2a_1y_1x_2a_2, \quad Y = y_1x_2a_2y_2x_3, \quad X_1 = x_1Xa_2a_1Y{a_3}^2a_2a_1Ya_3, \quad Y_1 = Ya_3y_3.$$

Therefore, $b_1b_2b_3 \in I_{a_3a_2a_1}$. Similarly, we can show that $a_3a_2a_1 \in I_{b_1b_2b_3}$.

Thus it is clear that $I_{a_3a_2a_1} = I_{b_1b_2b_3}$. Now

$$(i) \qquad abc = abcabcabc = a(bca)bcabc \in I_{bca}.$$

Similarly, we can show that $bca \in I_{abc}$. This implies that $I_{abc} = I_{bca}$. Again

$$\begin{aligned} (ii) \qquad abc &= abcabcabc = a(bcabc)(bcabc)(bcabc)abc \\ &= (abcab)(cbcabcbca)(bcabc) = (abcab)(cbcxa)(bcabc) \\ &= (abcab)(cbcxa)(cbcxa)(cbcxa)(bcabc) = (abcabcbcx)acb(cxacbcxabcabc), \end{aligned}$$

where $x = abcbc$. Therefore, $abc \in I_{acb}$. Hence $I_{abc} = I_{acb}$. Thus

$$I_{abc} = I_{acb} = I_{bac} = I_{bca} = I_{cab} = I_{cba}.$$

This shows that $I_{a_1a_2a_3} = I_{b_1b_2b_3}$. Therefore, ρ is a ternary congruence relation on B.

Now B/ρ be the set of all equivalence classes of the congruence relation and the elements are denoted by $\bar{a}$ for $a \in B$. Define a ternary operation on B/ρ by $\bar{a}\bar{b}\bar{c} = \overline{abc}$.

Now we show that B/ρ is a ternary semilattice w.r.t. above defined ternary operation. This is clear from the above discussion that B/ρ is a commutative ternary semigroup. Again since B is ternary band,

$$\bar{a}\bar{a}\bar{a} = \overline{aaa} = \bar{a^3} = \bar{a}.$$

Thus B/ρ is also a ternary band. Now

$$a^2b = a^2b^3 = a(ab^2)b \in I_{ab^2}, \quad ab^2 = a^3b^2 = a(a^2b)b \in I_{a^2b}.$$

Therefore, $I_{a^2b} = I_{ab^2}$. This implies that $\overline{a^2b} = \overline{ab^2}$. Hence $\bar{a}^2\bar{b} = \bar{a}\bar{b}^2$. Thus B/ρ is a ternary semilattice.

Now we define a mapping $\sigma : B \longrightarrow B/\rho$ such that $\sigma(a) = \bar{a}$. Then σ is an epimorphism. If we consider $S = B/\rho$ then there exists a ternary semilattice which is homomorphic image of B. $\square$

Lemma 10. *Let B be a ternary $*$-band and S be a ternary semilattice image of B. If $X, Y \in Ch(B)$ are such that $X < Y$ and $\sigma(X) \twoheadrightarrow \sigma(Y)$ [resp. $\sigma(X) \longrightarrow \sigma(Y)$] holds in $P(S)$ then $X \twoheadrightarrow Y$ [resp. $X \longrightarrow Y$], where S is the semilattice image of B and σ is the corresponding epimorphism from B to S.*

P r o o f. Let $\sigma : B \longrightarrow S$ be an epimorphism. Then

$$\sigma(X) \twoheadrightarrow \sigma(Y) \Longrightarrow \sigma(X) < \sigma(Y).$$

Suppose $Z \in \mathcal{S}(B)$ be such that $X < Z < Y$. Now $Z \in \mathcal{S}(B)$ implies that $\sigma(Z) \in \mathcal{S}(S)$, since $(\sigma(Z))^3 = \sigma(Z^3) = \sigma(Z)$. Therefore,

$$X = X^2Z = XZ^2 = ZX^2 = Z^2X.$$

This implies that

$$\sigma(X) = \sigma(X)^2\sigma(Z) = \sigma(X)\sigma(Z)^2 = \sigma(Z)\sigma(X)^2 = \sigma(Z)^2\sigma(X).$$

Hence $\sigma(X) < \sigma(Z)$. Thus it follows that $\sigma(X) < \sigma(Z) < \sigma(Y)$. This contradicts $\sigma(X) \twoheadrightarrow \sigma(Y)$.

Hence $\sigma(X) \twoheadrightarrow \sigma(Y) \Longrightarrow X \twoheadrightarrow Y$.

Again let $X, Y \in Ch(B)$ such that $\sigma(X) \longrightarrow \sigma(Y)$. If possible, there exists $Z \in Ch(B)$ such that $X < Z < Y$. This implies $\sigma(X) < \sigma(Z) < \sigma(Y)$. Since $Z \in Ch(B)$ implies that $\sigma(Z) \in Ch(S)$, this contradicts that $\sigma(X) \longrightarrow \sigma(Y)$. Hence $X \longrightarrow Y$. □

Lemma 11. *Let B be a ternary $*$-band and $X \in Ch(B)$. If $x \in X$ is not a maximal element of X then $X \twoheadrightarrow X \setminus \{x\}$.*

P r o o f. Let $X \in Ch(B)$. This implies that $X^3 = X$. Now

$$X^2(X \setminus \{x\}) = X(X \setminus \{x\})^2 \subseteq X^3 = X.$$

Let $h \in X$ and $h \neq x$. Then $h = h^3 \in X^2(X \setminus \{x\})$. Again if $h = x$ then there exists some $y \in X$ such that $x = x^2y = xy^2$. Therefore, $h = x = x^2y \in X^2(X \setminus \{x\})$. This implies that $X \subseteq X^2(X \setminus \{x\})$. Hence $X^2(X \setminus \{x\}) = X$. Thus $X < X \setminus \{x\}$. Since $X \in Ch(B)$, $X \setminus \{x\} \in Ch(B)$. Since $\{x\}$ is not a maximal element of X, $\sigma(\{x\})$ is also not a maximal element of $\sigma(X)$. Then from [8], we can write

$$\sigma(X) \twoheadrightarrow \sigma(X) \setminus \sigma(\{x\}) = \sigma(X \setminus \{x\}).$$

Hence by Lemma 10, it follows that $X \twoheadrightarrow X \setminus \{x\}$. □

Lemma 12. *Let B be a ternary $*$-band and $X \in Ch(B)$. If X has a greatest element x_1 and there exists a projection $y \in B$ such that $x_1 \longrightarrow y$, then $X \longrightarrow X \cup \{y\}$.*

P r o o f. Let B be a ternary $*$-band. Then $y \in B$ implies that

$$(y^*)^* = y, \quad yy^*y = y, \quad y^*yy^* = y^*.$$

Let $X \in Ch(B)$. This implies that X is a chain of projections. If y be a projection of B such that $x_1 \longrightarrow y$ then it is clear that $X \cup \{y\}$ is also a chain of projections. Hence $X \cup \{y\} \in Ch(B)$. Now

$$X^2(X \cup \{y\}) = X^3 \cup X^2\{y\} = X.$$

Similarly, $(X \cup \{y\})X^2 = X$. Again

$$X(X \cup \{y\})^2 = X^3 \cup X^2\{y\} \cup X\{y\}^2 \cup X\{y\}X = X, \quad (X \cup \{y\})^2X = X.$$

Therefore, $X < (X \cup \{y\})$.

If possible, there exists $Y \in Ch(B)$ such that $X < Y < (X \cup \{y\})$. Thus

$$X^2Y = YX^2 = X \quad \text{and} \quad Y^2(X \cup \{y\}) = (X \cup \{y\})Y^2 = Y.$$

Therefore, $Y^2X \cup Y^2\{y\} = Y$. This implies that

$$X \cup Y^2\{y\} = Y \Longrightarrow X \subseteq Y.$$

If $X \neq Y$ then there exists $z \in Y$ such that $z \notin X$. Again let $z \in Y \setminus X$. If $z < x_1$ then

$$z = zx_1^2 = z^2x_1 = x_1z^2 = x_1^2z \in YX^2 = X.$$

So $z \in X$. This contradicts our assumption that $X \neq Y$.

Hence $x_1 < z$. Since

$$X \cup Y^2\{y\} = X \cup \{y\}Y^2 = Y, \quad z \in Y^2\{y\} = \{y\}Y^2.$$

Therefore, $z = y_1y_2y = yy_3y_4$ for some $y_1, y_2, y_3, y_4 \in Y$.

This implies that

$$zy^2 = y_1y_2yy^2 = y_1y_2y = z, \quad y^2z = y^2(yy_3y_4) = y^3y_3y_4 = yy_3y_4 = z.$$

Hence $z = zy^2 = y^2z$. Therefore, $z < y$.

Thus we get $x_1 < z < y$. This contradicts the relation $x_1 \longrightarrow y$. Hence our assumption is not true and so $X \longrightarrow X \cup \{y\}$. □

Lemma 13. *Let B be a ternary $*$-band and let $x \in B$ be a projection. If $\{x\} \longrightarrow Y$ for some $Y \in Ch(B)$ then $Y = \{x, y\}$ with $x \longrightarrow y$.*

P r o o f. Since $\{x\} \longrightarrow Y$, we get $\{x\}^2Y = Y\{x\}^2 = \{x\}$. Hence for any $y \in Y$, $x^2y = yx^2 = x$. This implies that $x \leq y$ for all $y \in Y$.

Let $Y_1 = \{x\} \cup Y$. Now

$$Y^2Y_1 = Y^2\{x\} \cup Y^3 = \{x\} \cup Y = Y_1 = YY_1{}^2, \quad Y_1Y^2 = \{x\}Y^2 \cup Y^3 = \{x\} \cup Y = Y_1 = Y_1{}^2Y.$$

This implies that $Y_1 \leq Y$.

Again

$$Y_1\{x\}^2 = \{x\}^3 \cup Y\{x\}^2 = \{x\} = Y_1{}^2\{x\}, \quad \{x\}^2Y_1 = \{x\}^3 \cup \{x\}^2Y = \{x\} = \{x\}Y_1{}^2.$$

This implies that $\{x\} < Y_1$. Therefore, $\{x\} < Y_1 \leq Y$. This contradicts the relation $\{x\} \longrightarrow Y$. Hence $Y_1 = Y$. This implies $x \in Y$.

Next we assume that $z \in Y \setminus \{x\}$ is an arbitrary element. Consider the set

$$Z = \{y \in Y : y \leq z\}.$$

Since $Y \in Ch(B)$, Z is also a chain of projections. Now $z \in Z$ implies that Z is nonempty. Hence

$$x^2Z = \{x^2y : y \leq z\} = \{x\}.$$

Similarly, $xZ^2 = \{x\}$. Therefore, $\{x\} \leq Z$.

Again let $u \in Z^2Y = ZY^2$. Therefore, $u = z_1z_2y_1$ for some $y_1 \in Y$ and $z_1, z_2 \in Z$. This implies either $y_1 \leq z$ or $y_1 > z$.

Case 1. Let $y_1 \leq z$ then $y_1z^2 = {y_1}^2z = z^2y_1 = z{y_1}^2 = y_1$. Hence

$$z^2u = z^2z_1z_2y_1 = z_1z_2y_1 = u, \quad uz^2 = z_1z_2y_1z^2 = z_1z_2y_1 = u.$$

This implies $u \leq z$.

Case 2. Let $y_1 > z$ then $y_1z^2 = {y_1}^2z = z^2y_1 = z{y_1}^2 = z$. Hence

$$z^2u = z^2z_1z_2y_1 = z_1z_2y_1 = u, \quad uz^2 = z_1z_2y_1z^2 = z_1z_2z^2y_1 = z_1z_2y_1 = u.$$

Therefore, $u \leq z$. Thus $u \in Z$. Hence $Z^2Y = ZY^2 \subseteq Z$.

Similarly, $v \in YZ^2 = Y^2Z \Longrightarrow v = y_1z_1z_2 = y_2y_3z_3$. Then either $y_1 \leq z$ or $y_1 > z$.

Case 1. Let $y_1 \leq z$. Then

$$y_1 = y_1z^2 = {y_1}^2z = z^2y_1 = z{y_1}^2,$$
$$z^2v = z^2y_1z_1z_2 = y_1z_1z_2 = v = zv^2, \quad vz^2 = y_1z_1z_2z^2 = y_1z_1z_2 = v = v^2z.$$

Hence $v \leq z$.

Case 2. Let $y_1 > z$. Therefore,

$$z = y_1z^2 = {y_1}^2z = z^2y_1 = z{y_1}^2,$$
$$vz^2 = y_1z_1z_2z^2 = y_1z_1z_2 = v, \quad z^2v = z^2y_1z_1z_2 = y_1z^2z_1z_2 = y_1z_1z_2 = v.$$

Hence $v \leq z$. This implies that $v \in Z$. Thus $YZ^2 = Y^2Z \subseteq Z$.

Conversely, $Z = Z^3 \subseteq Y^2Z = YZ^2$ and $Z = Z^3 \subseteq Z^2Y = ZY^2$.
Hence

$$Z = Y^2Z = YZ^2 = ZY^2 = Z^2Y.$$

This implies that $Z \leq Y$. Therefore, $\{x\} \leq Z \leq Y$. This contradicts the relation $\{x\} \longrightarrow Y$. Hence $Y = Z$. Since z is an arbitrary element, Y has only two elements say $\{x, y\}$. It is clear that $x < y$. If possible $x \not\twoheadrightarrow y$. Then there exists $z \in Ch(B)$ such that $x < z < y$. Then

$$x^2\{x, z\} = x\{x, z\}^2 = \{x, z\}^2x = \{x, y\}x^2 = \{x\},$$
$$\{x, z\}Y^2 = xY^2 \cup zY^2 = \{x\} \cup \{z\} = \{x, z\} = \{x, z\}^2Y = Y^2\{x, z\} = Y\{x, z\}^2.$$

Therefore, $\{x\} < \{x, z\} < Y$. This is a contradiction. Hence $Y = \{x, y\}$ and $x \longrightarrow y$. $\square$

Proposition 1. *Let B be a ternary $*$-band and $X \in Ch(B)$ such that $|X| \geq 3$. Then X has a topknot.*

P r o o f. Let $X \in Ch(B)$ and $|X| \geq 3$. Then there exist $x, y, z \in X$ such that $x < y < z$. Since $\{x\}$ and $\{y\}$ are not maximal elements of X, $X \twoheadrightarrow X \setminus \{x\}$ and $X \setminus \{y\}$, by Lemma 11. Again $X \setminus \{x\} \twoheadrightarrow X \setminus \{x, y\}$ and $X \setminus \{y\} \twoheadrightarrow X \setminus \{x, y\}$. Therefore, we have the following topknot:

$$\begin{tikzcd}
 & X \setminus \{x\} \arrow[dr] & \\
X \arrow[ur] \arrow[dr] & & X \setminus \{x, y\} \\
 & X \setminus \{y\} \arrow[ur] &
\end{tikzcd}$$

$\square$

Proposition 2. *Let B be a ternary $*$-band. If $X \in Ch(B)$ and $|X| = 2$ then X has either a maximal hair of length* 1 *or a topknot.*

P r o o f. Let $X = \{x, y\}$, with $x < y$. By Lemma 11, it follows that $X \twoheadrightarrow X \setminus \{x\} = \{y\}$. If X has no maximal hair of length 1 then there is an element $z \in Ch(B)$ such that $y \longrightarrow z$. Again by Lemma 13, $\{y\} \longrightarrow \{y, z\}$. Also by Lemma 12, $X \longrightarrow X \cup \{z\} = \{x, y, z\}$. Then by Lemma 11, $\{x, y, z\} \twoheadrightarrow \{y, z\}$. Hence we can construct the following topknot:

$$\begin{tikzcd} & \{y\} \arrow[dr] & \\ X \arrow[ur, two heads] \arrow[dr] & & \{y,z\} \\ & \{x,y,z\} \arrow[ur, two heads] & \end{tikzcd}$$

$\square$

Proposition 3. *Let B is a ternary $*$-band and $X \in Ch(B)$. Then $|X| = 1$ if and only if X has neither maximal hair of length* 1 *nor topknots.*

P r o o f. Suppose that $X = \{x\}$ and if possible, X has a maximal hair of length 1. Then there exists $Y \in Ch(B)$ such that $X \twoheadrightarrow Y$. Thus by Lemma 13, we get $Y = \{x, y\}$ for some y with $x \longrightarrow y$. Since x is not maximal in Y, by Lemma 11, it is clear that $\{x\} \twoheadrightarrow Y \twoheadrightarrow Y \setminus \{x\}$. This contradicts our assumption that X has a maximal hair of length 1. Now suppose that X has a topknot as follows:

$$\begin{tikzcd} & Y \arrow[dr, two heads] & \\ \{x\} \arrow[ur, two heads] \arrow[dr, two heads] & & T \\ & Z \arrow[ur, two heads] & \end{tikzcd}$$

Again by Lemma 13, we have $Y = \{x, y\}$ with $x \longrightarrow y$ and $Z = \{x, z\}$ with $x \longrightarrow z$ and $y \neq z$ so that $y^2z = yz^2 = zy^2 = z^2y$.

Now $T \in Ch(B)$. Consider $W = \{x, y, z\}$. Since $y^2z = yz^2 = zy^2 = z^2y = x$, $W \notin Ch(B)$.

So $xyz = x$ implies that $W^3 = W$. Hence $W \in \mathcal{S}(S)$. Now

$$\{x, y\}^2 W = \{x^3, x^2y, x^2z, xyx, xy^2, xyz, yx^2, yxy, yxz, y^2x, y^3, y^2z\} = \{x, y\}.$$

Similarly,

$$\{x, y\}W^2 = W\{x, y\}^2 = W^2\{x, y\} = \{x, y\}.$$

This shows that $\{x, y\} < W$. Again $W^2T = WT^2 = W$. Thus $\{x, y\} < W < T$. This contradicts the existence of topknot. Hence X has no topknot.

Conversely, suppose that X has neither maximal hair of length 1 nor topknot. If possible, $|X| > 1$. Then Proposition 1 and Proposition 2 contradicts our assumption. Thus the result holds. $\square$

Theorem 4. *Ternary $*$-band is globally determined.*

P r o o f. Let B_1, B_2 be two ternary $*$-bands such that $\psi : P(B_1) \longrightarrow P(B_2)$ be an isomorphism. Let $\bar{B_1}$, $\bar{B_2}$ be the set of all singleton subsets of B_1 and B_2 respectively. Let us define $\psi_1 : B_1 \longrightarrow \bar{B_1}$ such that $\psi_1(x) = \{x\}$ and $\psi_2 : \bar{B_2} \longrightarrow B_2$ such that $\psi_2(\{y\}) = y$. Now from the construction of ψ_1, ψ_2 it follows that ψ_1 and ψ_2 be two isomorphisms from B_1 to $\bar{B_1}$ and from $\bar{B_2}$ to B_2 respectively.

If we are able to show that $\psi\,|_{\bar{B_1}}$ is a bijection from $\bar{B_1}$ to $\bar{B_2}$ then it follows that $\psi\,|_{\bar{B_1}} : \bar{B_1} \longrightarrow \bar{B_2}$ is an isomorphism.

Let $X \in Ch(B_1)$ such that $|X| = 1$. Then $X \in \bar{B_1}$. If possible, $\psi(X) = X' \notin \bar{B_2}$, i.e., $|X'| \geq 2$. Then by Proposition 1 and Proposition 2, it follows that X' has either a maximal hair of length 1 or a topknot.

Case 1. Suppose X' has a maximal hair of length 1. Then X has also a maximal hair of length 1. This contradicts that $X \in \bar{B_1}$.

Case 2. Suppose X' has a topknot as follows:

$$\begin{tikzcd}
& Y' \arrow[dr] & \\
X' \arrow[ur] \arrow[dr] & & W' \\
& Z' \arrow[ur] &
\end{tikzcd}$$

where $Y', Z', W' \in Ch(B_2)$ and $Y' \neq Z'$. Then there exists $Y, Z, W \in Ch(B_1)$ and $Y \neq Z$ such that $Y' = \psi(Y)$, $Z' = \psi(Z)$ and $W' = \psi(W)$. Then the above topknot can be written as follows:

$$\begin{tikzcd}
& \psi(Y) \arrow[dr] & \\
\psi(X) \arrow[ur] \arrow[dr] & & \psi(W) \\
& \psi(Z) \arrow[ur] &
\end{tikzcd}$$

Hence we have the following topknot:

$$\begin{tikzcd}
& Y \arrow[dr] & \\
X \arrow[ur] \arrow[dr] & & W \\
& Z \arrow[ur] &
\end{tikzcd}$$

From the Proposition 3, it follows that $X \notin \bar{B_1}$. This contradicts our assumption. Therefore, $|\psi(X)| = |X'| = 1$ and $X' \in Ch(B_2)$. Thus ψ is a bijection from the singleton subset of projection of B_1 to the singleton subset of projection of B_2.

Now let $x \in B_1$ such that x is not projection. Then $(x^2x^*)^* = xx^{*2} = x^2x^*$. Therefore, x^2x^* is a projection.

Similarly, x^*x^2 is also a projection and $(x^2x^*)(x^2x^*)(x^*x^2) = x$. Therefore, any element of B can be written as a product of three projections, say $x = lmn$, where l, m, n are projections. So

$$\psi(\{x\}) = \psi(\{lmn\}) = \psi(\{l\}\{m\}\{n\}) = \psi(\{l\})\psi(\{m\})\psi(\{n\}) = l_1m_1n_1 = x_1.$$

Therefore, $\{x\} \in \bar{B_1}$ implies that $\psi(\{x\}) \in \bar{B_2}$, i.e., $\psi(\bar{B_1}) \subseteq \bar{B_2}$. Similarly, $\psi^{-1}(\bar{B_2}) \subseteq \bar{B_1}$. This implies that $\psi(\psi^{-1}(\bar{B_2})) \subseteq \psi(\bar{B_1})$ and $\bar{B_2} \subseteq \psi(B_1)$. Hence $\psi(\bar{B_1}) = \bar{B_2}$. Therefore, ψ is an onto mapping from $\bar{B_1}$ to $\bar{B_2}$ and since ψ is an isomorphism from $P(B_1)$ to $P(B_2)$ and $\bar{B_1} \subseteq P(B_1)$, it follows that $\psi \mid_{\bar{B_1}}: \ \bar{B_1} \longrightarrow \bar{B_2}$ is an isomorphism.

Therefore, $\psi_2\psi \mid_{\bar{B_1}} \psi_1 : B_1 \longrightarrow B_2$ is an isomorphism. Hence $B_1 \cong B_2$. Thus we conclude that ternary $*$-bands are globally determined. □

4. Conclusion

Throughout this paper we investigated the on global determinism of ternary $*$-bands and successfully proved that ternary $*$-bands are globally determined. This research enriches the study of global determinism problem on different classes of ternary semigroup. In future we will be able to study the global determinism problem of another class of ternary semigroup with the help of those results that we have proved in this paper. We hope this work will flourish the field of ternary semigroup, specially the global determinism problem on various classes of ternary semigroupes.

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