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WEIGHTED S^p-PSEUDO S-ASYMPTOTICALLY PERIODIC SOLUTIONS FOR SOME SYSTEMS OF NONLINEAR DELAY INTEGRAL EQUATIONS WITH SUPERLINEAR PERTURBATION

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Abstract: This work is concerned with the existence of positive weighted pseudo S-asymptotically periodic solution in Stepanov-like sense for some systems of nonlinear delay integral equations. In this context, we will first be interested in establishing a suitable composition theorem, and then some existing results concerning the S-asymptotic periodicity in the scalar case are developed here for the vector case. We point out that, in this paper, we adopt some changes in the definitions, which, although slight, are necessary to accomplish the work.

Keywords: Weighted S^p-pseudo S-asymptotic periodicity, S-asymptotic periodicity, Systems of nonlinear delay integral equations, Equations with superlinear perturbation.

1. Introduction

The concept of S-asymptotically periodic functions was introduced in the literature by Henríquez et al. [10] in 2008. The concept turns out to generalize that of asymptotically periodic functions. For additional details on this topic, we refer the reader to [1, 5, 7, 8, 10, 11, 18] and the references therein. Since then, S-asymptotically periodic functions are widely investigated and used in the study of differential and integral equations.

However, the notion of weighted S^p-pseudo S-asymptotic periodicity, which was introduced by Xia [17] in 2015, is more general than that of asymptotic periodicity and all its various extensions, namely S-asymptotic periodicity, pseudo S-asymptotic periodicity and weighted pseudo S-asymptotic periodicity.

Motivated by the works on various kinds of systems of nonlinear delay integral equations (see, e.g., [13–16]), on S-asymptotically periodic functions and by the works [9, 17] on weighted Stepanov-like pseudo S-asymptotically periodic functions, we investigate the existence of positive weighted S^p-pseudo S-asymptotically ω-periodic solution ($\omega > 0$) for systems of nonlinear delay integral equations with superlinear perturbations of the following type:

$$\begin{aligned} x(s) &= \alpha_1(s)x^{\eta}(s-l) + \int_0^{\tau_1(s)} f\big(s,\sigma,x(s-\sigma-l),y(s-\sigma-l)\big)d\sigma, \\ y(s) &= \alpha_2(s)y^{\nu}(s-l) + \int_0^{\tau_2(s)} g\big(s,\sigma,x(s-\sigma-l),y(s-\sigma-l)\big)d\sigma. \end{aligned} \tag{1.1}$$

Let $\eta, \nu \geq 1$ and $l \geq 0$ be fixed numbers, and let $f, g : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $\alpha_1, \alpha_2 : \mathbb{R} \to \mathbb{R}^+$, and $\tau_1, \tau_2 : \mathbb{R} \to \mathbb{R}^+$ be suitable functions satisfying some appropriate conditions mentioned later in the assumptions.

First of all, it is interesting to highlight the biological context of our model. Note that, considering the equation

$$x(s) = \alpha(s)x^{\eta}(s-l) + \int_0^{\tau(s)} f(s,\sigma,x(s-\sigma-l))d\sigma,$$

we have the scalar case of system (1.1), which generalizes the model studied in 2016 by Zhao et al. [18], if one changes the variable $s-\sigma = u$ and takes $l = 0$:

$$x(s) = \alpha(s)x^{\eta}(s-l) + \int_{s-\tau(s)}^{s} f(\sigma,x(\sigma))d\sigma,$$

which in turn generalizes the model

$$x(s) = \int_{s-\tau}^{s} f(\sigma,x(\sigma))d\sigma$$

published in 1976 by Cooke and Kaplan [4] to explain the spread of some infectious diseases or the population growth of single species.

The work consists of four sections and a conclusion. In the next section, we introduce some basic concepts, definitions, and notation required in what follows. Section 3 is devoted to proving several lemmas and a composition theorem needed to prove our existence result. In Section 4, we give sufficient conditions that ensure the existence and uniqueness of a weighted S^p-pseudo S-asymptotically ω-periodic solution to system (1.1).

2. Some definitions and preliminaries

Throughout the paper, we use the following notation. Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{R} = (-\infty,+\infty)$, $\mathbb{R}^* = (-\infty,0)\cup(0,+\infty)$, $\mathbb{R}_+ = [0,+\infty)$, $\mathbb{R}_+^n = \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$ (n times), and let, for $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$,

$$\|x\| = \sum_{i=1}^{n} |x_i|.$$

Let $BC(\mathbb{R},\mathbb{R}^n)$ (resp. $BC(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+^n,\mathbb{R}^n)$) be the space of continuous bounded functions $f:\mathbb{R}\to\mathbb{R}^n$ (resp. $f:\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+^n\to\mathbb{R}^n$). Then, endowed with the sup norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|,$$

$BC(\mathbb{R},\mathbb{R}^n)$ is a Banach space. For $1 \le p \le +\infty$, $L^p(\mathbb{R},\mathbb{R}^n)$ denotes the Lebesgue space and $L^p_{Loc}(\mathbb{R},\mathbb{R}^n)$ denotes the space of all equivalence classes of measurable functions $f:\mathbb{R}\longrightarrow\mathbb{R}^n$ such that the restriction of f to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R},\mathbb{R}^n)$. Let $L^{p,1}_{Loc}(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}^n)$ denote the space of all equivalence classes of measurable functions $f:\mathbb{R}\times\mathbb{R}_+\longrightarrow\mathbb{R}^n$, $(s,\sigma)\longrightarrow f(s,\sigma)$ such that the restriction of f to every bounded subset of $\mathbb{R}\times\mathbb{R}_+$ is in $L^{p,1}(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}^n) = L^p\left(\mathbb{R},L^1(\mathbb{R}_+,\mathbb{R}^n)\right)$.

Furthermore, in the general case when $x=(x_1,\ldots,x_n):\mathbb{R}\longrightarrow\mathbb{R}_+^n$, $\tau=(\tau_1,\ldots,\tau_n):\mathbb{R}\longrightarrow\mathbb{R}_+^n$, and $f=(f_1,\ldots,f_n):\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+^n\longrightarrow\mathbb{R}_+^n$ are appropriate functions, we use the notation

$$\int_0^{\tau(s)} f(s,\sigma,x(s-\sigma-l))d\sigma$$

for the vector of $\mathbb{R}^n$ whose components are

$$\int_0^{\tau_i(s)} f_i(s,\sigma,x_1(s-\sigma-l),\ldots,x_n(s-\sigma-l))d\sigma, \quad i=1,2,\ldots,n.$$

Definition 1 [18]. *A function $f \in BC(\mathbb{R}, \mathbb{R}^n)$ is said to be S-asymptotically ω-periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty} \|f(t+\omega) - f(t)\| = 0$. In this case, we say that ω is an asymptotic period of f. We denote by $SAP_\omega(\mathbb{R}, \mathbb{R}^n)$ the set of all such functions.*

Lemma 1 [18]. *Let $f, g \in SAP_\omega(\mathbb{R}, \mathbb{R}^n)$. Then the following assertions hold*:

(i) *the function $t \to f(t+s)$ lies in $SAP_\omega(\mathbb{R}, \mathbb{R}^n)$ for every $s \in \mathbb{R}$;*

(ii) *the product $f \cdot g$ lies in $SAP_\omega(\mathbb{R}, \mathbb{R}^n)$;*

(iii) *equipped with the sup norm*

$$\|f\|_\infty = \sup_{s\in\mathbb{R}} \|f(s)\|,$$

$SAP_\omega(\mathbb{R}, \mathbb{R}^n)$ turns out to be a Banach space.

Let U denote the collection of all functions (weights) $\rho : \mathbb{R}^* \longrightarrow (0, +\infty)$ locally integrable over $(-\infty, 0)$ and $(0, +\infty)$ such that $\rho(t) > 0$ for almost all $t \in \mathbb{R}^*$. For $\rho \in U$ and $r > 0$, we set

$$m^-(r, \rho) = \int_{-r}^{0} \rho(s)ds \quad \text{and} \quad m^+(r, \rho) = \int_{0}^{r} \rho(s)ds.$$

Throughout this paper, the set of weights U_∞ stands for

$$U_\infty = \left\{\rho \in U : \lim_{r\to+\infty} m^-(r, \rho) = +\infty \text{ and } \lim_{r\to+\infty} m^+(r, \rho) = +\infty\right\}.$$

Obviously, $U_\infty \subset U$, with strict inclusions.

Definition 2. *Let $\rho \in U_\infty$ and $f \in BC(\mathbb{R}, \mathbb{R}^n)$. If*

$$\lim_{r\to+\infty} \frac{1}{m^-(r, \rho)} \int_{-r}^{0} \|f(s-\omega) - f(s)\| \rho(s)ds = 0,$$
$$\lim_{r\to+\infty} \frac{1}{m^+(r, \rho)} \int_{0}^{r} \|f(s+\omega) - f(s)\| \rho(s)ds = 0,$$

for some $\omega > 0$, then we call f weighted pseudo S-asymptotically ω-periodic. The collection of such functions is denoted by $PSAP_\omega(\mathbb{R}, \mathbb{R}^n, \rho)$. In particular, we use the notation $PSAP_\omega(\mathbb{R}, \mathbb{R}^n)$ when $\rho \equiv 1$. Equipped with the sup norm

$$\|f\|_\infty = \sup_{s\in\mathbb{R}} \|f(s)\|,$$

$PSAP_\omega(\mathbb{R}, \mathbb{R}^n, \rho)$ turns out to be a Banach space.

Definition 3 [6]. *The Bochner transform $f^b(t, s),\ t \in \mathbb{R},\ s \in [0, 1]$, of a function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$, is defined as*

$$f^b(t, s) := f(t + s).$$

Remark 1. Note that a function $\varphi(t, s),\ t \in \mathbb{R},\ s \in [0, 1]$, is the Bochner transform of a certain function $f(t)$,

$$\varphi(t, s) = f^b(t, s),$$

if and only if $\varphi(t + \tau, s - \tau) = \varphi(s, t)$ for all $t \in \mathbb{R},\ s \in [0, 1]$, and $\tau \in [s - 1, s]$.

Definition 4 [6]. *The Bochner transform $f^b(t,s,\sigma,u)$, $t\in\mathbb{R}$, $s\in[0,1]$, $(\sigma,u)\in\mathbb{R}\times\mathbb{R}^n$, of a function $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$, is defined as*

$$f^b(t,s,\sigma,u):=f(t+s,\sigma,u).$$

Definition 5 [12]. *Let $p\in[1,+\infty)$.*

(i) *The space $BS^p(\mathbb{R},\mathbb{R}^n)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in $\mathbb{R}^n$ such that $f^b\in L^\infty\big(\mathbb{R},L^p([0,1],\mathbb{R}^n)\big)$. This is a Banach space with the norm*

$$\|f\|_{S^p}=\|f^b\|_{L^\infty(\mathbb{R},L^p)}=\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(s)\|^p ds\right)^{1/p}.$$

(ii) *The space $BS^p\big(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^n_+,\mathbb{R}^n\big)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f:\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^n_+\longrightarrow\mathbb{R}^n$ such that*

$$f^b(\cdot,\cdot,\sigma,u)\in L^\infty\left(\mathbb{R},L^p([0,1],\mathbb{R}^n)\right),\quad t\to f^b(t,\cdot,\sigma,u)\in L^p\left([0,1],\mathbb{R}^n\right),$$

for every $t\in\mathbb{R}$ and every $(\sigma,u)\in\mathbb{R}_+\times\mathbb{R}^n_+$.

One can see that, for every $f\in L^p_{Loc}(\mathbb{R},\mathbb{R}^n)$, the function f^b is continuous (by construction). Then, the space $BS^p(\mathbb{R},\mathbb{R}^n)$ may also be written as

$$BS^p(\mathbb{R},\mathbb{R}^n)=\big\{f\in L^p_{Loc}(\mathbb{R},\mathbb{R}^n):f^b\in BC(\mathbb{R}),\ L^p([0,1],\mathbb{R}^n)\big\}.$$

In fact, for $p\geq 1$, we have

$$\left(BC(\mathbb{R},\mathbb{R}^n),\|\cdot\|_{BC}\right)\text{ is continuously embeded in }\left(BS^p(\mathbb{R},\mathbb{R}^n),\|\cdot\|_{S^p}\right).$$

Also, it is well known that $L^p(\mathbb{R},\mathbb{R}^n)\subset BS^p(\mathbb{R},\mathbb{R}^n)\subset L^p_{Loc}(\mathbb{R},\mathbb{R}^n)$ and $BS^p(\mathbb{R},\mathbb{R}^n)\subset BS^q(\mathbb{R},\mathbb{R}^n)$ for $p\geq q\geq 1$.

Definition 6. *Let $\rho\in U_\infty$ and $f\in BS^p(\mathbb{R},\mathbb{R}^n)$. If*

$$\lim_{r\to+\infty}\frac{1}{m^-(r,\rho)}\int_{-r}^0\rho(t)\left(\int_{t-1}^t\|f(s-\omega)-f(s)\|^p\,ds\right)^{1/p}dt=0,$$
$$\lim_{r\to+\infty}\frac{1}{m^+(r,\rho)}\int_0^r\rho(t)\left(\int_t^{t+1}\|f(s+\omega)-f(s)\|^p\,ds\right)^{1/p}dt=0$$

for some $\omega>0$, then we call f weighted S^p-pseudo S-asymptotically ω-periodic. Such function space is denoted by $PSAP^p_\omega(\mathbb{R},\mathbb{R}^n,\rho)$. In particular, we use the notation $PSAP^p_\omega(\mathbb{R},\mathbb{R}^n)$ when $\rho\equiv 1$.

Remark 2. The above definition has a slight difference from [17, Definition 3.1], where a weighted S^p-pseudo S-asymptotically ω-periodic function is defined on $\mathbb{R}_+$.

Similarly to [9], we give an example illustrating that $PSAP^p_\omega(\mathbb{R},\mathbb{R}^n)\neq PSAP^p_\omega(\mathbb{R},\mathbb{R}^n,\rho)$.

Example 1. Define a function $f:\mathbb{R}\to\mathbb{R}$ as follows:

$$f(t)=\begin{cases}-n^5(t-n^3-1/n)^2+n^3, & t\in[n^3,n^3+2/n],\quad n\in\mathbb{N},\\ -n^5(t+n^3+1/n)^2+n^3, & t\in[-n^3-2/n,-n^3],\quad n\in\mathbb{N},\\ 0, & \text{otherwise.}\end{cases}$$

Then, for all $\omega > 0$, there exists integer n_0 such that $f(s+\omega) = 0$ (resp. $f(s-\omega) = 0$) for all $n \geq n_0$ and $s \in [n^3, n^3 + 2/n]$ (resp. for all $n \geq n_0$ and $s \in [-n^3 - 2/n, -n^3]$). Let $p = 1$, let $r > 0$ be a sufficiently large number, and let k be the largest integer satisfying the inequality

$$n_0^3 + \frac{2}{n_0} \leq k^3 + \frac{2}{k} \leq r.$$

If the function (weight) $\rho \equiv 1$, then, by the same calculation as in [9, Example 2.2], we obtain

$$\begin{aligned}
\frac{1}{r}\int_{-r}^{0}\left(\int_{t-1}^{t}\|f(s-\omega)-f(s)\|ds\right)dt &= \frac{1}{r}\int_{-r}^{0}\left(\int_{-1}^{0}\|f(t+s-\omega)-f(t+s)\|ds\right)dt \\
&= \int_{-1}^{0}\left(\frac{1}{r}\int_{-r}^{0}\|f(t+s-\omega)-f(t+s)\|dt\right)ds \\
&\geq \int_{-1}^{0}\frac{1}{(k+1)^3+2/(k+1)}\left(\sum_{n=n_0}^{k}\int_{-n^3-2/n}^{-n^3}\left[-n^5\left(t+n^3+\frac{1}{n}\right)^2+n^3\right]dt\right)ds \\
&= \frac{1}{(k+1)^3+2/(k+1)}\sum_{n=n_0}^{k}\frac{4n^2}{3} \to \frac{4}{9} \quad (k \to +\infty)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{r}\int_{0}^{r}\left(\int_{t}^{t+1}\|f(s+\omega)-f(s)\|ds\right)dt \\
&\geq \int_{0}^{1}\frac{1}{(k+1)^3+2/(k+1)}\left(\sum_{n=n_0}^{k}\int_{n^3}^{n^3+2/n}\left[-n^5\left(t-n^3-\frac{1}{n}\right)^2+n^3\right]dt\right)ds \\
&= \frac{1}{(k+1)^3+2/(k+1)}\sum_{n=n_0}^{k}\frac{4n^2}{3} \to \frac{4}{9} \quad (k \to +\infty).
\end{aligned}$$

This implies that $f \notin PSAP_{\omega}^{p}(\mathbb{R},\mathbb{R})$.

Now, take $\rho(t) = 1/t^4$ and $t \neq 0$. Again, by the same calculation as in [9, Example 2.2], we obtain $f \in PSAP_{\omega}^{p}(\mathbb{R},\mathbb{R},\rho)$.

Theorem 1 [9]. *$PSAP_{\omega}^{p}(\mathbb{R},\mathbb{R}^n,\rho)$, where $\rho \in U_{\infty}$, with the norm $\|.\|_{S^p}$ is a Banach space.*

P r o o f. The proof is similar to that of [9, Theorem 3.2], where weighted S^p-pseudo S-asymptotically periodic function is defined on $\mathbb{R}_+$, so it is omitted here. □

Definition 7. *Let $\rho \in U_{\infty}$. A function $f : \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+^n \to \mathbb{R}$ is called weighted S^p-pseudo S-asymptotically ω-periodic in $s \in \mathbb{R}$ for all $(\sigma,x) \in \mathbb{R}_+\times\mathbb{R}_+^n$ if $f(\cdot,\sigma,x) \in BS^p(\mathbb{R},\mathbb{R}^n)$ and*

$$\begin{aligned}
\lim_{r\to+\infty}\frac{1}{m^+(r,\rho)}\int_{0}^{r}\rho(t)\left(\int_{t}^{t+1}\|f(s+\omega,\sigma,x)-f(s,\sigma,x)\|^p\,ds\right)^{1/p}dt = 0, \\
\lim_{r\to+\infty}\frac{1}{m^-(r,\rho)}\int_{-r}^{0}\rho(t)\left(\int_{t-1}^{t}\|f(s-\omega,\sigma,x)-f(s,\sigma,x)\|^p\,ds\right)^{1/p}dt = 0
\end{aligned}$$

for all $(\sigma,x) \in \mathbb{R}_+\times\mathbb{R}_+^n$. Denote by $PSAP_{\omega}^{p}\left(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+^n,\mathbb{R},\rho\right)$ the set of all such functions.

3. Composition theorem

To study the existence of solutions to system (1.1), we reduce the problem to a fixed point problem of a nonlinear operator. For this, we must prove a composition theorem adapted to our case.

Let $BPSAP^p_\omega(\mathbb{R},\mathbb{R}^n,\rho)$ be the subset of $PSAP^p_\omega(\mathbb{R},\mathbb{R}^n,\rho)$ consisting of all bounded functions x, that is,

$$\|x\|_\infty = \sup_{s\in\mathbb{R}} \|x(s)\| < \infty.$$

It is clear that $BPSAP^p_\omega(\mathbb{R},\mathbb{R}^n,\rho)$ is a Banach space with respect to the norm $\|\cdot\|_{S^p}$.

Let $PSAP^{p,1}_\omega\left(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^n_+,\mathbb{R}^n,\rho\right)$ be the subset of the space $PSAP^p_\omega\left(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^n_+,\mathbb{R}^n,\rho\right)$ consisting of all functions f such that $f(\cdot,\cdot,u)\in L^{p,1}_{Loc}(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}^n)$ for all $u\in\mathbb{R}^n_+$. For $\rho\in U_\infty$, we further assume that (see [2])

$$(H_\rho)\qquad \text{for all } \sigma\in\mathbb{R},\quad \limsup_{|s|\longrightarrow+\infty}\frac{\rho(s+\sigma)}{\rho(s)}<+\infty.$$

Note that hypothesis (H_ρ) implies that, for all $\sigma\in\mathbb{R}_+$,

$$\limsup_{r\longrightarrow+\infty}\frac{m^+(r+\sigma,\rho)}{m^+(r,\rho)}<+\infty \quad\text{and}\quad \limsup_{r\longrightarrow+\infty}\frac{m^-(r+\sigma,\rho)}{m^-(r,\rho)}<+\infty.$$

Lemma 2. *Let $\rho\in U_\infty$ satisfy hypothesis (H_ρ). If $f\in PSAP^p_\omega(\mathbb{R},\mathbb{R}^n,\rho)$, then $f_{-\sigma}\in PSAP^p_\omega(\mathbb{R},\mathbb{R}^n,\rho)$ for all $\sigma\in\mathbb{R}_+$, where $f_{-\sigma}(s)=f(s-\sigma)$.*

P r o o f. Fix $\sigma\in\mathbb{R}_+$. From assumption (H_ρ), there exist constants $k,s_0>0$ such that, for $|s|\geq s_0$,

$$\frac{\rho(s-\sigma)}{\rho(s)}\leq k,\quad \frac{\rho(s+\sigma)}{\rho(s)}\leq k,\quad \frac{m^-(r+\sigma,\rho)}{m^-(r,\rho)}\leq k,\quad\text{and}\quad \frac{m^+(r+\sigma,\rho)}{m^+(r,\rho)}\leq k.$$

Thus, for $r>s_0+\sigma$,

$$\begin{aligned}
&\frac{1}{m^-(r,\rho)}\int_{-r}^{-\sigma}\rho(t)\left(\int_{t-1}^{t}\|f_{-\sigma}(s-\omega)-f_{-\sigma}(s)\|^p\,ds\right)^{1/p}dt\\
&=\frac{1}{m^-(r,\rho)}\int_{-r}^{-s_0-\sigma}\rho(t)\left(\int_{t-1}^{t}\|f_{-\sigma}(s-\omega)-f_{-\sigma}(s)\|^p\,ds\right)^{1/p}dt\\
&\quad+\frac{1}{m^-(r,\rho)}\int_{-s_0-\sigma}^{-\sigma}\rho(t)\left(\int_{t-1}^{t}\|f_{-\sigma}(s-\omega)-f_{-\sigma}(s)\|^p\,ds\right)^{1/p}dt.
\end{aligned}$$

It is clear that the following integral is defined:

$$\int_{-s_0-\sigma}^{-\sigma}\rho(t)\left(\int_{t-1}^{t}\|f_{-\sigma}(s-\omega)-f_{-\sigma}(s)\|^p\,ds\right)^{1/p}dt.$$

Therefore, since

$$\lim_{r\to+\infty}m^-(r,\rho)=+\infty,$$

$$\lim_{r\to+\infty}\frac{1}{m^-(r,\rho)}\int_{-s_0-\sigma}^{-\sigma}\rho(t)\left(\int_{t-1}^{t}\|f_{-\sigma}(s-\omega)-f_{-\sigma}(s)\|^p\,ds\right)^{1/p}dt=0.$$

Also, we have

$$\begin{aligned}
&\frac{1}{m^-(r,\rho)}\int_{-r}^{-s_0-\sigma}\rho(t)\left(\int_{t-1}^{t}\|f_{-\sigma}(s-\omega)-f_{-\sigma}(s)\|^p\,ds\right)^{1/p}dt\\
&=\frac{m^-(r+\sigma,\rho)}{m^-(r,\rho)}\frac{1}{m^-(r+\sigma,\rho)}\int_{-r-\sigma}^{-s_0-2\sigma}\frac{\rho(t+\sigma)}{\rho(t)}\rho(t)\left(\int_{t-1}^{t}\|f(s-\omega)-f(s)\|^p\,ds\right)^{1/p}dt\\
&\le\frac{k^2}{m^-(r+\sigma,\rho)}\int_{-(r+\sigma)}^{0}\rho(t)\left(\int_{t-1}^{t}\|f(s-\omega)-f(s)\|^p\,ds\right)^{1/p}dt.
\end{aligned}$$

Since $f\in PSAP^p_\omega(\mathbb{R},\mathbb{R}^n,\rho)$, we have

$$\lim_{r\to+\infty}\frac{k^2}{m^-(r+\sigma,\rho)}\int_{-(r+\sigma)}^{0}\rho(t)\left(\int_{t-1}^{t}\|f(s-\omega)-f(s)\|^p\,ds\right)^{1/p}dt=0.$$

Thus,

$$\lim_{r\to+\infty}\frac{1}{m^-(r,\rho)}\int_{-r}^{0}\rho(t)\left(\int_{t-1}^{t}\|f_{-\sigma}(s-\omega)-f_{-\sigma}(s)\|^p\,ds\right)^{1/p}dt=0.$$

Similarly, we obtain

$$\lim_{r\to+\infty}\frac{1}{m^+(r,\rho)}\int_{0}^{r}\rho(t)\left(\int_{t}^{t+1}\|f(s+\omega)-f(s)\|^p\,ds\right)^{1/p}dt=0.$$

We deduce that $f_{-\sigma}\in PSAP^p_\omega(\mathbb{R},\mathbb{R}^n,\rho)$ for all $\sigma\in\mathbb{R}_+$ (see [9, Theorem 3.1] for more details). □

Now, let us put forward the following hypothesis, which will be helpful throughout the rest of this paper.

(H_0) For every compact subset $K\subset\mathbb{R}^n_+\setminus\{0\}$, there exist constants $L_K,M_K>0$ such that

(i) for all $x,u\in K$ and all $(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+$,

$$\|f(s,\sigma,x)-f(s,\sigma,u)\|\le L_K\|x-u\|;$$

(ii) for all $x\in K$ and all $(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+$,

$$\|f(s,\sigma,x)\|\le M_K\|x\|.$$

Lemma 3. *Let $\rho\in U_\infty$. Assume that $f\in PSAP^{p,1}_\omega\left(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^n_+,\mathbb{R}^n,\rho\right)$ satisfies (H_0), and K_1 and K_2 are compact subsets of $\mathbb{R}^n_+\setminus\{0\}$. Then*

$$\lim_{r\to+\infty}\frac{1}{m^+(r,\rho)}\int_0^r\rho(t)\left[\int_t^{t+1}\left(\sup_{(\tau,x)\in K}\left\|\int_0^\tau[f(s+\omega,\sigma,x)-f(s,\sigma,x)]d\sigma\right\|\right)^p ds\right]^{1/p}dt=0,$$

$$\lim_{r\to+\infty}\frac{1}{m^-(r,\rho)}\int_{-r}^0\rho(t)\left[\int_{t-1}^{t}\left(\sup_{(\tau,x)\in K}\left\|\int_0^\tau[f(s-\omega,\sigma,x)-f(s,\sigma,x)]d\sigma\right\|\right)^p ds\right]^{1/p}dt=0,$$

where $K=K_1\times K_2$ is a compact subset of $\mathbb{R}^n_+\times\mathbb{R}^n_+$.

P r o o f. Fix $\varepsilon > 0$. Then, there exist $(\tau_1, x_1), ..., (\tau_m, x_m) \in K = K_1 \times K_2$ such that

$$K \subset \bigcup_{i=1}^{m} B\Big((\tau_i, x_i), \frac{\varepsilon}{|K|}\Big),$$

where

$$|K| = \sup_{(\tau,x)\in K} \{\|\tau\| + \|x\|\}.$$

For the above $\varepsilon > 0$, there exists $r_0 > 0$ such that

$$\frac{1}{m^+(r,\rho)} \int_0^r \rho(t) \Big(\int_t^{t+1} \|f(s+\omega,\sigma,x_i) - f(s,\sigma,x_i)\|^p \, ds \Big)^{1/p} dt < \frac{\varepsilon}{m} \tag{3.1}$$

for $r > r_0$, $\sigma \geq 0$, and $i \in \{1, 2, \dots, m\}$.

Now, let $(\tau, x) \in K$. Then there exists $i_0 \in \{1, 2, \dots, m\}$ such that

$$\|\tau - \tau_{i_0}\| < \frac{\varepsilon}{|K|} \quad \text{and} \quad \|x - x_{i_0}\| < \frac{\varepsilon}{|K|}.$$

Using (H_0), for all $r > r_0$, we have

$$\begin{aligned}
&\Big\| \int_0^{\tau} [f(s+\omega,\sigma,x) - f(s,\sigma,x)] d\sigma \Big\| \\
&\leq \Big\| \int_0^{\tau} f(s+\omega,\sigma,x) d\sigma - \int_0^{\tau_{i_0}} f(s+\omega,\sigma,x_{i_0}) d\sigma \Big\| + \Big\| \int_0^{\tau_{i_0}} [f(s+\omega,\sigma,x_{i_0}) - f(s,\sigma,x_{i_0})] d\sigma \Big\| \\
&\quad + \Big\| \int_0^{\tau_{i_0}} f(s,\sigma,x_{i_0}) d\sigma - \int_0^{\tau} f(s,\sigma,x) d\sigma \Big\| \\
&\leq \Big\| \int_0^{\tau_{i_0}} [f(s+\omega,\sigma,x) - f(s+\omega,\sigma,x_{i_0})] d\sigma \Big\| + \int_0^{\|\tau_{i_0}\|} \big\| f(s+\omega,\sigma,x_{i_0}) - f(s,\sigma,x_{i_0}) \big\| \\
&\quad + \Big\| \int_0^{\tau} [f(s,\sigma,x_{i_0}) - f(s,\sigma,x)] d\sigma \Big\| + \Big\| \int_{\tau_{i_0}}^{\tau} f(s+\omega,\sigma,x) d\sigma \Big\| + \Big\| \int_{\tau}^{\tau_{i_0}} f(s,\sigma,x_{i_0}) d\sigma \Big\| \\
&\leq \int_0^{\|\tau_{i_0}\|} \|f(s+\omega,\sigma,x_{i_0}) - f(s,\sigma,x_{i_0})\| + 2(L_{K_2} + M_{K_2})\varepsilon.
\end{aligned}$$

Minkowski's inequality, Hölder's inequality (see, for instance, [3, Theorem 4.6 and Theorem 4.7]), and (3.1) imply that, for all $r > r_0$,

$$\begin{aligned}
&\frac{1}{m^+(r,\rho)} \int_0^r \rho(t) \Big[\int_t^{t+1} \Big(\sup_{(\tau,x)\in K} \Big\| \int_0^{\tau} [f(s+\omega,\sigma,x) - f(s,\sigma,x)] d\sigma \Big\| \Big)^p ds \Big]^{1/p} dt \\
&\leq \sum_{i=1}^{m} \|\tau_i\|^{(p-1)/p} \frac{1}{m^+(r,\rho)} \int_0^r \rho(t) \Big[\int_0^{\|\tau_i\|} \int_t^{t+1} \|f(s+\omega,\sigma,x_i) - f(s,\sigma,x_i)\|^p ds d\sigma \Big]^{1/p} dt \\
&\quad + 2(L_{K_2} + M_{K_2})\varepsilon \\
&< \sum_{i=1}^{m} \|\tau_i\|^{(p-1)/p} \|\tau_i\|^{1/p} \frac{\varepsilon}{m} + 2(L_{K_2} + M_{K_2})\varepsilon \leq \big[|K| + 2(L_{K_2} + M_{K_2})\big]\varepsilon.
\end{aligned}$$

This proves the former limit. By the same considerations, we prove the latter limit. □

Theorem 2. *Let $\rho \in U_\infty$ satisfy (H_ρ). Assume that $\tau, x \in BPSAP^p_\omega\left(\mathbb{R}, \mathbb{R}^n_+, \rho\right)$, $\inf_{s\in\mathbb{R}} x(s) > 0$, and $f \in PSAP^{p,1}_\omega\left(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^n_+, \mathbb{R}^n, \rho\right)$ satisfy (H_0). Then, the function $Tx : \mathbb{R} \to \mathbb{R}^n$ defined as*

$$Tx(s) = \int_0^{\tau(s)} f(s,\sigma,x(s-\sigma-l))d\sigma, \quad l \geq 0,$$

belongs to $BPSAP^p_\omega\left(\mathbb{R}, \mathbb{R}^n_+, \rho\right)$.

P r o o f. Since $\tau, x \in BPSAP^p_\omega\left(\mathbb{R}, \mathbb{R}^n_+, \rho\right)$, using (H_0) (ii), one can easily show that $Tx(\cdot) \in BS^p\left(\mathbb{R}, \mathbb{R}^n\right)$. In addition,

$$\begin{aligned}
&\left(\int_t^{t+1} \left\|Tx(s+\omega) - Tx(s)\right\|^p ds\right)^{1/p} \\
&= \left(\int_t^{t+1} \left\|\int_0^{\tau(s+\omega)} f(s+\omega,\sigma,x(s+\omega-\sigma-l))d\sigma - \int_0^{\tau(s)} f(s,\sigma,x(s-\sigma-l))d\sigma\right\|^p ds\right)^{1/p} \\
&\leq \left(\int_t^{t+1} \left\|\int_0^{\tau(s+\omega)} \left[f(s+\omega,\sigma,x(s+\omega-\sigma-l)) - f(s,\sigma,x(s+\omega-\sigma-l))\right]d\sigma\right\|^p ds\right)^{1/p} \\
&\quad + \left(\int_t^{t+1} \left\|\int_0^{\tau(s)} \left[f(s,\sigma,x(s+\omega-\sigma-l)) - f(s,\sigma,x(s-\sigma-l))\right]d\sigma\right\|^p ds\right)^{1/p} \\
&\quad + \left(\int_t^{t+1} \left\|\int_{\tau(s)}^{\tau(s+\omega)} f(s,\sigma,x(s+\omega-\sigma-l))d\sigma\right\|^p ds\right)^{1/p}.
\end{aligned}$$

Let

$$K_1 = \overline{\{\tau(s) : s \in \mathbb{R}\}}, \quad K_2 = \overline{\{x(s) : s \in \mathbb{R}\}},$$

and $K = K_1 \times K_2$. Then, we have

$$\begin{aligned}
&\int_0^r \rho(t)\left(\int_t^{t+1} \left\|Tx(s+\omega) - Tx(s)\right\|^p ds\right)^{1/p} dt \\
&\leq \int_0^r \rho(t)\left[\int_t^{t+1} \left(\sup_{(\tau,x)\in K} \left\|\int_0^{\tau} [f(s+\omega,\sigma,x) - f(s,\sigma,x)]d\sigma\right\|\right)^p ds\right]^{1/p} dt \\
&+ \|\tau\|_\infty^{(p-1)/p} \int_0^r \rho(t)\left[\int_t^{t+1}\int_0^{\|\tau\|_\infty} \|f(s,\sigma,x(s+\omega-\sigma-l)) - f(s,\sigma,x(s-\sigma-l))\|^p d\sigma ds\right]^{1/p} dt \\
&+ M\|x\|_\infty \int_0^r \rho(t)\left(\int_t^{t+1} \left\|\tau(s+\omega) - \tau(s)\right\|^p ds\right)^{1/p} dt.
\end{aligned}$$

From (H_0), Lemma 2, and Lemma 3, we obtain

$$\lim_{r\to+\infty} \frac{1}{m^+(r,\rho)} \int_0^r \rho(t)\left[\int_t^{t+1} \|Tx(s+\omega) - Tx(s)\|^p ds\right)^{1/p} dt = 0.$$

Similarly, we get

$$\lim_{r\to+\infty} \frac{1}{m^-(r,\rho)} \int_{-r}^0 \rho(t)\left[\int_{t-1}^{t} \|Tx(s-\omega) - Tx(s)\|^p ds\right)^{1/p} dt = 0.$$

□

We close this section with the following lemma, which, together with Lemma 2 and Theorem 2, are necessary for the sequel.

Lemma 4. *Let $\rho \in U_\infty$. Assume that $f, g \in BPSAP^p_\omega(\mathbb{R},\mathbb{R},\rho)$, then the product $f \cdot g$ belongs to $BPSAP^p_\omega(\mathbb{R},\mathbb{R},\rho)$.*

P r o o f. Since $f, g \in PSAP^p_\omega(\mathbb{R},\mathbb{R},\rho)$ are bounded, we have

$$\begin{aligned}
&\frac{1}{m^+(r,\rho)}\int_0^r \rho(t)\left(\int_t^{t+1}|f(s+\omega)g(s+\omega)-f(s)g(s)|^p ds\right)^{1/p} dt\\
&\le \frac{1}{m^+(r,\rho)}\int_0^r \rho(t)\left(\int_t^{t+1}|f(s+\omega)g(s+\omega)-f(s+\omega)g(s)|^p ds\right)^{1/p} dt\\
&\quad+\frac{1}{m^+(r,\rho)}\int_0^r \rho(t)\left(\int_t^{t+1}|f(s+\omega)g(s)-f(s)g(s)|^p ds\right)^{1/p} dt\\
&\quad\le \frac{\|f\|_\infty}{m^+(r,\rho)}\int_0^r \rho(t)\left(\int_t^{t+1}|g(s+\omega)-g(s)|^p ds\right)^{1/p} dt\\
&\quad+\frac{\|g\|_\infty}{m^+(r,\rho)}\int_0^r \rho(t)\left(\int_t^{t+1}|f(s+\omega)-f(s)|^p ds\right)^{1/p} dt.
\end{aligned}$$

Thus,

$$\lim_{r\to+\infty}\frac{1}{m^+(r,\rho)}\int_0^r \rho(t)\left(\int_t^{t+1}|f(s+\omega)g(s+\omega)-f(s)g(s)|^p ds\right)^{1/p} dt = 0,$$

and similarly we get

$$\lim_{r\to+\infty}\frac{1}{m^-(r,\rho)}\int_{-r}^0 \rho(t)\left(\int_{t-1}^{t}|f(s-\omega)g(s-\omega)-f(s)g(s)|^p ds\right)^{1/p} dt = 0.$$

□

4. Existence theorem

In this section, we give sufficient conditions for system (1.1) to have a solution in the Banach space $BPSAP^p_\omega(\mathbb{R},\mathbb{R},\rho)\times BPSAP^p_\omega(\mathbb{R},\mathbb{R},\rho)$. Suppose that $\rho \in U_\infty$ satisfies assumption (H_ρ). We put forward the following hypotheses on the components of system (1.1), which are essential in the proof of our existence result.

(H_1) $\tau_i, \alpha_i \in BPSAP^p_\omega(\mathbb{R},\mathbb{R},\rho)$ $(i=1,2)$ are nonnegative functions.

(H_2) $F=(f,g)\in PSAP^{p,1}_\omega\left(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^2_+,\mathbb{R}^2_+,\rho\right)$ is such that, for every $(s,\sigma,x,y)\in\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+$, $f(s,\sigma,\cdot,y)$ and $g(s,\sigma,x,\cdot)$ are nondecreasing, and $f(s,\sigma,x,\cdot)$ and $g(s,\sigma,\cdot,y)$ are nonincreasing.

(H_3) There exist positive-valued functions ξ on $(0,1)$ and φ_i on $(0,1)\times\mathbb{R}_+\times\mathbb{R}_+$ $(i=1,2)$ such that

(i) $\xi:(0,1)\to(0,1)$ is a surjection;

(ii) for all $x,y\in(0,+\infty)$, all $(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+$, and all $\gamma\in(0,1)$,

$$\begin{aligned}
f\left(s,\sigma,\xi(\gamma)x,\frac{1}{\xi(\gamma)}y\right)&\ge\varphi_1(\gamma,x,y)f(s,\sigma,x,y),\\
g\left(s,\sigma,\frac{1}{\xi(\gamma)}x,\xi(\gamma)y\right)&\ge\varphi_2(\gamma,x,y)g(s,\sigma,x,y).
\end{aligned}$$

(H_4) There exist constants $M > \varepsilon > 0$ and $N > \delta > 0$ such that, for all $s \in \mathbb{R}$,

$$\varepsilon \leq \alpha_1(s)\varepsilon^{\eta} + \int_0^{\tau_1(s)} f(s,\sigma,\varepsilon,N)d\sigma \leq \alpha_1(s)M^{\eta} + \int_0^{\tau_1(s)} f(s,\sigma,M,\delta)d\sigma \leq M$$

and

$$\delta \leq \alpha_2(s)\delta^{\nu} + \int_0^{\tau_2(s)} g(s,\sigma,M,\delta)d\sigma \leq \alpha_2(s)N^{\nu} + \int_0^{\tau_2(s)} g(s,\sigma,\varepsilon,N)d\sigma \leq N.$$

(H_5) For every $\gamma \in (0,1)$,

$$\underline{\varphi_1}(\gamma) = \inf_{\substack{x\in[\varepsilon^2/M,M],\\ y\in[\delta^2/N,N]}} \varphi_1(\gamma,x,y) > \xi(\gamma) + r_1\left[\xi(\gamma) - (\xi(\gamma))^{\eta}\right],$$
$$\underline{\varphi_2}(\gamma) = \inf_{\substack{x\in[\varepsilon^2/M,M],\\ y\in[\delta^2/N,N]}} \varphi_2(\gamma,x,y) > \xi(\gamma) + r_2\left[\xi(\gamma) - (\xi(\gamma))^{\nu}\right],$$

where

$$r_1 = \frac{\overline{\alpha}_1 M^{\eta}}{\inf_{s\in\mathbb{R}} \int_0^{\tau_1(s)} f(s,\sigma,\varepsilon^2/M,N)d\sigma} < +\infty, \quad r_2 = \frac{\overline{\alpha}_2 N^{\nu}}{\inf_{s\in\mathbb{R}} \int_0^{\tau_2(s)} g(s,\sigma,M,\delta^2/N)d\sigma} < +\infty,$$
$$\text{and} \quad \overline{\alpha}_i = \sup_{s\in\mathbb{R}} \alpha_i(s), \quad i = 1,2.$$

Theorem 3. *Let $F = (f,g) \in PSAP^{p,1}_{\omega}\left(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^2_+,\mathbb{R}^2_+,\rho\right)$ be a function satisfying (H_0). Assume that (H_1)–(H_5) hold. Then system (1.1) has a bounded positive weighted S^p-pseudo S-asymptotically periodic solution (x^*,y^*), that is, $x^*,y^* \in BPSAP^p_{\omega}(\mathbb{R},\mathbb{R},\rho)$ are such that $\inf_{s\in\mathbb{R}} x^*(s) > 0$ and $\inf_{s\in\mathbb{R}} y^*(s) > 0$.*

P r o o f. Consider the following set in the Banach space $PSAP^p_{\omega}(\mathbb{R};\mathbb{R},\rho)$:

$$K = \{x \in PSAP^p_{\omega}(\mathbb{R};\mathbb{R},\rho) : \inf_{s\in\mathbb{R}} x(s) > 0\}.$$

Consider nonlinear operators $B = (B_1,B_2)$ and $C = (C_1,C_2)$ defined as

$$B_1(x,y)(s) = \int_0^{\tau_1(s)} f(s,\sigma,x(s-\sigma-l),y(s-\sigma-l))d\sigma,$$
$$B_2(x,y)(s) = \int_0^{\tau_2(s)} g(s,\sigma,x(s-\sigma-l),y(s-\sigma-l))d\sigma,$$
$$C_1(x)(s) = \alpha_1(s)x^{\eta}(s-l) \quad \text{and} \quad C_2(y)(s) = \alpha_2(s)y^{\nu}(s-l),$$

for all $(x,y) \in K\times K$ and all $s \in \mathbb{R}$. Let

$$A_1(x,y)(s) = B_1(x,y)(s) + C_1(x)(s),$$
$$A_2(x,y)(s) = B_2(x,y)(s) + C_2(y)(s),$$

and

$$A(x,y)(s) = (A_1(x,y)(s),\ A_2(x,y)(s))$$

for all $x,y \in K$ and all $s \in \mathbb{R}$.

Now, for $(x,y) \in K \times K$ such that

$$\frac{\varepsilon^2}{M} \leq x(s) \leq M \quad \text{and} \quad \frac{\delta^2}{N} \leq y(s) \leq N$$

for all $s \in \mathbb{R}$, we have

$$C_1(x)(s) \leq \overline{\alpha}_1 M^{\eta} = r_1 \inf_{s\in\mathbb{R}} \int_0^{\tau_1(s)} f\left(s, \sigma, \frac{\varepsilon^2}{M}, N\right) d\sigma \leq r_1 B_1(x,y)(s), \quad s \in \mathbb{R},$$

and

$$C_2(y)(s) \leq \overline{\alpha}_2 N^{\nu} = r_2 \inf_{s\in\mathbb{R}} \int_0^{\tau_2(s)} g\left(s, \sigma, M, \frac{\delta^2}{N}\right) d\sigma \leq r_2 B_2(x,y)(s), \quad s \in \mathbb{R}.$$

It follows that, for all $(x,y) \in K \times K$ such that

$$\frac{\varepsilon^2}{M} \leq x(s) \leq M, \quad \frac{\delta^2}{N} \leq y(s) \leq N, \quad s \in \mathbb{R},$$

and all $\gamma \in (0,1)$,

$$\begin{aligned}
A_1\left(\xi(\gamma)x, \frac{1}{\xi(\gamma)}y\right)(s) &= B_1\left(\xi(\gamma)x, \frac{1}{\xi(\gamma)}y\right)(s) + C_1\left(\xi(\gamma)x\right)(s) \\
&\geq \underline{\varphi_1}(\gamma) B_1(x,y)(s) + (\xi(\gamma))^{\eta} C_1(x)(s) \\
&= \xi(\gamma) A_1(x,y)(s) + \underline{\varphi_1}(\gamma) B_1(x,y)(s) + (\xi(\gamma))^{\eta} C_1(x)(s) - \xi(\gamma) A_1(x,y)(s) \\
&\geq \xi(\gamma) A_1(x,y)(s) + \left[\underline{\varphi_1}(\gamma) - \xi(\gamma)\right] B_1(x,y)(s) - \left[\xi(\gamma) - (\xi(\gamma))^{\eta}\right] r_1 B_1(x,y)(s) \\
&\geq \left[\xi(\gamma) + \frac{\underline{\varphi_1}(\gamma) - \xi(\gamma) - \left[\xi(\gamma) - (\xi(\gamma))^{\eta}\right] r_1}{1 + r_1}\right] A_1(x,y)(s) = \psi_1(\gamma) A_1(x,y)(s).
\end{aligned}$$

Similarly, we obtain

$$A_2\left(\frac{1}{\xi(\gamma)}x, \xi(\gamma)y\right)(s) \geq \psi_2(\gamma) A_2(x,y)(s),$$

where

$$\begin{aligned}
\psi_1(\gamma) &= \xi(\gamma) + \frac{\underline{\varphi_1}(\gamma) - \xi(\gamma) - \left[\xi(\gamma) - (\xi(\gamma))^{\eta}\right] r_1}{1 + r_1} > \xi(\gamma), \\
\psi_2(\gamma) &= \xi(\gamma) + \frac{\underline{\varphi_2}(\gamma) - \xi(\gamma) - \left[\xi(\gamma) - (\xi(\gamma))^{\nu}\right] r_2}{1 + r_2} > \xi(\gamma)
\end{aligned}$$

for all $\gamma \in (0,1)$ by (H_5). Take

$$\begin{aligned}
x_0(s) &= \varepsilon, \quad u_0(s) = M, \\
y_0(s) &= \delta, \quad v_0(s) = N
\end{aligned}$$

and consider the sequences

$$\begin{aligned}
x_k(s) &= A_1(x_{k-1}, v_{k-1})(s), \quad u_k(s) = A_1(u_{k-1}, y_{k-1})(s), \\
y_k(s) &= A_2(u_{k-1}, y_{k-1})(s), \quad v_k(s) = A_2(x_{k-1}, v_{k-1})(s).
\end{aligned}$$

From (H_4) and the monotony of the functions f and g assumed in (H_2), it is easy to show by induction that, for all $s \in \mathbb{R}$,

$$\begin{aligned}
&\varepsilon \leq x_1(s) \leq x_2(s) \leq \cdots \leq x_k(s) \leq \cdots \leq u_k(s) \leq \cdots \leq u_2(s) \leq u_1(s) \leq M, \\
&\delta \leq y_1(s) \leq y_2(s) \leq \cdots \leq y_k(s) \leq \cdots \leq v_k(s) \leq \cdots \leq v_2(s) \leq v_1(s) \leq N.
\end{aligned}$$

Now, let

$$\mu_k = \sup\big\{\mu > 0 :\ x_k(s) \geq \mu u_k(s) \text{ and } y_k(s) \geq \mu v_k(s),\ s \in \mathbb{R}\big\}.$$

Then $x_k(s) \geq \mu_k u_k(s)$ and $y_k(s) \geq \mu_k v_k(s)$ for all $k \geq 0$.

It follows that

$$\begin{aligned} x_{k+1}(s) &\geq x_k(s) \geq \mu_k u_k(s) \geq \mu_k u_{k+1}(s), \\ y_{k+1}(s) &\geq y_k(s) \geq \mu_k v_k(s) \geq \mu_k v_{k+1}(s) \end{aligned}$$

for all $s \in \mathbb{R}$, which implies that $\mu_{k+1} \geq \mu_k$ and

$$\max\left(\frac{\varepsilon}{M}, \frac{\delta}{N}\right) \leq \mu_k \leq 1, \quad k \geq 0.$$

Therefore, $(\mu_k)_k$ is a convergent sequence. Let us set $\mu^* = \lim_{k\to+\infty} \mu_k$ and prove that $\mu^* = 1$. Indeed, if we suppose to the contrary that $\mu^* < 1$, then by (H_3) (i), there exist $\gamma^* \in (0,1)$ such that $\mu^* = \xi(\gamma^*)$. We distinguish two cases.

Case 1. There exists integer k_0 such that $\mu_{k_0} = \mu^*$. Then, $\mu_k = \mu^*$ for all $k \geq k_0$. Hence, for all $k \geq k_0$ and all $s \in \mathbb{R}$,

$$x_{k+1} = A_1(x_k, v_k)(s) \geq A_1\left(\mu_k u_k, \frac{1}{\mu_k} y_k\right)(s) = A_1\left(\xi(\gamma^*)u_k, \frac{1}{\xi(\gamma^*)} y_k\right)(s) \geq \psi_1(\gamma^*) u_{k+1}(s).$$

We also conclude that $y_{k+1}(s) \geq \psi_2(\gamma^*) v_{k+1}(s)$ for all $s \in \mathbb{R}$.

Thus,

$$\mu_{k+1} = \mu^* \geq \max\{\psi_1(\gamma^*), \psi_2(\gamma^*)\} > \xi(\gamma^*) = \mu^*.$$

This is a contradiction.

Case 2. For all integer k, $\mu_k < \mu^*$. Again, by (H_3) (i), there exist $\gamma_k \in (0,1)$ such that

$$\xi(\gamma_k) = \frac{\mu_k}{\mu^*} \in (0,1).$$

Then, for all $s \in \mathbb{R}$, we have

$$\begin{aligned} x_{k+1}(s) = A_1(x_k, v_k)(s) &\geq A_1\left(\mu_k u_k, \frac{1}{\mu_k} y_k\right)(s) = A_1\left(\frac{\mu_k}{\mu^*}\mu^* u_k, \frac{\mu^*}{\mu^k}\frac{1}{\mu^*} y_k\right)(s) \\ &= A_1\left(\xi(\gamma_k)\mu^* u_k, \frac{1}{\xi(\nu_k)}\frac{1}{\mu^*} y_k\right)(s) \geq \psi_1(\gamma_k)\psi_1(\gamma^*) u_{k+1}(s). \end{aligned}$$

Similarly, we obtain

$$y_{k+1}(s) \geq \psi_2(\gamma_k)\psi_2(\gamma^*) v_{k+1}(s).$$

Thus, by the definition of μ_k, we have

$$\mu_{k+1} \geq \max\big\{\psi_1(\gamma_k)\psi_1(\gamma^*), \psi_2(\gamma_k)\psi_2(\gamma^*)\big\} \geq \max\left\{\frac{\mu_k}{\mu^*}\psi_1(\gamma^*), \frac{\mu_k}{\mu^*}\psi_2(\gamma^*)\right\}.$$

Let $k \to +\infty$, then

$$\mu^* \geq \max\{\psi_1(\gamma^*), \psi_2(\gamma^*)\} > \xi(\gamma^*) = \mu^*.$$

This is also a contradiction.

On the other hand, using hypotheses (H_1) and (H_2) combined with Lemma 2, Theorem 2, and Lemma 4, one can show that $x_k, u_k, y_k, v_k \in BPSAP^p_\omega(\mathbb{R};\mathbb{R},\rho)$ for all integer k.

In addition, for integer i and j such that $i > j$ and for all $s \in \mathbb{R}$, we have

$$0 \le x_i(s) - x_j(s) \le u_i(s) - x_j(s) \le u_j(s) - x_j(s) \le (1-\mu_j)u_j(s) \le (1-\mu_j)M,$$
$$0 \le y_i(s) - y_j(s) \le v_i(s) - y_j(s) \le v_j(s) - y_j(s) \le (1-\mu_j)v_j(s) \le (1-\mu_j)N.$$

It follows that

$$\|x_i - x_j\|_{S^p} \le (1-\mu_j)M \to 0, \quad \|y_i - y_j\|_{S^p} \le (1-\mu_j)N \to 0 \quad (\text{as} \quad j \to +\infty).$$

This means that $(x_k)_k$ and $(y_k)_k$ are Cauchy sequences in $BPSAP^p_\omega(\mathbb{R};\mathbb{R},\rho)$, and thus, there exist $x^*, y^* \in BPSAP^p_\omega(\mathbb{R};\mathbb{R},\rho)$ such that $x_k \to x^*$ and $y_k \to y^*$ in $BPSAP^p_\omega(\mathbb{R};\mathbb{R},\rho)$ as $k \to +\infty$. Also, one can easily see that $u_k \to x^*$ and $v_k \to y^*$ in $BPSAP^p_\omega(\mathbb{R};\mathbb{R},\rho)$ as $k \to +\infty$. Moreover, for all integer k and all $s \in \mathbb{R}$,

$$x_k(s) \le x^*(s) \le u_k(s) \quad \text{and} \quad y_k(s) \le y^*(s) \le v_k(s).$$

Finaly, we have

$$x_{k+1}(s) = A_1(x_k, v_k)(s) \le A_1(x^*, y^*)(s) \le A_1(u_k, y_k)(s) = u_{k+1}(s),$$
$$y_{k+1}(s) = A_2(u_k, y_k)(s) \le A_2(x^*, y^*)(s) \le A_2(x_k, v_k)(s) = v_{k+1}(s).$$

If $k \to +\infty$, we get

$$A(x^*, y^*) = (A_1(x^*, y^*), \quad A_2(x^*, y^*)) = (x^*, y^*).$$

That is, (x^*, y^*) is a positive solution of system (1.1) in $BPSAP^p_\omega(\mathbb{R};\mathbb{R},\rho) \times BPSAP^p_\omega(\mathbb{R};\mathbb{R},\rho)$. The proof is complete. □

Example 2. Let us choose

$$\eta = \frac{3}{2}, \quad \nu = \frac{4}{3}, \quad \alpha_1 := \frac{1}{10}, \quad \alpha_2 := \frac{1}{6}, \quad \tau_1 = \tau_2 := 1.$$

Consider functions $a, b \in PSAP^{p,1}_\omega(\mathbb{R},\mathbb{R},\rho)$ such that

$$\frac{9}{10}\sqrt{\frac{12}{19}} \le \inf_{s\in\mathbb{R}} a(s) \le \sup_{s\in\mathbb{R}} a(s) \le \frac{8}{5}\sqrt{\frac{9}{11}},$$
$$\frac{5}{6}\Big(\frac{5}{2}\Big)^{1/5} \le \inf_{s\in\mathbb{R}} b(s) \le \sup_{s\in\mathbb{R}} b(s) \le \frac{4}{3}\Big(\frac{2}{3}\Big)^{1/5}$$

and take

$$f(s,\sigma,x,y) = a(s-\sigma)\sqrt{x + \frac{1}{4} + \frac{1}{y+1}}, \quad g(s,\sigma,x,y) = b(s-\sigma)\sqrt[5]{\frac{y+1}{x^2+1}}.$$

Then, using the Mean value Theorem, one easily verifies that f and g satisfy $(H_0)(i)$, furthermore $(H_0)(ii)$ is obvious. Also, (H_1) and (H_2) are easy to check.

Hypothesis (H_3) is satisfied for

$$\xi(\lambda) := \lambda, \quad \varphi_1(\lambda,x,y) := \sqrt{\lambda}, \quad \text{and} \quad \varphi_2(\lambda,x,y) := \sqrt[5]{\lambda^3},$$

whenever $\lambda \in (0,1)$ and $x, y \in (0,+\infty)$.

Finally, (H_4) and (H_5) are satisfied for $\varepsilon = \delta = 1$, $M = N = 2$,

$$r_1 = \frac{2^{3/2}}{10\inf_{s\in\mathbb{R}}\int_0^1 f(s,\sigma,1/2,2)d\sigma} \le 1, \quad \text{and} \quad r_2 = \frac{2^{4/3}}{6\inf_{s\in\mathbb{R}}\int_0^1 g(s,\sigma,2,1/2)d\sigma} \le 1.$$

Thus, all the assumptions of Theorem 3 hold. Therefore, system (1.1) with the above data has the desired solution.

5. Conclusion

We have extended for the first time the study of a nonlinear integral equation in certain spaces to multidimensional systems in the space of weighted S^p-pseudo S-asymptotically ω-periodic functions. Moreover, we have made a change to the definition of this type of function, especially in the domain of definition, which we considered as $\mathbb{R}$ instead of $\mathbb{R}^+$. Our perspective in the future is to extend such a study to the abstract case where the dimension is infinite.

REFERENCES

1. Blot J., Cieutat P., N'Guérékata G. M. *S*-asymptotically ω-periodic functions and applications to evolution equations. *Afr. Diaspora J. Math.*, 2011. Vol. 12, No. 2. P. 113–121.
2. Blot J., Mophou G. M., N'Guérékata G. M., Pennequin D. Weighted pseudo almost automorphic functions and applications to abstract differential equations. *Nonlinear Anal.*, 2009. Vol. 71. P. 903–909. DOI: 10.1016/j.na.2008.10.113
3. Brezis H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations,* 1st edition. NY: Springer, 2010. 600 p. DOI: 10.1007/978-0-387-70914-7
4. Cooke K. L., Kaplan J. L. A periodicity threshold theorem for epidemics and population growth. *Math. Biosci.*, 1976. Vol. 31, No. 1–2. P. 87–104. DOI: 10.1016/0025-5564(76)90042-0
5. Cuevas C., de Souza J. C. *S*-asymptotically ω-periodic solutions of semilinear fractional integro-differential equations. *Appl. Math. Lett.*, 2009. Vol. 22, No. 6. P. 865–870. DOI: 10.1016/j.aml.2008.07.013
6. Diagana T., Mophou G. M., N'Guérékata G. M. Existence of weighted pseudo-almost periodic solutions to some classes of differential equations with S^p-weighted pseudo-almost periodic coefficients. *Nonlinear Anal.*, 2010. Vol. 72, No. 1. P. 430–438. DOI: 10.1016/j.na.2009.06.077
7. Dimbour W., Mado J.-C. *S*-asymptotically ω-periodic solution for a nonlinear differential equation with piecewise constant argument in a Banach space. *CUBO. Math. J.*, 2014. Vol. 16, No. 3. P. 55–65. URL: https://hal.science/hal-02067049
8. Dimbour W., Manou-Abi S. M. Asymptotically ω-periodic functions in the Stepanov sense and its application for an advanced differential equation with piecewise constant argument in a Banach space. *Mediterr. J. Math.*, 2018. Vol. 15. Art. no. 25. DOI: 10.1007/s00009-018-1071-6
9. He B., Wang Q.-R., Cao J.-F. Weighted S^p-pseudo *S*-asymptotic periodicity and applications to Volterra integral equations. *Appl. Math. Comput.*, 2020. Vol. 380. P. 125–275. DOI: 10.1016/j.amc.2020.125275
10. Henríquez H. R., Pierri M., Táboas P. On *S*-asymptotically ω-periodic functions on Banach spaces and applications. *J. Math. Anal. Appl.*, 2008. Vol. 343. P. 1119–1130. DOI: 10.1016/j.jmaa.2008.02.023
11. Lee H. M., Jang H. H., Yun C. M. *S*-asymptotically ω-periodic mild solutions for the systems of differential equations with piecewise constant argument in Banach spaces. *J. Chungcheong Math. Soc.*, 2018. Vol. 31, No. 1. P. 13–27. DOI: 10.14403/jcms.2018.31.1.13
12. N'Guérékata G. M., Pankov A. Stepanov-like almost automorphic functions and monotone evolution equations. *Nonlinear Anal.*, 2008. Vol. 68, No. 9. P. 2658–2667. DOI: 10.1016/j.na.2007.02.012
13. Sadrati A., Zertiti A. A study of systems of nonlinear delay integral equations by using the method of upper and lower solutions. *Int. J. Math. Comput.*, 2012. Vol. 17, No. 4. P. 93–102.
14. Sadrati A., Zertiti A. A topological methods for Existence and multiplicity of positive solutions for some systems of nonlinear delay integral equations. *Int. J. Math. Stat.*, 2013. Vol. 13, No. 1. P. 47–55.
15. Sadrati A., Zertiti A. Existence and uniqueness of positive almost periodic solution for systems of nonlinear delay integral equations. *Electron. J. Diff. Equ.*, 2015. Vol. 2015, No. 116. P. 1–12.
16. Sadrati A., Zertiti A. The Existence and uniqueness of positive weighted pseudo almost automorphic solution for some systems of neutral nonlinear delay integral equations. *Int. J. Appl. Math.*, 2016. Vol. 29, No. 3. P. 331–347. DOI: 10.12732/ijam.v29i3.5
17. Xia Z. N. Weighted pseudo asymptotically periodic mild solutions of evolution equations. *Acta. Math. Sin.-English Ser.*, 2015. Vol. 31, No. 8. P. 1215–1232. DOI: 10.1007/s10114-015-4727-1
18. Zhao J.-Y., Ding H.-S., N'Guérékata G. M. *S*-asymptotically periodic solutions for an epidemic model with superlinear perturbation. *Adv. Differ. Equ.*, 2016. Vol. 2016. Art. no. 221. DOI: 10.1186/s13662-016-0954-8