URAL MATHEMATICAL JOURNAL, Vol. 8, No. 2, 2022, pp. 127–132

DOI: 10.15826/umj.2022.2.010

ON DISTANCE-REGULAR GRAPHS OF DIAMETER 3 WITH EIGENVALUE $\theta = 1$

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Abstract: For a distance-regular graph Γ of diameter 3, the graph Γ_i can be strongly regular for $i = 2$ or 3. J. Kulen and co-authors found the parameters of a strongly regular graph Γ_2 given the intersection array of the graph Γ (independently, the parameters were found by A.A. Makhnev and D.V. Paduchikh). In this case, Γ has an eigenvalue $a_2 - c_3$. In this paper, we study graphs Γ with strongly regular graph Γ_2 and eigenvalue $\theta = 1$. In particular, we prove that, for a Q-polynomial graph from a series of graphs with intersection arrays $\{2c_3 + a_1 + 1, 2c_3, c_3 + a_1 - c_2; 1, c_2, c_3\}$, the equality $c_3 = 4(t^2 + t)/(4t + 4 - c_2^2)$ holds. Moreover, for $t \leq 100000$, there is a unique feasible intersection array $\{9, 6, 3; 1, 2, 3\}$ corresponding to the Hamming (or Doob) graph $H(3, 4)$. In addition, we found parametrizations of intersection arrays of graphs with $\theta_2 = 1$ and $\theta_3 = a_2 - c_3$.

Keywords: Strongly regular graph, Distance-regular graph, Intersection array.

1. Introduction

We consider undirected graphs without loops and multiple edges.

Let Γ be a connected graph. The *distance* $d(a, b)$ between two vertices a, b of Γ is the length of a shortest path between a and b in Γ. For a vertex a of Γ, denote by $\Gamma_i(a)$ the induced subgraph on the set of all vertices at distance i from a in Γ. Let Γ be a graph with diameter d and let a and b be vertices of Γ at distance i $(0 \leq i \leq d)$. Then the number of vertices that are at distance j from a and h from b is denoted by $p^i_{jh}(a, b)$ $(0 \leq i, j, h \leq d)$ and is called an intersection number of Γ. Note that $p^i_{jh}(a, b) = |\Gamma_j(a) \cap \Gamma_h(b)|$. Consider the numbers $c_i(a, b) = p^i_{i'1,1}(a, b)$, $a_i(a, b) = p^i_{i1}(a, b)$, and $b_i(a, b) = p^i_{i+1,1}(a, b)$. If the intersection numbers do not depend on the choice of a and b but only on i, then these numbers are denoted simply by p^i_{jh} $(0 \leq i, j, h \leq d)$. In this case, Γ of diameter d is called a *distance-regular* graph with intersection array $(b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d)$.

If a and b are vertices of the graph Γ, then we denote by $d(a, b)$ the distance between a and b. Given a vertex a in a graph Γ, we denote by $\Gamma_i(a)$ the subgraph induced by Γ on the set of all vertices at the distance i from a. The subgraph $\Gamma_1(a)$ is called the *neighbourhood of the vertex a* and is denoted by $[a]$, if the graph Γ is fixed.

Let Γ be a graph of diameter d and $i \in \{1, 2, 3, \ldots, d\}$. The graph Γ_i have the same set of vertices, and vertices u and w are adjacent in Γ_i if $d_\Gamma(u, w) = i$. For a subset of vertices Y from Γ, we denote by $\Gamma_i(Y)$ the subgraph with the set of vertices Y in which PI vertices u and w are adjacent if $d_\Gamma(u, w) = i$.

An incidence system with a set of points P and a set of lines $\mathcal{L}$ is called an *α-partial geometry of order* (s, t) if each line contains exactly $s+1$ points, each point lies exactly on $t+1$ lines, any two points lie on at most one line, and, for any antiflag $(a, l) \in (P, \mathcal{L})$, there is exactly α lines passing through a and intersecting l (the notation is $pG_\alpha(s,t)$).

A point graph of a geometry of points and lines is a graph whose vertices are points of the geometry, and two different vertices are adjacent if they lie on a common line. It is easy to see that a point graph of a partial geometry $pG_\alpha(s,t)$ is strongly regular with parameters $v = (s+1)(1+st/\alpha)$, $k = s(t+1)$, $\lambda = (s-1)+(\alpha-1)t$, and $\mu = \alpha(t+1)$. A strongly regular graph having the above parameters for some positive integers α, s, and t is called a *pseudogeometric graph* for $pG_\alpha(s,t)$.

The direct problem in the theory of distance-regular graphs is, given an intersection array, to find the parameters of a symmetric structure corresponding to a graph with this intersection array. The inverse problem is finding the intersection array of a distance-regular graph given the parameters of the corresponding symmetric structure.

If, for a distance-regular graph Γ of diameter 3, the graph Γ_3 is strongly regular, then, by [1, Lemma 3], the graph $\bar{\Gamma}_3$ is pseudogeometric for $pG_{c_3}(k, b_1/c_2)$. Conversely, for the graph $\bar{\Gamma}_3$, which is pseudogeometric for $pG_\alpha(l,t)$, the graph Γ has an intersection array $\{l, tc_2, l-\alpha+1; 1, c_2, \alpha\}$, where $l > tc_2 \geq l-\alpha+1$ and $c_2 \leq \alpha$.

Let Γ be a non-bipartite distance-regular graph of diameter 3. By [2, Lemma 3.1], the graph Γ_2 is strongly regular if and only if Γ has the eigenvalue $\theta = a_2 - c_3$.

The inverse problem was solved by A.A. Makhnev and D.V. Paduchickh. Let Γ be a distance-regular graph of diameter 3, for which Γ_2 is a strongly regular graph with parameters $(v, \kappa, \lambda, \mu)$ and eigenvalues κ, r, and $-s$. Then for $x = b_2 + c_2 \leq rs$ and $\mu x \neq rs(r+1)(s-1)$ the parameters of the intersection array of the graph Γ are expressed in terms of $\kappa, \mu, r, -s$, and x ([3, Theorem 2]).

We continue the study of distance-regular graphs Γ of diameter 3 with strongly regular graph Γ_2 and eigenvalue $\theta_2 = 1$.

The following result is obtained in [2, Lemma 4.5].

Proposition 1. *Let Γ be a non-bipartite distance-regular graph of diameter 3 with eigenvalue $\theta_2 = a_2 - c_3 = 1$. The following statements hold:*

(1) *the eigenvalues θ_1 and θ_3 are integer, $\theta_1 + \theta_3 = a_1$;*

(2) $c_3(c_2+2) = -(\theta_1+1)(\theta_3+1)$;

(3) *Γ has the intersection array $\{2c_3 + a_1 + 1, 2c_3, c_3 + a_1 - c_2; 1, c_2, c_3\}$.*

By Proposition 1, the graph Γ with $\theta_2 = a_2 - c_3 = 1$ and $n = a_1^2 + 4(c_2+2)c_3 + 4a_1 + 4$ has non-principal eigenvalues 1 and $a_1/2 \pm \sqrt{n}$, where the multiplicity of 1 is equal to

$$(2a_1 - c_2 + 4c_3 + 2)(a_1 + 2c_3 + 1)c_3/(c_2c_3 + 2a_1 + 2c_3).$$

This implies that n is a square and the multiplicity of $a_1/2 \pm \sqrt{n}$ is equal to

$$\begin{aligned}4(2a_1 - c_2 + 4c_3 + 2)(a_1 - c_2 + c_3)(a_1 + 2c_3 + 1)(a_1 + 2c_3)/\big((2a_1^3 - a_1^2c_2 + 2a_1^2c_3 \\ +8a_1c_2c_3 - 4c_2^2c_3 + 8c_2c_3^2 + \sqrt{n}(2a_1^2 - a_1c_2 + 2a_1c_3 + 2c_2c_3 + 2c_2) \\ +8a_1^2 - 4a_1c_2 + 24a_1c_3 - 8c_2c_3 + 16c_3^2 + 8a_1 - 4c_2 + 8c_3)c_2\big).\end{aligned}$$

Theorem 1. *Let Γ be a Q-polynomial distance-regular graph of diameter 3 with strongly regular graph Γ_2. If Γ has an eigenvalue $\theta = a_2 - c_3 = 1$, then $c_3 = 4(t^2+t)/(4t+4-c_2^2)$ and Γ has the intersection array $\big\{(c_2^2+4c_2+4t+4)(t+1)/(4t+4-c_2^2), 8(t+1)t/(4t+4-c_2^2), (c_2+t+2)c_2^2/(4t+4-c_2^2); 1, c_2, 4(t^2+t)/(4t+4-c_2^2)\big\}$.*

For $t \leq 100000$, there is only one feasible intersection array $\{9, 6, 3; 1, 2, 3\}$ ($t = c_2 = 2$) corresponding to the Hamming graph $H(3, 4)$ or the Doob graph with the same parameters.

We found parametrizations of distance-regular graphs of diameter 3 with eigenvalues $\theta_2 = 1 \neq \theta_3 = a_2 - c_3$.

Theorem 2. *Let Γ be a distance-regular graph of diameter 3 with strongly regular graph Γ_2. If Γ has the eigenvalue $\theta_2 = 1 \neq a_2 - c_3$, then Γ has the intersection array $\{(2n+r)t+1, 2(n-1)t, r(t-1); 1, n+r+1, 2nt\}$ or $\{(2n+r)t+n+r+1, (n-1)(2t+1), r(2t-1); 1, n+2r+1, n(2t+1)\}$.*

The following examples of graphs with eigenvalues $\theta_2 = 1 \neq \theta_3 = a_2 - c_3$ are known:

(1) $\{21, 10, 3; 1, 6, 15\}$, half 7-cube with spectrum $21^1, 9^7, 1^{21}, -3^{35}$, $v = 1+21+35+7 = 64$, and Γ_2 is a graph with parameters $(64, 35, 18, 20)$;

(2) $\{111, 88, 9; 1, 12, 99\}$ with spectrum $111^1, 21^{148}, 1^{444}, -9^{407}$, $v = 1 + 111 + 814 + 74 = 1000$, and Γ_2 is a strongly regular graph with parameters $(1000, 814, 663, 660)$.

For graphs from Theorem 2 for $n < 350, t < 1000$, we have only feasible intersection arrays $\{21, 10, 3; 1, 6, 15\}$, $\{111, 88, 9; 1, 12, 99\}$, $\{561, 448, 54; 1, 12, 504\}$, and $\{561, 448, 75; 1, 21, 480\}$.

2. Proof of Theorem 1

Let Γ be a Q-polynomial distance-regular graph of diameter 3 with eigenvalue $\theta_2 = a_2 - c_3 = 1$. By Proposition 1, the graph Γ has integer eigenvalues.

Lemma 1. *$a_1 = (c_2+2)c_3/t - t - 2$ for some positive integer t.*

P r o o f. We have

$$(a_1^2 + 4(c_2+2)c_3 + 4a_1 + 4) = u^2,$$

where u is a positive integer. Solving the Diophantine equation

$$u^2 - (a_1+2)^2 = 4(c_2+2)c_3,$$

we get

$$u = (c_2+2)c_3/t + t, \quad a_1 = (c_2+2)c_3/t - t - 2$$

for some positive integer t. □

Lemma 2. *The inequality $c_3 > t$ holds.*

P r o o f. We have

$$k = (c_2c_3 + 2c_3t - t^2 + 2c_3 - t)/t,$$

hence

$$(c_2c_3 + 2c_3t - t^2 + 2c_3 - t) > 0.$$

Further,

$$k_3 = 2(c_2c_3 + 2c_3t - t^2 + 2c_3 - t)(c_2 + t + 2)(c_3 - t)/(c_2t^2),$$

hence $c_3 > t$. □

Lemma 3. *The graph Γ is not Q-polynomial with respect to E_2.*

P r o o f. Suppose that Γ is a Q-polynomial graph with respect to E_2. Then, by [4], the equality

$$-2(c_2c_3+2c_3t-t^2+2c_3-2t)(c_2+2t+2)(2c_3-t)(c_3+1)/((c_2c_3+2c_3-2t)(t+2)t)$$
$$=-(c_2c_3+2c_3t-t^2+2c_3-2t)(c_2+2t+2)(2c_3-t)(c_3+1)/((c_2c_3+2c_3-2t)(t+2)t)$$

holds and either $c_3=(t^2+2t)/(c_2+2t+2)$, or $c_3=t/2$, or $c_3=-1$.

In any case, we have a contradiction. □

Lemma 4. *If Γ is not Q-polynomial with respect to E_1, then $c_3=4(t^2+t)/(4t+4-c_2^2)$.*

P r o o f. Let Γ be a Q-polynomial graph with respect to E_1. Then, by [4], the following equality holds:

$$\begin{aligned}&-(c_2^2c_3^2-c_2^2c_3t-c_2c_3t^2+4c_2c_3^2-4c_2c_3t+2c_3t^2-2t^3+4c_3^2-4c_3t)(c_2c_3+2c_3t\\&-t^2+2c_3-2t)(c_2+2t+2)(2c_3-t)/((c_2c_3+t^2+2c_3)(c_2c_3+2c_3-2t)c_2t^2)\\&=-(c_2^4c_3^3+4c_2^3c_3^3t-5c_2^3c_3^2t^2+4c_2^2c_3^3t^2+c_2^3c_3t^3-10c_2^2c_3^2t^3+4c_2^2c_3t^4-4c_2c_3^2t^4\\&+4c_2c_3t^5+8c_2^3c_3^3-6c_2^3c_3^2t+24c_2^2c_3^3t-42c_2^2c_3^2t^2+16c_2c_3^3t^2+16c_2^2c_3t^3-40c_2c_3^2t^3\\&+24c_2c_3t^4+24c_2^2c_3^3-36c_2^2c_3^2t+48c_2c_3^3t+12c_2^2c_3t^2-108c_2c_3^2t^2+16c_3^3t^2\\&+68c_2c_3t^3-40c_3^2t^3-8c_2t^4+32c_3t^4-8t^5+32c_2c_3^3-72c_2c_3^2t+32c_3^3t+48c_2c_3t^2\\&-88c_3^2t^2-8c_2t^3+80c_3t^3-24t^4+16c_3^3-48c_3^2t+48c_3t^2-16t^3)(c_2c_3+2c_3t-t^2+2c_3\\&-2t)(2c_3-t)/\big((c_2c_3+2c_3t-2t^2+2c_3-2t)(c_2c_3+t^2+2c_3)(c_2c_3+2c_3-2t)c_2t^2\big).\end{aligned}$$

Hence,

$$c_3\in\big\{4(t^2+t)/(4t+4-c_2^2),(2t^3+(t^2+2t)c_2+4t^2+4t)/(c_2^2+2c_2(t+2)+2t^2+4t+4),$$
$$(t^2+2t)/(c_2+2t+2),1/2t\big\}.$$

The latter three cases contradict Lemma 2. □

Theorem 1 is proved. □

3. Proof of Theorem 2

Let Γ be a non-bipartite distance-regular graph of diameter 3 with eigenvalues

$$\theta_1=a_1-1,\quad \theta_2=1,\quad \theta_3=a_2-c_3.$$

By [2, Lemma 3.1(v)], we have $b_1=(a_2-c_3+1)c_3/(a_2-c_3)$. This implies the following statement.

Lemma 5. *One of the following equalities holds*:

(1) $c_3=(c_3-a_2)m$, *where m is a positive integer not exceeding* 1;

(2) $k=b_2+c_2+c_3+1$;

(3) $k=b_2+c_2+c_3-1$.

In the second case, we have $a_2 - c_3 = 1$. In the third case, we have $a_2 - c_3 = -1$, a contradiction with [2, Lemma 3.1(b)].

Hence,

$$c_3 = (c_3 - a_2)m, \quad a_2 m = c_3(m-1), \quad a_2 = (m-1)n,$$

$b_1 = mn - m$ for some positive integer n greater than 1.

The non-principal eigenvalues $a_1 - 1$ and 1 are roots of the quadratic equation

$$x^2 - (b_2 + c_2 + m - n - 1)x + c_2 m - (m-1)n - b_2 - c_2 = 0.$$

Hence,

$$a_1 = k - a_2 + m - n - 1$$

and

$$a_1 - 1 = c_2 m - (m-1)n - k + a_2.$$

Hence

$$k = a_1 + 1 + mn - m, \quad k + a_1 - 1 = c_2 m, \quad 2a_1 = m(c_2 - n + 1).$$

If $m = 2t$, then $c_2 = n + r + 1$, $a_1 = t(r+2)$, $b_1 = 2t(n-1)$, and Γ has the intersection array

$$\{t(2n+r)+1, 2t(n-1), rt - r; 1, n+r+1, 2nt\}$$

and the non-principal eigenvalues $rt + 2t - 1, 1$, and $-n$ of multiplicities

$$(2nt+rt+n+1)(2nt+rt+1)(2n+r)(t-1)(n-1)/((rt+n+2t-1)(rt+2t-2)(n+r+1)n),$$
$$(2nt+rt+n+1)(2nt+rt+1)(nt-t+1)(n-1)r/((rt+2t-2)(n+r+1)(n+1)n),$$
$$2(2nt+rt+1)(nt-t+1)(2n+r)t/((rt+n+2t-1)(n+1)n),$$

respectively.

If $m = 2t+1$, then

$$c_2 = n + 2r + 1, \quad a_1 = (2t+1)r, \quad b_1 = (2t+1)(n-1),$$

and Γ has the intersection array

$$\big\{(2t+1)(n+r-1)+1, (2t+1)(n-1), 2rt - r; 1, n+2r+1, 2nt+n\big\}$$

and the non-principal eigenvalues $r(2t+1)+2t, 1$, and $-n$ of multiplicities

$$(2nt+2rt+2n+r+1)(2nt+2rt+n+r+1)(n+r)(n-1)(2t-1)/((2rt+n+r+2t) \times(2rt+r+2t-1)(n+2r+1)n),$$
$$(2nt+2rt+2n+r+1)(2nt+2rt+n+r+1)(2nt+n-2t+1)(n-1)r/((2rt+r+2t-1) \times(n+2r+1)(n+1)n),$$
$$(2nt+2rt+n+r+1)(2nt+n-2t+1)(n+r)(2t+1)/((2rt+n+r+2t)(n+1)n),$$

respectively.

Theorem 2 is proved. $\square$

REFERENCES

1. Makhnev A. A, Nirova M. S. Distance-regular Shilla graphs with $b_2 = c_2$. *Math. Notes*, 2018. Vol. 103, No. 5. P. 780–792. DOI: 10.1134/S0001434618050103
2. Iqbal Q., Koolen J. H., Park J., Rehman M. U. Distance-regular graphs with diameter 3 and eigenvalue $a_2 - c_3$. *Linear Algebra Appl.*, 2020. Vol. 587. P. 271–290. DOI: 10.1016/j.laa.2019.10.021
3. Makhnev A. A., Paduchikh D. V. Inverse problems in the theory of distance-regular graphs. *Trudy Inst. Mat. i Mekh. UrO RAN*, 2018. Vol. 24, No. 3. P. 133–144. DOI: 10.21538/0134-4889-2018-24-3-133-144
4. Terwilliger P. A new unequality for distance-regular graphs. *Discrete Math.*, 1995. Vol. 137, No. 1–3. P. 319–332. DOI: 10.1016/0012-365X(93)E0170-9