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# APPROXIMATION OF POSITIONAL IMPULSE CONTROLS FOR DIFFERENTIAL INCLUSIONS

**Ivan A. Finogenko**

Matrosov Institute for System Dynamics and Control Theory,
Siberian Branch of Russian Academy of Sciences,
134 Lermontova st., Irkutsk, 664033, Russian Federation

fin@icc.ru

**Alexander N. Sesekin**

Ural Federal University,
19 Mira str., Ekaterinburg, 620002, Russian Federation

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620108, Russian Federation

a.n.sesekin@urfu.ru

**Abstract:** Nonlinear control systems presented as differential inclusions with positional impulse controls are investigated. By such a control we mean some abstract operator with the Dirac function concentrated at each time. Such a control ("running impulse"), as a generalized function, has no meaning and is formalized as a sequence of correcting impulse actions on the system corresponding to a directed set of partitions of the control interval. The system responds to such control by discontinuous trajectories, which form a network of so-called "Euler's broken lines." If, as a result of each such correction, the phase point of the object under study is on some given manifold (hypersurface), then a slip-type effect is introduced into the motion of the system, and then the network of "Euler's broken lines" is called an impulse-sliding mode. The paper deals with the problem of approximating impulse-sliding modes using sequences of continuous delta-like functions. The research is based on Yosida's approximation of set-valued mappings and some well-known facts for ordinary differential equations with impulses.

**Keywords:** Positional impulse control, Differential inclusion, Impulse-sliding mode.

## 1. Introduction

We consider a dynamic system of the form

$$\dot{x}(t) \in F(t, x(t)) + B(t, x(t))u, \quad x(t_0) = x_0, \tag{1.1}$$

where $F : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is a set-valued function whose values are convex compact sets in the space $\mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$, the matrix function $B(t, x)$ of dimension $n \times m$ is continuous in a set of variables, the column vector $u = (u_1, \ldots, u_m)$ is some function that describes the control action on the system.

We make the following assumptions about $F(t, x)$:

(B1) The set-valued mapping $F(t, x)$ is upper semicontinuous at each point $(t, x)$. This means that, for an arbitrary $\epsilon > 0$, there exists $\delta = \delta(t, x, \epsilon) > 0$ such that $F(t', x') \subset F^\epsilon(t, x)$ for all $(t', x') \in W_\delta(t, x)$, where $F^\epsilon(t, x)$ is an $\epsilon$-neighborhood of the set $F(t, x)$ and $W^\delta(t, x)$ is a $\delta$-neighborhood of the point $(t, x)$.

(B2) The set-valued mapping $F(t,x)$ satisfies the condition of sublinear growth: the inequality $\|w\| \leq L(t)(1+\|x\|)$ with some continuous function $L(t)$ holds for any $(t,x) \in \mathbb{R}^{n+1}$ and $w \in F(t,x)$.

Conditions (B1) and (B2) ensure the existence of a solution to the differential inclusion

$$\dot{x} \in F(t,x), \tag{1.2}$$

on any segment $I = [t_0, \vartheta]$ (see, for example, [3]).

It is assumed that the matrix $B(t,x)$ satisfies the Frobenius condition

$$\sum_{\nu=1}^{n} \frac{\partial b_{ij}(t,x)}{\partial x_\nu} b_{\nu l}(t,x) = \sum_{\nu=1}^{n} \frac{\partial b_{il}(t,x)}{\partial x_\nu} b_{\nu j}(t,x).$$

This condition will ensure the uniqueness of the reaction of system (1.1) for impulse control $u$ [13, 14].

By a positional impulse control, we mean some abstract operator $(t,x) \longrightarrow U(t,x)$ that maps the space of variables $(t,x)$ into the space $m$ of vector distributions [14] according to the rule: $U(t,x) = r(t,x)\,\delta_t$, where $r(t,x)$ is a vector function with values in $\mathbb{R}^m$ and $\delta_t$ is the Dirac impulse function concentrated at the point $t$. The expression $r(t,x)\,\delta_t$ ("running impulse") has no meaning as a generalized function and means only the fact that the system has impulse control, which implies a discrete implementation of the "running impulse" in the form of a sequence of correcting impulses concentrated at points of some partition $h\colon t_0 < t_1 < \ldots < t_N = \vartheta$ of the segment $I$. The result of such a sequential correction is a discontinuous curve $x^h(\cdot)$, here called "Euler's broken line."

According to [14], we define a network of "Euler's broken lines" $x^h(\cdot)$ corresponding to the set of partitions $h:\ t_0 < t_1 < ... < t_p = \vartheta$ of the segment $I$. To do this, we first define a jump function by the equations

$$S(t,x,r(t,x)) = z(1) - z(0), \quad \dot{z}(\xi) = B(t, z(\xi)) r(t,x), \quad z(0) = x. \tag{1.3}$$

Here we take into account that there are dependencies $S = S(t,x,r)$ and $z = z(\xi,t,x,r)$. Note also that the jump function is a vector function $S = (S^1, \ldots, S^n)$.

The jumps of the "Euler broken lines" at the points of the partitions $h$ of the segment $I$ are determined by the equations

$$S(t_i, x^h(t_i), r(t_i, x^h(t_i))) = z(1) - z(0), \quad \dot{z}(\xi) = B(t_{t_i}, z(\xi)) r(t_i, x^h(t_i)),$$

with the initial conditions $z(0) = x^h(t_i)$.

On each interval $(t_i, t_{i+1}]$, "Euler's broken line" $x^h(t)$ is constructed as a function that coincides with the solution of the differential inclusion (1.2) with the initial conditions

$$x(t_i) = x^h(t_i) + S(t_i, x^h(t_i), r(t_i, x^h(t_i))), \quad x^h(t_0) = x_0, \quad i = 0, \ldots, p-1.$$

In this case, the following relations are valid:

$$x^h(t_i + 0) = x^h(t_i) + S(t_i, x^h(t_i), r(t_i, x^h(t_i))), \quad S = 0 \Leftrightarrow r = 0.$$

We assume that the following condition holds for all $(t,x)$:

$$r(t, x + S(t,x,r(t,x))) = 0. \tag{1.4}$$

This means that, after an impulsive action on the system at time $t$, the phase point $x(t)$ turns out to be on the manifold

$$\Phi = \{(t,x) : r(t,x) = 0\}.$$

In this case, the "Euler broken line" is called the impulse-sliding mode. We also assume that the functions $S(t,x,r)$ and $r(t,x)$ are continuously differentiable.

Under some assumptions, one can consider a subsequence of the sequence of Euler broken lines convergent as $d(h) = \max(t_{k+1} - t_k) \to 0$, whose limit is on the surface $\Phi$. This is called the ideal impulse-sliding mode. The purpose of the impulse control is to keep the phase point on the manifold $\Phi$.

In [10], a differential inclusion of an ideal impulse-sliding mode was obtained in the form

$$\begin{gathered}\dot{x} \in \frac{\partial S(t,x,r(t,x))}{\partial t} + \frac{\partial S(t,x,r(t,x))}{\partial r}\frac{\partial r(t,x)}{\partial t} + \\ + \left(E + \frac{\partial S(t,x,r(t,x))}{\partial x} + \frac{\partial S(t,x,r(t,x))}{\partial r}\frac{\partial r(t,x)}{\partial x}\right)F(t,x), \\ x(t_0+0) = x(t_0) + S(t_0, x(t_0), r(t_0, x(t_0))).\end{gathered} \tag{1.5}$$

In [11], a differential inclusion with discontinuous positional controls under constraints on control resources was constructed, for which the ideal impulse-sliding mode of inclusion (1.5) is an ordinary sliding mode in the sense of the theory of discontinuous systems. This makes it possible to use combinations of positional impulse and conventional discontinuous controls for controlled systems in situations without enough control resources for the latter. Note that the sliding mode of controlled systems with discontinuous feedback is the primary mode of operation and allows solving such problems as stabilization, complete controllability, and tracking (movement along a predetermined trajectory). Many studies were devoted to these issues.

Here we continue our research from the papers mentioned above and consider the approximation of Euler broken lines for system (1.1) with positional impulse control.

There are various ways to describe discontinuous trajectories (generalized solutions) of dynamical systems. One of them is to establish rules by which the trajectory jumps (see, for example, [4, 9]). If the jump function is somehow defined, then to describe the solution of the differential equation

$$\dot{x} = f(t,x) + g(t,x)\delta(t) \tag{1.6}$$

with the $\delta$-function $\delta(t)$ concentrated at a point (for convenience, at zero), one can justify the passage to the limit on the solutions of this equation after the replacement of the ideal momentum $\delta(t)$ in it by a sequence of its smooth or continuous approximations. The precise definitions are given below. For convenience, we will call them $\delta$-shaped functions.

Note that different approaches also give different concepts of a generalized solution. Even within the framework of the approximation approach, which goes back to the study of Kurzweil [12], the concept of a solution is not uniquely defined and depends on the nature of the passage to the limit (see, for example, [5, pp. 34–37]).

In this paper, we construct the approximation of "Euler broken lines" for the differential inclusion (1.1) using $\delta$-shaped functions and the jump function (1.3). The research method is based on Yosida's approximations of set-valued mappings from [7, 8] and theorems from [5] on differential equations (1.6) with $\delta$-functions in the coefficients.

## 2. Yosida's approximations

We consider Yosida's approximations for the set-valued mapping $F(t,x)$ under the following assumption.

**Condition $\mathcal{A}$.** *For any points $(t, x, y)$, the inequality*

$$(x - y)^T A(t, x)(u - v) \leq l\|x - y\|^2 \tag{2.1}$$

*holds for any $u \in F(t, x)$ and $v \in F(t, y)$, where $l > 0$ is a constant and $A(t, x) = [a_{ij}(t, x)]_{i,j=1}^n$ is a symmetrical, positive definite, and continuously differentiable matrix whose eigenvalues are from a segment $[c, d]$, $0 < c \leq d < +\infty$.* (In (2.1), we understand vectors as columns, and $T$ means transpose.)

Denote by $z = J_\lambda(t, x)$ the solution of the inclusion $z \in x + \lambda F(t, z)$. Let $F_\lambda(t, x) = (J_\lambda(t, x) - x)/\lambda$. Note that $J_\lambda(t, x)$ and $F_\lambda(t, x)$ are the resolvent and the Yosida approximation, respectively, for the set-valued mapping $x \to -F(t, x)$ (see [1]) for every fixed $t$. Therefore, here we also call the function $F_\lambda(t, x)$ Yosida's approximation for the set-valued mapping $(t, x) \to F(t, x)$.

*Remark 1.* Provided that $A(t, x) \equiv E$, inequality (2.1) is called the condition of right Lipschitz property and is used to study the property of right uniqueness of solutions to differential equations [5, p. 8]. In particular, it follows from the usual Lipschitz condition, which no longer gives the right uniqueness of solutions for set-valued mappings. Provided that the right-hand side of inequality (2.1) is equal to zero and the mapping $F$ does not depend on the variable $t$, Condition $\mathcal{A}$ turns into a condition of monotonicity type for set-valued mappings, which ensures the existence and some properties of the Yosida approximants [1]. Inequality (2.1) is more general than the right Lipschitz condition and the monotonicity condition. The use of the matrix $A(t, x)$ in it is convenient for studying differential equations with a matrix at the derivatives, for example, in Lagrange equations of the second kind when describing mechanical systems with discontinuous nonlinearities (dry friction and discontinuous feedbacks).

In what follows, we suppose that assumptions (B1) and (B2) and condition $\mathcal{A}$ are satisfied. The following Assertions 1–3 follow from Lemma 1 and Theorems 1 and 3 from [8].

**Assertion 1.** *For any segment $I = [a, b]$ and any bounded region $\Omega \subset \mathbb{R}^n$, there is a number $\lambda' > 0$ such that, for all $\lambda \in [0, \lambda']$ and $(t, x) \in \Omega$, the Yosida approximation $F_\lambda(t, x)$ with the following properties is uniquely defined*:

(1) *the mapping $(\lambda, t, x) \to F_\lambda(t, x)$ is continuous in $(\lambda, t, x)$ and Lipschitz in $x$. The latter means that, for each fixed $\lambda \in (0, \lambda']$, there exists a constant $L_\lambda$ such that*

$$\|F_\lambda(t, x) - F_\lambda(t, y)\| \leq L_\lambda \|x - y\|$$

*holds for any $(t, x)$, $(t, y) \in I \times \Omega$;*

(2) *there are constants $l_1 > 0$ and $L > 0$ such that the following inequality holds for any $(t, x)$, $(t, y) \in I \times \Omega$ and $\lambda \in (0, \lambda']$:*

$$(x - y)^T A(t, x)\big(F_\lambda(t, x) - w\big) \leq l_1\|x - y\|^2 + \lambda L; \tag{2.2}$$

(3) *for every fixed point $(t, x)$, $F_\lambda(t, x) \to m(F(t, x))$ as $\lambda \to +0$, where $m(F(t, x)) \in F(t, x)$ is the minimum point of the quadratic form $z^T A(t, x) z$ on the set $F(t, x)$.*

We define a mapping $F_\lambda(t, x)$ for $\lambda = 0$, setting $F_0(t, x) = m(F(t, x))$ for any $(t, x) \in \Omega$ and consider a one-parameter family of equations

$$\dot{x} = F_\lambda(t, x). \tag{2.3}$$

**Assertion 2.** *The following statements are valid*:

(1) *for any initial state* $(t_0, x_0)$, *equation* (2.3) *has a unique solution* $x_\lambda(t)$ *for all sufficiently small values* $\lambda > 0$;

(2) *for* $\lambda = 0$, *a solution* $x_0(t)$ *of equation* (2.3) *exists and is the right-unique solution to the differential inclusion* (1.2), *i.e., any two solutions can merge but cannot fork as* $t$ *increases.* (*Such solutions are called slow in* [1]);

(3) *for solutions* $x_\lambda(t)$ *of equations* (2.3) *with the same initial conditions,* $x_\lambda(t) \to x_0(t)$ *uniformly on any segment* $[t_0, t_1]$ *on which these solutions exist; more precisely,*

$$\|x_\lambda(t) - x_0(t)\|^2 = O(\lambda) \quad \text{for all} \quad t \in [t_0, t_1].$$

**Assertion 3.** *Let the mapping* $F$ *on the right-hand side of the inclusion* (1.2) *and the matrix* $A$ *in inequality* (2.1) *do not depend on the variable* $t$. *Then*

(1) *for any solution* $x(t)$ *of the differential inclusion* (1.2) *defined in an interval, the function* $t \to m(F(x(t))$ *is right-continuous;*

(2) *any solution* $x(t)$ *to the inclusion* (1.2) *for* $\lambda = 0$ *is right-handed. This means that, for all* $t$ *from the domain of this solution,* $D^+x(t) = m(F(x(t))$, *where* $D^+x(t)$ *is the right-hand derivative of the function* $x(t)$.

Assertions 1–3 used below define some qualitative properties of Euler broken lines and can be useful in developing algorithms for numerical calculations.

Consider the differential inclusion

$$\dot{x} \in F(t, x) + \delta_*(t) g(t, x(t - \tau)) \tag{2.4}$$

and the equation

$$\dot{x} = F_\lambda(t, x) + \delta_*(t) g\big(t, x(t - \tau)\big), \tag{2.5}$$

where $F_\lambda(t, x)$ is Yosida's approximation of the mapping $F(t, x)$ and $\tau > 0$ is a positive parameter.

**Lemma 1.** *Let* $g(t, x)$ *be a continuous vector function satisfying the Lipschitz condition in* $x$ *with constant* $L_p$, *and let* $\delta_*(t)$ *be a continuous scalar function.*

*Then there are positive constants* $K_1$, $K_2$, $K_3$, *and* $\lambda'$ *such that, for any solutions* $x_\lambda(t)$ *and* $x(t)$ *to equations* (2.4) *and the inclusion* (2.5), *respectively, defined on the segment* $[t_0 - \tau, t_0 + T]$ *with the initial functions* $x_\lambda(t) = x_\lambda(t_0)$ *and* $x(t) = x(t_0)$ *on the segment* $[t_0 - \tau, t_0]$, *the following inequality holds for all* $t \in [t_0, t_0 + T]$ *and* $\lambda \in (0, \lambda']$:

$$\|x_\lambda(t) - x(t)\|^2 \le (K_1\lambda + K_2\|x_\lambda(t_0) - x(t_0)\|)e^{\int_{t_0}^{t_0+T} K_3|\delta_*(s)|ds}. \tag{2.6}$$

P r o o f. According to Assertion 1, there are numbers $\lambda' > 0$, $L > 0$, and $l_1 > 0$ such that, for all $\lambda \in (0, \lambda']$, a mapping $F_\lambda(t, x)$ is defined, which is continuous and Lipschitz in $x$. Define

$$\Gamma_\lambda(t, x, x') = F_\lambda(t, x) + \delta_*(t) g(t, x')$$

and take an arbitrary $w(t, y, y') \in F(t, y) + \delta_*(t) g(t, y')$. Then there is $u(t, y) \in F(t, y)$ such that $w(t, y, y') = u(t, y) + \delta_*(t) g(y')$. From inequality (2.2), we get

$$\begin{gathered}
(x - y)^T A(t, x)\big(\Gamma_\lambda(t, x, x') - w(t, y, y')\big) = \\
= (x - y)^T A(t, x)\Big(F_\lambda(t, x) - u(t, y) + \delta_*(t)\big(p(t, x') - p(t, y')\big)\Big) = \\
= (x - y)^T A(t, x)\big(F_\lambda(t, x) - u(t, y)\big) + \delta_*(t)(x - y)^T A(t, x)\big(p(t, x') - p(t, y')\big) \le \\
\le l_1\|x - y\|^2 + L\lambda + |\delta_*(t)| L_p \|A(t, x)\| \|x - y\| \|x' - y'\|.
\end{gathered} \tag{2.7}$$

Let $y(t) = x(t) - x_\lambda(t)$ and

$$\xi(t) = \frac{1}{2}\big(y(t)\big)^T A\big(t, x(t)\big) y(t).$$

Then

$$\dot{\xi}(t) = \big(y(t)\big)^T A\big(t, x_(t)\big)\dot{y}(t) + \frac{1}{2}\big(y(t)\big)^T \dot{A}\big(t, x(t)\big) y(t)$$

for almost all $t \in [t_0, t_0 + T]$.

In our proof, we will use some quite obvious estimates and Lipschitz conditions for functions and matrices that assumed or follow from the continuous differentiability of the matrix $A(t,x)$, as well as some well-known inequalities related to the properties and norms of matrices. In particular, we will use the following property of quadratic forms with symmetric positive definite matrices (see, for example, [2, p. 13]):

$$c\|x - y\|^2 \le (x-y)^T A(t,x)(x-y) \le d\|x-y\|^2, \tag{2.8}$$

where the segment $[c, d]$ contains all eigenvalues of the matrix $A(t,x)$ for all $(t,x)$.

Setting

$$x = x(t), \quad y = x_\lambda(t), \quad x' = x(t-\tau), \quad y' = x_\lambda(t-\tau)$$

in inequalities (2.7) and (2.8), we obtain

$$\dot{\xi}(t) \le l_2\xi(t) + L\lambda + l_3\|\delta_*(t)\|\sqrt{\xi(t)\xi(t-\tau)} \tag{2.9}$$

with some positive constants $l_2$ and $l_3$.

Let

$$\eta(t) = \max\{\xi(s) : t_0 \le s \le t\}.$$

Then $\xi(s) \le \eta(t)$ for all $s \in [t_0 - \tau, t]$, $\xi(t') = \eta(t)$ for some $t' \in [t_0, t]$, and (2.9) implies

$$\dot{\xi}(t) \le (l_4 + l_5|\delta_*(t)|)\eta(t) + L\lambda \tag{2.10}$$

with some positive constants $l_4$ and $l_5$. Integrating (2.10), we get

$$\eta(t) = \xi(t') = \eta(t_0) + \int_{t_0}^{t'} \big((l_4 + l_5|\delta_*(s)|)\eta(s) + L\lambda\big)\, ds \le \eta(t_0) + \int_{t_0}^{t} \big((l_4 + l_5|\delta_*(s)|)\eta(s) + L\lambda\big)\, ds.$$

Now Granwall's lemma (see, for example, [3, p. 122]) implies

$$\eta(t) \le (\eta(t_0) + TL\lambda)e^{l_4 T} e^{\int_{t_0}^{t_0+T} l_5|\delta_*(t)|\, dt}.$$

Since $\xi(t) \le \eta(t)$ and $\eta(t_0) = \xi(t_0)$, using this inequality and inequalities (2.8) for quadratic forms, it is easy to find constants $K_1$, $K_2$, and $K_3$ such that inequality (2.6) holds.

The lemma is proved. □

## 3. Differential inclusions with delay and delta functions involved in coefficients

Consider a problem written in the form

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big) + \delta(t) g\big(t, x(t-0)\big), \\ x(t_0) = x_0, \end{cases} \tag{3.1}$$

where $\delta(t)$ is the Dirac $\delta$-function concentrated at the point $t=0$, and the sequence of problems

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big) + \delta_i(t) g\big(t, x(t-\tau_i)\big), & i = 1, 2, \ldots, \\ x(t_0) = x_{i0} \end{cases} \tag{3.2}$$

where $x_{i0} \to x_0$ and $\delta_i(t)$ form a sequence of continuous ($\delta$-shaped) functions satisfying the conditions

(D1) $\delta_i(t) = 0$ ($t \le \alpha_i$, $t \ge \beta_i$), $\delta_i(t) \ge 0$ ($\alpha_i < t < \beta_i$), where $\alpha_i \to 0$, $\beta_i \to 0$, and $\beta_i - \alpha_i \le \tau_i \to 0$ as $i \to +\infty$;

(D2) $\int_{\alpha_i}^{\beta_i} \delta_i(t) dt \to 1$ as $i \to +\infty$.

Let us introduce auxiliary problems:

$$\dot{u} \in F(t, u), \quad u(t_0) = x_0, \quad t \in [t_0, 0]; \tag{3.3}$$

$$\dot{z} \in F(t, z), \quad z(0) = u(0) + g\big(t, u(0)\big), \quad t \in [0, t_0 + T]. \tag{3.4}$$

**Theorem 1.** *Let $g(t, x)$ be a function continuous and Lipschitz in $x$, and let the functions $\delta_i(t)$ satisfy conditions* (D1)–(D2). *Then, for any sequence of solutions $x_i(t)$ of problems* (3.2), *the following holds as $i \to +\infty$:*

$$\begin{aligned} &x_i(t) \to u(t), \quad t_0 \le t < 0; \\ &x_i(t) \to z(t), \quad 0 < t \le t_0 + T, \end{aligned}$$

*where $u(t)$ and $z(t)$ are the solutions of the inclusions* (3.3) *and* (3.4), *respectively.*

P r o o f. First, we consider the sequence of ordinary differential equations

$$\begin{cases} \dot{x}_i = F_\lambda(t, x_i(t)) + \delta_i(t) g\big(t, x_i(t-\tau_i)\big), \\ x_i(t_0) = x_{i0} \end{cases} \tag{3.5}$$

and related auxiliary problems

$$\begin{cases} \dot{u}^\lambda = F_\lambda(t, u^\lambda), \\ u^\lambda(t_0) = x_0, \quad t_0 \le t \le 0; \end{cases} \tag{3.6}$$

$$\begin{cases} \dot{z}^\lambda = F_\lambda(t, z^\lambda), \\ z^\lambda(0) = u^\lambda(0) + g\big(t, u^\lambda(0)\big), \quad 0 \le t \le t_0 + T. \end{cases} \tag{3.7}$$

Here $F_\lambda(t, x)$ is a continuous and Lipschitz in $x$ approximation of the Yosida set-valued mapping $F(t, x)$. Therefore, we can use the well-known results for ordinary differential equations with $\delta$-functions in the coefficients. Note that, by Assertion 1 and 2, problems (3.2)–(3.4) have right-unique solutions, and problems (3.5)–(3.7) have unique solutions for their initial data.

Let $x_i^\lambda(t)$, $i = 1, 2, \ldots$, be solutions of equations (3.5). From Theorem 4 [5, pp. 36–37] and remarks to it there, we obtain the following for any fixed $0 < \lambda < \lambda'$ as $i \to +\infty$:

$$\begin{aligned} &x_i^\lambda(t) \to u^\lambda(t), \quad t_0 \le t < 0, \\ &x_i^\lambda(t) \to z^\lambda(t), \quad 0 < t \le t_0 + T, \end{aligned} \tag{3.8}$$

where $u^\lambda(t)$ and $z^\lambda(t)$ are solutions to equations (3.6) and (3.7), respectively.

By Lemma 1, for arbitrary $\epsilon > 0$, there exist a number $\eta$ and an index $N_1$ such that, for all $t \in [t_0, t_0 + T]$, $0 < \lambda < \eta$, and $i \ge N_1$, we have

$$\|x_i(t) - x_i^\lambda(t)\| \le K\sqrt{\lambda}, \quad \forall i = 1, 2, \ldots \tag{3.9}$$

where $x_i(t)$ are the solutions to problems (3.2), and

$$\|u^\lambda(t) - u(t)\| \le K\sqrt{\lambda} \tag{3.10}$$

for all $t \in [t_0, 0]$, where $u(t)$ is the solution of the differential inclusion (3.6). The first line of (3.8) implies that, for any $t \in [t_0, 0)$ and the same value $\lambda$, there exists a number $N_2 \ge N_1$ such that

$$\|x_i^\lambda(t) - u^\lambda(t)\| < \frac{\varepsilon}{3} \tag{3.11}$$

for all $i \ge N_2$.

Let $\sqrt{\lambda} < \varepsilon/3$. Then, from (3.9)–(3.11), we get $\|x_i(t) - u(t)\| < \varepsilon$ for any fixed $t \in [t_0, 0)$ for all $i \ge N_2$.

Therefore, it is established that $x_i(t) \to u(t)$ as $i \to +\infty$ for any fixed $t \in [t_0, 0)$.

It follows from inequality (2.6) that, for the solutions $z^\lambda(t)$ of equations (3.7) and solution $w(t)$ of the inclusion (3.4), there exists $0 < \eta < \lambda'$ such that

$$\|z^\lambda(t) \to z(t)\| \le \frac{\varepsilon}{3}$$

for all $0 < \lambda < \eta$ and $t \in [0, t_0 + T]$. The second line of (3.8) implies that, for any fixed $0 < \lambda < \lambda'$ and $t \in (0, t_0 + T]$, there exists a number $N_3$ such that

$$\|x_i^\lambda(t) - w^\lambda(t)\| < \frac{\varepsilon}{3}$$

for all $i \ge N_3$. Now, in view of (3.9), similarly to the above, for any $\varepsilon > 0$, there exists a positive integer $N_4 \ge N_3$ such that

$$\|x_i(t) - w(t)\| \le \|w(t) - w^\lambda(t)\| + \|w^\lambda(t) - x_i^\lambda(t)\| + \|x_i^\lambda(t) - x_i(t)\| < \varepsilon$$

for any fixed $t \in (0, t_0 + T]$ for all $i \ge N_4$.

Hence, $x_i(t) \to z(t)$ as $i \to +\infty$ for any fixed $t \in (0, t_0 + T]$. □

**Definition 1.** *By a generalized solution of inclusion* (3.1) *we mean a function* $x(t)$ *satisfying the differential inclusion* (3.3) *on the segment* $[t_0, 0]$ *and the differential inclusion* (3.4) *on* $(0, t_0+T]$ *with the initial condition* $x(+0) = x(0) + p\big(t, x(0)\big)$.

By this definition, Theorem 1 ensures the existence and structure of generalized solutions of the inclusion (3.1). A convenient convention for us is to extend the generalized solution $x(t)$ at the discontinuity point $t = 0$ by a limit on the left (which obviously exists).

Consider a differential equation of the form

$$\dot{x}(t) = F_\lambda\big(t, x(t)\big) + \delta(t)p\big(t, x(t-0)\big), \tag{3.12}$$

where $F_\lambda(t, x)$ is the Yoshida approximation of the set-valued mapping $F(t, x)$.

**Corollary 1.** *Let the assumptions of Theorem 1 be satisfied. Then there exist positive constants* $\lambda'$ *and* $K$ *such that, for any generalized solutions* $x(t)$ *and* $x^\lambda(t)$ *of problems* (3.1) *and* (3.6), *respectively, the following holds*:

$$\|x(t) - x_\lambda(t)\| \le K(\sqrt{\lambda} + \|x(t_0) - x_\lambda(t_0)\|)$$

*for all* $t \in I$ *and* $\lambda \in (0, \lambda']$.

**Corollary 2.** *Let the assumptions of Theorem 1 be satisfied, let $x_i^\lambda(t)$ be solutions of equations (3.5) such that $x_i^\lambda(t_0) \to x_0$ for $\lambda \to +0$, $i \to +\infty$, and let $x(t)$ be a generalized solution of inclusion (3.2) with the initial condition $x(t_0) = x_0$. Then, for any $\varepsilon > 0$, there are numbers $\lambda' > 0$ and $N$ such that*

$$\|x_i^\lambda(t) - x(t)\| < \varepsilon$$

*for all $0 < \lambda < \lambda'$ and $i \geq N$.*

*Remark 2.* Theorem 1 and its corollaries are formulated for differential inclusions with impulsive action at time $t = 0$. However, this does not limit the generality of the results since the change of variable $s = t - t'$ allows us to consider inclusions with impulsive action at the time $t = t'$.

*Remark 3.* In Definition 1 of the generalized solution of the differential inclusion (3.1), the set-valued mapping $F(t,x)$ can be replaced by the mapping $m(F(t,x))$ from Assertion 1. In the case when the matrix $A(t,x)$ in Condition $\mathcal{A}$ is identity, $m(F(t,x))$ is the point of the set $F(t,x)$ closest to the origin in the Euclidean norm.

## 4. Euler broken line approximation

Euler broken line approximations for system (1.1) are based on Theorem 1 and its corollaries. Suppose that assumptions (B1) and (B2) and Condition $\mathcal{A}$ are satisfied and consider a jump function $g(t,x) = S(t,x,r(t,x))$ defined by the equality (1.3).

We define a partition $h\colon t_0 < t_1 < \ldots < t_N = t_0 + T$ of the segment $I = [t_0, t_0 + T]$, and Euler broken lines $x^h(t)$, which on each interval $(t_k, t_{k+1}]$ coincide with the solutions of the Cauchy problems for the differential inclusion

$$\dot{x} \in F(t,x), \quad x(t_i) = x^h(t_k) + g\big(t_k, x^h(t_k)\big), \quad k = 0, \ldots, N-1.$$

In this case, the following conditions are satisfied for $k = 0, \ldots, N-1$:

$$x^h(t_0) = x_0, \quad x^h(t_k + 0) = x^h(t_k) + g\big(t_k, x^h(t_k)\big).$$

For the partition $h$ of the segment $I$, we introduce the sequence of problems

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big) + g\big(t, x(t - \tau_i^k)\big) \sum\limits_{k=0}^{N-1} \delta_i^k(t - t_k), \quad i = 1, 2, \ldots, \\ x(t_0) = x_0 + g(t_0, x_0) \end{cases} \tag{4.1}$$

as $i \to +\infty$. For each fixed $k = 1, \ldots, N-1$, we impose the following conditions on the functions $\delta_i^k(t)$:

(D1k) $\delta_i^k(t) = 0$ $(t \leq \alpha_i^k, t \geq \beta_i^k)$ and $\delta_i^k(t) \geq 0$ $(\alpha_i^k < t < \beta_i^k)$, where $\alpha_i^k \to 0$, $\beta_i^k \to 0$, and $\beta_i^k - \alpha_i^k \leq \tau_i^k \to 0$ as $i \to +\infty$;

(D2k) $\int\limits_{\alpha_i^k}^{\beta_i^k} \delta_i^k(t)dt \to 1$ for all $i = 1, 2, \ldots$.

Since $\alpha_i^k \to 0$, $\beta_i^k \to 0$, and $\tau_i^k \to 0$ as $i \to +\infty$, we initially consider these quantities to be so small that the intervals $(t_k + \alpha_i^k, t_k + \beta_i^k)$, $k = \overline{1, N-1}$, are pairwise disjoint.

**Theorem 2.** *Let $F(t,x)$ and $g(t,x)$ satisfy the assumptions of Lemma 1, and let the functions $\delta_i^k(t)$ satisfy conditions* (D1k)–(D2k)*. Then, for every fixed partition $h$ of the segment $I$, the sequence of solutions $x_i^h(t)$ of problems* (4.1) *converges as $i \to +\infty$ to the Euler broken line $x^h(t)$ at every point $t \in I$ such that $t \neq t_k$, $k = \overline{0, N-1}$.*

P r o o f. Taking into consideration Remark 2, we apply Theorem 1 to the inclusion (4.1) on the segment $I_1^\varepsilon = [t_0, t_2 - \varepsilon]$ for an arbitrary $\varepsilon > 0$ so small that $t_1 \in I_1^\varepsilon$. Here we take into account that, starting from some number $i$, $\delta_i^2(t) = 0$ will hold for all $t \in I_1^\varepsilon$. As a result, we get

$$x_i^h(t) \to x^h(t) \tag{4.2}$$

for any $t \in I_1^\varepsilon$, $t \neq t_1$ and $t \neq t_0$. Then, in view of the right uniqueness of the solutions of the inclusion $\dot{x} \in F(t,x)$ and the arbitrariness of $\varepsilon > 0$, we conclude that (4.2) holds at all points of the segment $[t_0, t_2]$ except for the points $t_k$, $k = 0, 1, 2$.

Now, as initial data, we take some point $s \in (t_1, t_2)$ (for example, the midpoint of this interval) and the value $x_i^h(s)$ of the Euler broken line at this point. Applying similar reasoning to the segment $I_2^\varepsilon = [s, t_3 - \varepsilon]$ and taking into account the right uniqueness of the solutions, we conclude that (4.2) holds at all points of the segment $[t_0, t_3]$ except for the points $t_k$, $k = 0, 1, 2, 3$. Here we took into account that, starting from some number $i$, $\delta_i^k(t) = 0$ holds for all $t \in I_2^\varepsilon$, $k = 1, 2$. This process continues up to the point $t_{N-1}$, and at this last step, we consider the segment $[s, t_0 + T]$, where $s$ is the midpoint of the segment $[t_{N-2}, t_{N-1}]$. □

Consider the problems

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big) + p\big(x(t-0)\big) \sum\limits_{k=1}^{N-1} \delta(t - t_k), \\ x(t_0) = x_0 + g(x_0). \end{cases} \tag{4.3}$$

$$\begin{cases} \dot{x}(t) = F_\lambda\big(t, x(t)\big) + p\big(x(t-0)\big) \sum\limits_{k=1}^{N-1} \delta(t - t_k), \\ x(t_0) = x_0 + g(x_0), \end{cases} \tag{4.4}$$

where $F_\lambda(t,x)$ is the Yoshida approximation for $F(t,x)$.

For (4.3) and (4.4), the concepts of generalized solutions $x(t)$ and $x_\lambda(t)$ are introduced by analogy with Definition 1.

**Corollary 3.** *Let all assumptions of Theorem 2 be satisfied. Then, for any fixed partition $h$ of the segment $I$, there exists a constant $K$ depending on the number $N$ of points of the partition $h$ such that, for any generalized solutions $x(t)$ and $x_\lambda(t)$ of the inclusions* (4.3) *and equations* (4.4)*, respectively, the inequality*

$$\|x(t) - x_\lambda(t)\| \leq K\sqrt{\lambda}$$

*holds for any $0 < \lambda < \lambda'$ and $t \in I$.*

P r o o f. The proof follows from the successive application of Corollary 3 to the segments $[t_{k-1}, t_k]$ and the initial conditions $x(t_{k-1} + 0)$ and $x_\lambda(t_{k-1} + 0)$ for $k = \overline{1, N-1}$. □

**Corollary 4.** *Let all assumptions of Theorem 2 be satisfied. Then, for any fixed partition $h$ of the segment $I$, there exists a constant $K$ depending on the number $N$ of points of the partition $h$*

*such that, for any generalized solutions $x_\lambda(t)$ of equation (4.4) and "Euler's broken lines" $x^h(t)$ of inclusions $\dot{x} \in F(t,x)$, the inequality*

$$\|x^h(t) - x_\lambda(t)\| \le K\sqrt{\lambda}$$

*holds for any $0 < \lambda < \lambda'$ and $t \in (t_0, t_0 + T]$.*

P r o o f. The definition of the Euler broken line $x^h(t)$ implies that, on the interval $(t_0, t_0+T]$, it coincides with the generalized solution to the inclusion (4.3), and then the corollary follows from Corollary 3. □

*Remark 4.* Let condition (1.4) be satisfied. Then, using Theorem 2 and its corollaries, we can formulate statements about the approximation of ideal impulse-sliding modes, which satisfy the inclusion (1.5).

## 5. Conclusion

Let us make a number of concluding remarks.

1. The Yosida approximation has a rather complex structure, and for its applications, it is necessary to calculate the resolution $J_\lambda$, which reduces to finding fixed points of set-valued mappings. It is not always possible to solve such a problem in an analytical form in the general case. At the same time, the results of Sections 2–4 remain valid for any other continuous approximations of set-valued mappings for which inequality (2.2) holds. For a set-valued function $u(x) = \operatorname{sgn} x$ equal to $-1$ for $x > 0$, $1$ for $x < 0$, and segment $[-1, 1]$ for $x = 0$, the Yosida approximation is

$$u_\lambda(x) = \begin{cases} x/\lambda, & |x| \le \lambda; \\ \operatorname{sgn} x, & |x| > \lambda. \end{cases}$$

   But instead of it, for example, the function $u_\lambda(x) = 2/\pi \cdot \arctan(\lambda x)$ can be used.

2. Consider the system

$$P(t,x)\dot{x} \in R(t,x) - H(t,x)\operatorname{sgn} x, \tag{5.1}$$

   where $P(t,x)$ is a symmetric, positive definite $n \times n$ matrix, $H(t,x)$ is a diagonal $n \times n$ matrix with nonzero elements, and $\operatorname{sgn} x = (\operatorname{sgn} x_1 \times \cdots \times \operatorname{sgn} x_n)$ is a set-valued function. If the function $R(t,x)$ and the elements of the matrices $P$ and $H$ are continuous and locally Lipschitz functions in $x$, then the mapping $F(t,x) = P^{-1}(t,x)(R(t,x) - H(t,x)\operatorname{sgn} x)$ satisfies assumptions (B1) and (B2) and Condition $\mathcal{A}$ with matrix $A(t,x) = P(t,x)$, and all the results of Sections 2–4 are valid. In this case, to approximate $\operatorname{sgn} x_i$, one can use functions of the form $u_\lambda(x_i)$.

   If the matrix $P(t,x)$ and the function $H(t,x)$ are constant, then, using the scheme of the proof of Lemma 1, we can obtain the estimate

$$v(t) \le \frac{H\gamma\lambda}{2L}(e^{2L/\gamma(t-t_0)} - 1),$$

   where $L$ is the Lipschitz constant of the function $f(t,x)$ in the variable $x$, $\gamma$ is the smallest eigenvalue of the matrix $P$, $H = \sum_{i=1}^n H_i$, and

$$v(t) = (x(t) - x_\lambda(t))^T P(x(t) - x_\lambda(t)), \quad x(t_0) = x_\lambda(t_0).$$

3. A large class of systems that lead to differential inclusions are differential equations with a discontinuous right-hand side. As stated in [6], near the discontinuity points of the function $F(t,x)$, the approximate solution obtained by some numerical method usually differs from the exact one by $O(h)$, where $h$ is an integration step, regardless of the order of accuracy of the approximate method. The guaranteed in Lemma 1 accuracy of the approximation of solutions to the differential inclusion (1.2) by solutions to the approximating inclusions (2.3) is of order $O(\sqrt{\lambda})$. Numerical experiments using computers and graphical visualization of the integration results have shown that if an integration step is much smaller than the parameter $\lambda$, then the qualitative behavior of the approximate solutions to the approximating equations is closer to the exact sliding modes of the original discontinuous equations; in the presence of discontinuities of the signature type, there are no sawtooth curves characteristic of approximate solutions to discontinuous systems.

4. Note that discontinuous characteristics, as a rule, are included in systems of equations in the form of terms or factors with continuous functions. Therefore, for some classes of systems, it is expedient to approximate discontinuous characteristics directly.

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