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ON SOME VERTEX–TRANSITIVE DISTANCE–REGULAR ANTIPODAL COVERS OF COMPLETE GRAPHS[1]

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Abstract: In the present paper, we classify abelian antipodal distance-regular graphs Γ of diameter 3 with the following property: $(*)$ Γ *has a transitive group of automorphisms* $\widetilde{G}$ *that induces a primitive almost simple permutation group* $\widetilde{G}^{\Sigma}$ *on the set* Σ *of its antipodal classes.* There are several infinite families of (arc-transitive) examples in the case when the permutation rank $\mathrm{rk}(\widetilde{G}^{\Sigma})$ of $\widetilde{G}^{\Sigma}$ equals 2; moreover, all such graphs are now known. Here we focus on the case $\mathrm{rk}(\widetilde{G}^{\Sigma})=3$. Under this condition the socle of $\widetilde{G}^{\Sigma}$ turns out to be either a sporadic simple group, or an alternating group, or a simple group of exceptional Lie type, or a classical simple group. Earlier, it was shown that the family of non-bipartite graphs Γ with the property $(*)$ such that $\mathrm{rk}(\widetilde{G}^{\Sigma})=3$ and the socle of $\widetilde{G}^{\Sigma}$ is a sporadic or an alternating group is finite and limited to a small number of potential examples. The present paper is aimed to study the case of classical simple socle for $\widetilde{G}^{\Sigma}$. We follow a classification scheme that is based on a reduction to *minimal* quotients of Γ that inherit the property $(*)$. For each given group $\widetilde{G}^{\Sigma}$ with simple classical socle of degree $|\Sigma|\leq 2500$, we determine potential minimal quotients of Γ, applying some previously developed techniques for bounding their spectrum and parameters in combination with the classification of primitive rank 3 groups of the corresponding type and associated rank 3 graphs. This allows us to essentially restrict the sets of feasible parameters of Γ in the case of classical socle for $\widetilde{G}^{\Sigma}$ under condition $|\Sigma|\leq 2500$.

Keywords: Distance-regular graph, Antipodal cover, Abelian cover, Vertex-transitive graph, Rank 3 group.

1. Introduction

Let Γ be an antipodal distance-regular graph of diameter three. Then Γ is an antipodal r-cover of a complete graph on $k+1$ vertices, and its intersection array has form $\{k,(r-1)\mu,1;1,\mu,k\}$, where k, r and μ denote the valency of Γ, the size of its antipodal classes and the number of common neighbours for each two vertices at distance two of Γ, respectively (e.g. see [2]); for brevity, we will refer to such a graph as an $(k+1,r,\mu)$-*cover*. We denote by $\mathcal{CG}(\Gamma)$ the group of all automorphisms of Γ fixing setwise each of its antipodal classes. If the group $\mathcal{CG}(\Gamma)$ is abelian and acts regularly on (every) antipodal class of Γ, then Γ is called an *abelian* $(k+1,r,\mu)$-*cover* (see [5]). There are some important links between abelian covers and other combinatorial or geometric objects (we refer to [9] and [5] for more background). The problem of finding new their constructions involves many natural questions on possible structure of such a graph, and one of them is to study vertex-transitive representatives.

In the present paper, we classify abelian $(k+1,r,\mu)$-covers Γ with the following property:

$(*)$ Γ *has a transitive group of automorphisms* $\widetilde{G}$ *that induces a primitive almost simple permutation group* $\widetilde{G}^{\Sigma}$ *on the set* Σ *of its antipodal classes.*

Without loss of generality, we may assume that $\widetilde{G}$ coincides with the full pre-image of $\widetilde{G}^{\Sigma}$ in $\mathrm{Aut}(\Gamma)$. When the permutation rank $\mathrm{rk}(\widetilde{G}^{\Sigma})$ of $\widetilde{G}^{\Sigma}$ equals 2, there are several infinite families of

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(arc-transitive) examples; moreover, all such graphs are now known. Here we focus on the case $\mathrm{rk}(\widetilde{G}^{\Sigma}) = 3$. Under this condition the socle of $\widetilde{G}^{\Sigma}$ turns out to be either a sporadic simple group, or an alternating group, or a simple group of exceptional Lie type, or a classical simple group (see [3, Ch. 11] for an overview on classification of primitive rank 3 permutation groups).

In [16] and [17], it was shown that the family of non-bipartite graphs Γ with the property $(*)$ such that $\mathrm{rk}(\widetilde{G}^{\Sigma}) = 3$ and the socle of $\widetilde{G}^{\Sigma}$ is a sporadic or alternating group is finite and limited to a small number of potential examples. The present paper is aimed to study the case of classical simple socle for $\widetilde{G}^{\Sigma}$. We follow a classification scheme that was proposed in [16] and that is based on a reduction to *minimal* quotients of Γ that inherit the property $(*)$. For each given group $\widetilde{G}^{\Sigma}$, we determine potential minimal quotients of Γ, applying the constraints for their spectrum and parameters obtained in [16] in combination with the classification of primitive rank 3 groups of the corresponding type (see [8], [11], and also [13]) and associated rank 3 graphs (see [3, Ch. 11]). This allows us to essentially restrict the sets of feasible parameters of Γ in the case of classical socle for $\widetilde{G}^{\Sigma}$ with $|\Sigma| \leq 2500$. In particular, we show that for most of these sets Γ must be a covering of a certain distance-transitive Taylor graph.

2. Preliminaries

We keep the notation and terminology from [16] and we refer the reader to [1] and [2] for basic definitions. Next we recall some of them. For a finite group G, we denote by $\mathrm{Soc}(G)$, $Z(G)$ and G' its socle, center and derived subgroup, respectively. If $G = G'$, then $\mathrm{M}(G)$ denotes its Schur multiplier. If $G \neq 1$, then we write "$\mathrm{d}_{\min}(G)$" to denote the number $|G : H|$, where H is a proper subgroup of G of the smallest possible index. Further, if G is a transitive permutation group on a finite set Ω and $\mathrm{O}rb_2(G)$ is the set of G-orbitals on Ω, then the number $|\mathrm{O}rb_2(G)|$, denoted by $\mathrm{rk}(G)$, is called the (*permutation*) *rank* of G. For each $Q \in \mathrm{O}rb_2(G)$, Q^* denotes the orbital paired with Q. If $Q^* = Q$ and $a \in \Omega$, then $Q(a)$ denotes the set of all points $b \in \Omega$ such that $(a, b) \in Q$.

In what follows, we consider only undirected graphs without loops or multiple edges. For a graph Γ by $\mathcal{V}(\Gamma)$ and $\mathcal{A}(\Gamma)$ we denote its vertex set and the arc set, respectively. An (n, r, μ)-cover is equivalently defined as a connected graph, whose vertex set admits a partition into n cells (called antipodal classes or fibres) of the same size $r \geq 2$ such that each cell induces an r-coclique, the union of any two distinct cells induces a perfect matching, and every two non-adjacent vertices that lie in distinct cells have exactly $\mu \geq 1$ common neighbours. Since an (n, r, μ)-cover is bipartite if and only if $r = 2$ and $\mu = n - 2$, and for each $n \geq 3$ there is a unique (abelian) $(n, 2, n-2)$-cover (see [2, Corollary 1.5.4]), we omit these from further consideration. We will say that the set of parameters (n, r, μ) of a non-bipartite abelian (n, r, μ)-cover Γ is *feasible* if it satisfies the known necessary conditions for the existence of Γ that are collected in [16, Proposition 1] (see [16] for detailed references) and [5, Lemma 3.5, Theorem 5.4]. In view of [5], for every (n, r, μ)-cover Γ and every subgroup N of $\mathcal{CG}(\Gamma)$ of order less than r, the *quotient* Γ^N that is defined as the graph on the set of N-orbits in which two vertices are adjacent if and only if there is an edge of Γ between the corresponding orbits, is a $(n, r/|N|, \mu|N|)$-cover. Hence if Γ is a non-bipartite abelian (n, r, μ)-cover, then, using decomposition $\mathcal{CG}(\Gamma) = O_p(\mathcal{CG}(\Gamma)) \times N$, where p is a prime divisor of r, we obtain that Γ possesses a quotient Γ^N that is a non-bipartite abelian $(n, p^l, \mu|N|)$-cover with $p^l = |O_p(\mathcal{CG}(\Gamma))|$. Clearly, the factor group $\mathrm{Aut}(\Gamma)/N$ acts as a group of automorphisms of Γ^N, and in case $\mathcal{CG}(\Gamma) > M > N$ other quotients Γ^M inherit a similar property when $M \trianglelefteq \mathrm{Aut}(\Gamma)$. Thus parameters of Γ may depend on the structure of $\mathcal{CG}(\Gamma)$. This is also demonstrated by the fact that for each non-bipartite abelian (n, r, μ)-cover, every prime divisor of r is also a divisor of n (see [5, Theorem 9.2] and also [6, Theorem 2.5]). These basic observations are crucial for our following arguments; they will be used further without any additional reference.

The next result from [16] distinguishes several types of quotients that an abelian non-bipartite

$(k+1,r,\mu)$-cover with the property $(*)$ may possess.

Proposition 1 [16, Proposition 2]. *Let Γ be a non-bipartite $(k+1,r,\mu)$-cover and Σ be the set of its antipodal classes. Suppose Γ has a transitive automorphism group G_1 which induces a primitive almost simple permutation group ${G_1}^{\Sigma}$ on Σ and put $T = \mathrm{Soc}({G_1}^{\Sigma})$. Let G be the full pre-image of the group T in G_1 and K be the kernel of the action of the group G on Σ. Then K contains a subgroup N that is normal in G_1 and satisfies one of the following conditions (below the symbol $^-$ denotes factorization with respect to N):*

(T1) *$\overline{K} \simeq E_{p^l}$ is an elementary abelian group of exponent p and either*

- *(i) $\overline{G} = \overline{K} \times \overline{G}'$ and $\overline{G}' \simeq T$, or*
- *(ii) $\overline{G}$ is a quasi-simple group with center $\overline{K}$;*

(T2) *$\overline{K} \simeq E_{p^l}$ is an elementary abelian group of exponent p, T acts faithfully on $\overline{K}$, i.e. $T \le GL_l(p)$, and $\mathrm{d}_{\min}(T) \le (p^l-1)/(p-1)$;*

(T3) *$\overline{K} \simeq S^l$, where S is a simple non-abelian group, and either*

- *(i) $\overline{G} = \overline{K} \times C_{\overline{G}}(\overline{K})$ and $C_{\overline{G}}(\overline{K}) \simeq T$, or*
- *(ii) $\overline{G} \le \mathrm{Aut}(\overline{K})$ and T contains a proper subgroup of index dividing l.*

Each graph Γ that satisfies the hypothesis of Proposition 1 will be referred to as a *minimal $(k+1,r,\mu)$-cover of type* (Tx) with $x = 1,2,3$ and denoted by $\Gamma(G_1,G,K)$ if $|K| = r$, the triple (G_1,G,K) satisfies the condition (Tx) from the conclusion of Proposition 1 and K is a minimal normal subgroup of G_1 (in particular, $N = 1$). Thus, for a minimal $(k+1,r,\mu)$-cover $\Gamma(G_1,G,K)$ the number r is a prime when $G_1 = G$ and $K \le Z(G)$.

From now on Γ is a non-bipartite abelian $(k+1,r,\mu)$-cover with property $(*)$, Σ is the set of its antipodal classes, $\widetilde{G}$ is a transitive group of automorphisms of Γ which induces a primitive almost simple permutation group $\widetilde{G}^{\Sigma}$ on Σ, $\mathrm{rk}(\widetilde{G}^{\Sigma}) = 3$, k_1 and k_2 are the non-trivial subdegrees of $\widetilde{G}^{\Sigma}$, $K = \mathcal{CG}(\Gamma) \le \widetilde{G}$ and G is the full pre-image of the group $\mathrm{Soc}(\widetilde{G}^{\Sigma})$ in $\widetilde{G}$.

Now we proceed with final technical definitions. For a vertex x of Γ, by $F(x)$ and $\Gamma_1(x)$ (or $[x]$) we denote, respectively, the antipodal class of Γ containing x, and its neighborhood in Γ. Put $\Omega = \mathcal{V}(\Gamma)$, and fix $a \in \Omega$ and $F = F(a)$. Let $M = \widetilde{G}_{\{F\}}$ and $H = \widetilde{G}_a$ (note $|K| = r$ implies $M = K : H$). Then $\mathcal{A}(\Gamma) = Q_1 \cup Q_2$ for some $Q_1, Q_2 \in Orb_2(G)$ with $Q_i = Q_i^*$ (see [16]), $|Q_i| = rk_i(k+1)$, and $|H : \widetilde{G}_{a,b_i}| = k_i$ for each arc $(a,b_i) \in Q_i$, so H has exactly two orbits on $\Gamma_1(a)$ (with representatives b_1 and b_2). For $i = 1,2$, let Φ_i denote the (rank 3) graph on Σ in which two vertices $F(x)$ and $F(y)$ are adjacent if and only if $(x,y) \in Q_i$. If $\mathrm{rk}(G^{\Sigma}) = 3$ then the group G^{Σ} is also primitive as $\mu(\Phi_i) \ne 0, k_i$ (see, for example, [1, 16.4]). Moreover, the parameters k_1, k_2 and λ satisfy the following equation (see [16])

$$(\lambda - \lambda_1)k_1 \quad = \quad (\lambda - \lambda_2)k_2,$$

where $\lambda_i = |\Gamma_1(b_i) \cap H(b_i)|$, $i = 1,2$. We will say that Γ admits an *H-uniform edge partition* (*with parameters* (μ_1,μ_2)) (see [16]), if for each $j = 1,2$ and for every two distinct vertices z_1, $z_2 \in F$, the number of edges between $Q_j(z_1)$ and $Q_j(z_2)$ is constant and equal to $k_j\mu_j$, where μ_j is a fixed integer.

Lemma 1 [16, Lemma 1]. *Suppose that $G_{\{F\}} = G_a \times K$ and $\mathrm{rk}(G^{\Sigma}) = 3$. If H acts transitively on $F \setminus \{a\}$ or $r \le 3$, then Γ admits an H-uniform edge partition.*

Theorem 1 [16, Theorem 1]. *Suppose that $G_{\{F\}} = G_a \times K$ and $\mathrm{rk}(G^\Sigma) = 3$. Then for each $x \in F \setminus \{a\}$ we have*

$$(\mu - \mu_1)k_1 = (\mu - \mu_2)k_2,$$

where $\mu_i = |\Gamma_1(b_i) \cap Q_i(x)|$, $i = 1, 2$. If, moreover, Γ admits an H-uniform edge partition with parameters (μ_1', μ_2'), then $\mu_i' = \mu_i$ (in particular, $k_i - 1 = \lambda_i + (r-1)\mu_i$) for every $i = 1, 2$ and $\gamma = -(\lambda - \lambda_1 - \lambda_2) + (\mu - \mu_1 - \mu_2) = r(\mu - \mu_1 - \mu_2) - 1$ is an eigenvalue of Γ.

3. Main results

Theorem 2. *Suppose that $\Gamma = \Gamma(\widetilde{G}, G, K)$ is a minimal abelian $(k+1, r, \mu)$-cover, $k+1 \leq 2500$, $\mathrm{rk}(\widetilde{G}^\Sigma) = 3$ and $T = \mathrm{Soc}(\widetilde{G}^\Sigma)$ is a classical simple group, isomorphic to the group $\widetilde{M}/Z(\widetilde{M})$, where $\widetilde{M} = Sp_{2n-2}(q)$, $\Omega_{2n}^\pm(q)$, $\Omega_{2n-1}(q)$ or $SU_n(q)$ for $n \geq 3$. Assume $\widetilde{G} = G$ whenever $\mathrm{rk}(T) = 3$. Then one of the following statements is true*:

(1) $T \simeq PSU_4(4)$, $\mathrm{rk}(T) = 3$, $k+1 = 1105$, $r = 5$ *and* $\mu = 210$;

(2) $T \simeq G' \simeq P\Omega_{2n}^\pm(2)$, $\mathrm{rk}(T) = 3$, $k+1 = (2^{2n-1} - \varepsilon 2^{n-1})$, *where* $\varepsilon = \pm 1$ *and* $n \leq 6$, $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$, $r = 2$ *and either* $G = G' \simeq Z_2.P\Omega_8^+(2)$, $\varepsilon = +1$, $k+1 = 120$, *and* $\mu \in \{64, 54\}$, *or the group* G' *is intransitive on* $\mathcal{V}(\Gamma)$;

(3) $T \simeq G' \simeq P\Omega_5(8) \simeq PSp_4(8)$, $\mathrm{rk}(T) = 5$, $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) \neq k - 1$, $k+1 = 2016$ *and* $r\mu \in \{2048, 1980\}$, *or* $k+1 = 2080$ *and* $r\mu \in \{2048, 2108\}$, *wherein either* $r = 4$ *and* G' *is intransitive on* $\mathcal{V}(\Gamma)$, *or* $r = 2$ *and* G' *is transitive on* $\mathcal{V}(\Gamma)$;

(4) $T \simeq P\Omega_m(q)$, $\mathrm{rk}(T) = 3$, $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$, $r = 2$ *and either*

  (*i*) $m = 5$, $q = 3$, $k+1 = 36$ *and* $\mu \in \{16, 18\}$, *or*

  (*ii*) $m = 5$, $q = 4$, *with* $k+1 = 120$ *and* $\mu \in \{54, 64\}$ *or* $k+1 = 136$ *and* $\mu \in \{64, 70\}$, *or*

  (*iii*) $m = 7$, $q = 4$, *with* $k+1 = 2016$ *and* $\mu \in \{990, 1024\}$ *or* $k+1 = 2080$ *and* $\mu \in \{1024, 1054\}$,

and in all cases (*i*)–(*iii*) *the group* G' *is intransitive on* $\mathcal{V}(\Gamma)$;

(5) $T \simeq G' \simeq SU_3(3)$, $\mathrm{rk}(T) = 4$, $k+1 = 36$, $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$, $r = 2$, $\mu \in \{16, 18\}$ *and* G' *is intransitive on* $\mathcal{V}(\Gamma)$;

(6) $T \simeq G' \simeq PSp_6(2) \simeq P\Omega_7(2)$, $\mathrm{rk}(T) = 3$, $k+1 = 120$, $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$, $r = 2$, $\mu \in \{54, 64\}$ *and* G' *is intransitive on* $\mathcal{V}(\Gamma)$.

Moreover, if $r = 2$ and $G' \simeq T$, then for any given pair of parameters k and μ, Γ is a unique (*up to isomorphism*) *distance-transitive $(k+1, 2, \mu)$-cover.*

P r o o f. Let $k+1 \leq 2500$. Under this condition $\mathrm{rk}(T) = 3$ for all k except the following cases (a)–(d) (note that in [16, Example] the case (d) is missing, and the subdegrees k_1, k_2 for the case (c) are mistyped):

(a) $k+1 = 36$, $k_1 = 14$, $k_2 = 21$, $T \simeq PSL_2(8)$, $\mathrm{rk}(T) = 5$, $\widetilde{G}^\Sigma \simeq P\Gamma L_2(8) = T.3$;

(b) $k+1 = 36$, $k_1 = 14$, $k_2 = 21$, $T \simeq PSU_3(3)$, $\mathrm{rk}(T) = 4$, $\widetilde{G}^\Sigma \simeq P\Gamma U_3(3) = T.2$;

(c) $k+1 = 2016$, $k_1 = 455$, $k_2 = 1560$, $G^\Sigma \simeq Sp_4(8)$, $\mathrm{rk}(G^\Sigma) = 5$ and $\widetilde{G}^\Sigma \simeq Sp_4(8).Z_3$

(*d*) $k+1=2080$, $k_1=567$, $k_2=1512$, $G^\Sigma\simeq Sp_4(8)$, $\mathrm{rk}(G^\Sigma)=5$ and $\widetilde{G}^\Sigma\simeq Sp_4(8).Z_3$.

Then, by [16, Propositions 2, 3], if $\mathrm{rk}(T)=3$, $T\not\leq \mathrm{Aut}(K)$ and $2(\lambda(\Phi_1)+\lambda(\Phi_2)+1)\neq k-1$, then either $G'\simeq T$ acts transitively on $\mathcal{V}(\Gamma)$, or G is a quasisimple group. Therefore, in order to find some necessary conditions for Γ to exist (as well as for its covers with property $(*)$), in case $\mathrm{rk}(T)=3$ it suffices to consider the case $\widetilde{G}=G$, and if, moreover, $K\leq Z(G)$, then one may assume that r is prime. Taking this into account, we further specify the possible structure of G for each potential pair $(\widetilde{G}^\Sigma,\Phi_1)$.

Throughout the rest of the proof, we put $N=G'$ and denote by θ and $-\tau$, respectively, the positive and negative eigenvalues of Γ, other than k and -1. We will consider the following possible combinations for T and complementary rank 3 graphs Φ_1 and Φ_2 associated with $\widetilde{G}^\Sigma$, applying their description from [8] and [3, Theorem 11.3.2].

(A) Let $k_1=q(q^{n-1}-1)(tq^{n-1}+1)/(q-1)$ and suppose the graph Φ_1 has parameters

$$\Big(\frac{q^n-1}{q-1}(tq^{n-1}+1),q\frac{q^{n-1}-1}{q-1}(tq^{n-2}+1),q^2\frac{q^{n-2}-1}{q-1}(tq^{n-3}+1)+q-1,\frac{q^{n-1}-1}{q-1}(tq^{n-2}+1)\Big),$$

where $t=q,1,q,q^2,q^{1/2},q^{3/2}$ for $\widetilde{M}=Sp_{2n}(q)$, $\Omega^+_{2n}(q)$, $\Omega_{2n+1}(q)$, $\Omega^-_{2n+2}(q)$, $SU_{2n}(\sqrt{q})$ or $SU_{2n+1}(\sqrt{q})$, respectively (see [3, Theorem 11.3.2(i)]).

By condition $k+1\leq 2500$, hence the equality $2(\lambda(\Phi_1)+\lambda(\Phi_2)+1)=k-1$ holds if and only if $t=1$, $q=3$, $n=2$ and $(v,k_1,\lambda(\Phi_1),\mu(\Phi_1))=(16,6,2,2)$, which contradicts the constraint $n\geq 3$ for $t=1$. If r is a power of a prime p, say $r=p^l$, then feasible sets of parameters k, r, and μ are described by Table 1, and Γ has no H-uniform edge partitions in the cases $t=1,q,q^2$ (this can be easily checked by complete enumeration in GAP [14], based on Theorem 1, [16, Proposition 1] and [5, Lemma 3.5, Theorem 5.4]).

(A1) Let $T\simeq PSp_{2n}(q)$ and $k+1=(q^{2n}-1)/(q-1)$. Then $\mathrm{rk}(T)=3$, while $\mathrm{d_{min}}(T)=k+1$, except for the cases when $q=2$, $2n\geq 6$ and $\mathrm{d_{min}}(T)=2^{n-1}(2^n-1)$ or $2n=4,q=3$ and $\mathrm{d_{min}}(T)=27$ (see [12, Theorem 2]). Moreover, $\mathrm{M}(T)=Z_{\gcd(2,q-1)}$ for $(q;n)\neq(2;2),(2;3)$ and $\mathrm{M}(T)=Z_2$ for $(q;n)=(2;2),(2;3)$, $\mathrm{Out}(T)=Z_{\gcd(2,q-1)}\cdot Z_e$, where $q=p^e$, p is a prime.

According to Table 1 $(q;n)\notin\{(2;3),(3;2)\}$. Hence $\mathrm{d_{min}}(T)=k+1$. It follows that $K\leq Z(G)$ and, as noted above, it suffices to consider the case of prime r.

Since Γ has no H-uniform edge partitions, we have $r\geq 5$. Also, $2(\lambda(\Phi_1)+\lambda(\Phi_2)+1)\neq k-1$. Hence, due to [16, Proposition 3] $N=G'\simeq T$ acts transitively on $\mathcal{V}(\Gamma)$. But then $G_a\simeq N_{\{F\}}$ contains a subgroup of index r and $G_{\{F\}}=G_aN_{\{F\}}$.

If $n=3=q$, then $(|N|)_5=5$ and hence $|N_{\{F\}}|$ is not divisible by 5, a contradiction.

Let $n=2$. Then $N_{\{F\}}$ is an extension of a group of order q^3 by a group of the form $((q-1)/2\times L_2(q)).2$ or $((q-1)\cdot L_2(q))$ (see, for example, [4] or [13]). In any case, $N_{\{F\}}$ does not contain subgroups of index 5, a contradiction.

(A2) Let $T\simeq O_{2n+1}(q)$, $t=q$ and $k+1=(q^{2n}-1)/(q-1)$. Recall that $PSp_4(q)\simeq O_5(q)$ for $n=2$, and also that $O_{2n+1}(q)\simeq PSp_{2n}(q)$ for even q (see, for example, [18]). Since the corresponding cases are considered in case (A1), we will further assume that $n\geq 3$ and q is odd. Then $\mathrm{rk}(T)=3$ and by [18, Theorem] $\mathrm{d_{min}}(T)=k+1$, except for the case $q=3$, in which $\mathrm{d_{min}}(T)=3^n(3^n-1)/2$. Moreover, $\mathrm{M}(T)=Z_{(2,q-1)}$ for $(q;n)\neq(3;3)$ and $\mathrm{M}(T)=Z_2\times Z_2\times Z_3$ for $q=3=n$ (e.g. see [7]).

As in case (A1) we have $q=n=3$ and since $\mathrm{d_{min}}(T)>r$ we conclude $K\leq Z(G)$. Hence we may assume that r is prime. But then Table 1 gives $r=2$ and hence by Lemma 1 and Theorem 1 Γ admits an H-uniform edge partition, a contradiction.

Table 1. Feasible parameters of Γ with $r = p^l$ in case (A)

	q, n	$k+1$	k_1, k_2	θ	$-\tau$	$r\mu$	r
Type $t = q$:							
	7, 2	400	56, 343	19	-21	400	$2, 4, 5, 8, 25$
				21	-19	396	2
	9, 2	820	90, 729	21	-39	836	2
				39	-21	800	$2, 4, 5, 8, 16, 25$
	3, 3	364	120, 243	11	-33	384	$2, 4, 8, 16$
				33	-11	340	2
Type $t = 1$:	$\varnothing$						
Type $t = q^2$:							
	4, 2	325	68, 256	9	-36	350	5
				12	-27	338	13
	3, 3	1066	336, 729	$\sqrt{1065}$	$-\sqrt{1065}$	1064	$2, 4$
	2, 4	495	238, 256	19	-26	500	$5, 25$
				26	-19	486	$3, 9, 27, 81, 243$
Type $t = \sqrt{q}$:							
	4, 2	45	12, 32	4	-11	50	5
				11	-4	36	$3, 9$
	9, 2	280	36, 243	9	-31	300	$2, 5$
				31	-9	256	$2, 4, 8, 16, 32, 64, 128$
	16, 2	1105	80, 1024	16	-69	1156	17
				69	-16	1050	$5, 25$
Type $t = \sqrt{q}^3$:	$\varnothing$						

(A3) Let $T \simeq O^+_{2n}(q)$, where $n \geq 3$, $t = 1$ and $k+1 = (q^n - 1)(q^{n-1}+1)/(q-1)$. Then condition $k+1 \leq 2500$ implies either $n = 3$ and $q \leq 5$, or $n = 4$ and $q \leq 3$, or $n = 5, 6$ and $q = 2$. According to Table 1 none of these cases is possible.

(A4) Let $T \simeq O^-_{2n}(q)$, where $n \geq 2$, $t = q^2$ and $k+1 = (q^n - 1)(q^{n+1}+1)/(q-1)$. Then $\mathrm{rk}(T) = 3$ and in view of Table 1 $n = 2, 3, 4$. Recall that $O^-_4(q) \simeq L_2(q^2)$ and $O^-_6(q) \simeq U_4(q)$ (e.g. see [18]).

If $n = 4$ and $q = 2$, then by [4] $\mathrm{d_{min}}(T) = 119$. By [12, Theorem 1, Theorem 3] $\mathrm{d_{min}}(T) = q^2 + 1 = 17$ for $n = 2$ and $\mathrm{d_{min}}(T) = (q^3 + 1)(q + 1) = 112$ for $n = 3 = q$. In each case $\mathrm{d_{min}}(T) > r$ and hence $K \leq Z(G)$. Arguing as in case (A3), we obtain that $r = 5$ and $N = G'$ acts transitively on $\mathcal{V}(\Gamma)$. But then $N \simeq L_2(16)$ or $O^-_8(2)$, and $|N|$ is not divisible by 25. This contradicts the fact that $N_{\{F\}}$ must contain a subgroup of index r.

(A5) Let $T \simeq PSU_{2n}(\sqrt{q})$. In view of Table 1 either $T \simeq PSU_4(2) \simeq PSp_4(3)$ and $\mathrm{d_{min}}(T) = 27$ or $T \simeq PSU_4(3) \simeq O^-_6(3) \not\leq GL_7(2)$ and $\mathrm{d_{min}}(T) = 112$, or $T \simeq PSU_4(4)$ and $\mathrm{d_{min}}(T) = 325$ (see [4] and [12, Theorem 3]). Hence $K \leq Z(G)$ and we may assume that r is a prime. If G is a quasi-simple group, then r divides $|\mathrm{M}(T)|$ and so by [7] $r = 2$ and $q = 9$. If $N \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$, then r^2 divides $|N|$ and so $r = 5$ for $q = 16$ and $r \leq 3$ for $q \leq 9$.

Suppose $q = 9$. Then $r = 2$, Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(15, 130), (20, 112)\}$. Enumeration of orbital graphs in GAP [14] shows that Γ does not exist when $N \simeq T$. But for $G = N$ the groups $(G_a)^{[a]}$ and $(G_{\{F\}})^{\Sigma - \{F\}}$ are permutation isomorphic. Moreover, for the vertex $b_1 \in Q_1(a)$ the group G_{a,b_1} has exactly two orbits of length 4 and one orbit of length 27 on $[a]$, which contradicts the fact $\lambda_1 \in \{15, 20\}$.

Suppose $q=4$. Then $r=3$, $N \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$, $(\lambda_1,\lambda_2)=(3,13)$ and Γ admits an H-uniform edge partition with parameters $(\mu_1,\mu_2)=(4,9)$. A complete enumeration of orbital graphs in GAP [14] shows that this case cannot occur.

(A6) In the case of $T \simeq PSU_{2n+1}(\sqrt{q})$ and $t=\sqrt{q}^3$ we have $n=2$ and $q=4,9$, but according to Table 1 none of the cases gives a feasible parameter set.

(B) Let us consider the cases $T \simeq \widetilde{M}/Z(\widetilde{M})$, where $\widetilde{M}=Sp_4(q)$, $SU_4(q)$, $SU_5(q)$, $\Omega_6^-(q)$, $\Omega_8^+(q)$ or $\Omega_{10}^+(q)$ from [3, Theorem 11.3.2(ii)] (see also [8]).

(B1) Let $k_1=t(q+1)$ and the graph Φ_1 have parameters

$$\big((t+1)(tq+1), t(q+1), t-1, q+1\big),$$

where $t=q$, q^2, $q^{1/2}$, $q^{3/2}$ for $\widetilde{M}=Sp_4(q)$, $\Omega_6^-(q)$, $SU_4(\sqrt{q})$ or $SU_5(\sqrt{q})$, respectively. If $r=p^l$ is a power of a prime p, then feasible sets of parameters k, r, and μ are described by Table 2, and Γ does not admit H-uniform edge partitions when $t=q,\sqrt{q}$ (this can be easily checked in GAP [14], applying Theorem 1, [16, Proposition 1] and [5, Lemma 3.5, Theorem 5.4]). Moreover, cases $t=q$, q^2, $\sqrt{q}$ correspond to the above cases (A1), (A5) and (A4), respectively.

Table 2. Feasible parameters of Γ with $r=p^l$ in case (B)

	q	$k+1$	k_1, k_2	θ	$-\tau$	$r\mu$	r
(B1), type $t=q$:							
	7	400	56, 343	19	−21	400	2, 4, 5, 8, 25
				21	−19	396	2
	9	820	90, 729	21	−39	836	2
				39	−21	800	2, 4, 5, 8, 16, 25
(B1), type $t=q^2$:							
	2	45	12, 32	4	−11	50	5
				11	−4	36	3, 9
	3	280	36, 243	9	−31	300	2, 5
				31	−9	256	2, 4, 8, 16, 32, 64, 128
	4	1105	80, 1024	16	−69	1156	17
				69	−16	1050	5, 25
(B1), type $t=\sqrt{q}$:							
	16	325	68, 256	9	−36	350	5
				12	−27	338	13
(B1), type $t=\sqrt{q}^3$:	∅						
(B2),(B3):	∅						

(B2)&(B3) Let $T=\Omega_8^+(q)$, $k_1=q(q^2+1)(q^3-1)/(q-1)$ and the graph Φ_1 have parameters

$$\big(1+q(q^2+1)\frac{q^3-1}{q-1}+q^6, q(q^2+1)\frac{q^3-1}{q-1}, q(q^2+1)\frac{q^3-1}{q-1}-q^5-1, (q^2+1)\frac{q^3-1}{q-1}\big),$$

(see [3, Theorem 2.2.17, Proposition 3.2.3]) or let $T=\Omega_{10}^+(q)$, $k_1=q(q^2+1)(q^5-1)/(q-1)$ and the graph Φ_1 have parameters

$$\big((q^4+1)(q^3+1)(q^2+1)(q+1), q(q^2+1)\frac{q^5-1}{q-1}, q-1+q^2(q+1)(q^2+q+1), (q^2+1)(q^2+q+1)\big).$$

As $k+1 \le 2500$, it follows that $q \le 3$ and hence either $k+1 = 135, 1120$ and $T = O_8^+(q)$ for $q = 2, 3$ respectively, or $q = 2$, $k+1 = 2295$ and $T = O_{10}^+(2)$. According to Table 2 in any case, none of the parameter sets k, r and μ is feasible (this was checked in GAP [14] using [16, Proposition 1]) and [5, Lemma 3.5, Theorem 5.4]).

(C) Now let us consider the cases $T \simeq \widetilde{M}/Z(\widetilde{M})$, where $\widetilde{M} = SU_m(2)$, $\Omega_{2m}^{\pm}(2)$, $\Omega_{2m}^{\pm}(3)$, $\Omega_{2m-1}(3)$, $\Omega_{2m-1}(4)$ or $\Omega_{2m-1}(8)$ for $m \ge 3$, from [3, Theorem 11.3.2 (iii,iv)] (see also [8]).

(C1) Let $T = U_n(2)$ (see [3, § 3.1.6]) and the graph $\Phi_1 = \mathrm{NU}_n(2)$ have parameters

$$\big(2^{n-1}(2^n - \varepsilon)/3, (2^{n-1} + \varepsilon)(2^{n-2} - \varepsilon), 2^{2n-5}3 - \varepsilon 2^{n-2} - 2, 2^{n-3}3(2^{n-2} - \varepsilon)\big),$$

where $\varepsilon = (-1)^n$.

In view of Table 3 we have $n = 5$ and $k + 1 = 176$, i.e. $T \simeq U_5(2)$. Since

$$2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) \ne k - 1$$

and r divides 4, then either $N \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$, or G is a quasisimple group and by [7] $K \le \mathrm{M}(T) = Z_2$. But in the first case, by [4], $L = N_{\{F\}} \simeq Z_3 \times U_4(2)$ has no subgroups of index r, a contradiction. In the second case $r = 2$ and Γ admits an H-uniform edge partition with parameters (μ_1, μ_2), and $\{(\mu_1, \mu_2), (\lambda_1, \lambda_2)\} = \{(78, 21), (56, 18)\}$. But then subdegrees of the group G_a on $Q_1(a)$ (recall that $|Q_1(a)| = k_1$) are as follows: $1^1, 6^1, 32^4, 36^1$ (the upper indices denote the multiplicities of the corresponding subdegrees). This contradicts the fact $\lambda_1 \in \{78, 56\}$.

(C2) Let $T = P\Omega_{2n}^{\pm}(2)$ (see [3, § 3.1.2]) and the graph $\Phi_1 = \mathrm{NO}_{2n}^{\varepsilon}(2)$ have parameters

$$\big(2^{2n-1} - \varepsilon 2^{n-1}, 2^{2n-2} - 1, 2^{2n-3} - 2, 2^{2n-3} + \varepsilon 2^{n-2}\big),$$

where $\varepsilon = \pm 1$. Since $k + 1 \le 2500$, $n \le 6$. Then

$$2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$$

for all n and ε (see also [17, Example 1]).

Suppose $n = 3$.

If $T \simeq P\Omega_6^+(2) \simeq L_4(2) \simeq \mathrm{Alt}_8$, then $r = 2$ and N is intransitive on $\mathcal{V}(\Gamma)$ (note that Γ is a graph from [17, Theorem 2(ii)]).

Let $T \simeq P\Omega_6^-(2) \simeq U_4(2) \simeq PSp_4(3)$. Then $k + 1 = 36$, $\mathrm{M}(T) = Z_2$ and $\mathrm{rk}(T) = 3$. Since $\mathrm{d_{min}}(U_4(2)) = 27$ (see [4]), we get $K \le Z(G)$.

Assume that N is transitive on $\mathcal{V}(\Gamma)$. Then $r = 2$, $N = G \simeq Sp_4(3)$ or $PSp_4(3)$. Consequently, $G_a \simeq SL_2(9)$ or $G_a \simeq \mathrm{Alt}_6$. In the first case $K = Z(G) \le G_a$, and in the second case the rank of the transitive representation N on $\mathcal{V}(\Gamma)$ is equal to 5. Both cases are impossible.

Let $n > 3$. Since $\mathrm{d_{min}}(T) = 2^{n-1}(2^n - 1)$ (see [18]) for $\varepsilon = +1$, $\mathrm{d_{min}}(T) = 119$ (see [4]) for $\varepsilon = -1$ and $n = 4$, $\mathrm{d_{min}}(T) = 495$ (see [4]) for $\varepsilon = -1$ and $n = 5$, and $\mathrm{d_{min}}(T) = 2015$ (see [13]) for $\varepsilon = -1$ and $n = 6$, we get $K \le Z(G)$. Then, by [16, Proposition 3], either $N \simeq T$ is intransitive on $\mathcal{V}(\Gamma)$, or N is transitive on $\mathcal{V}(\Gamma)$. Let us consider the second case. Recall that $\mathrm{M}(T) = Z_2 \times Z_2$ for $n = 4$, $\varepsilon = +1$ and $\mathrm{M}(T) = 1$ otherwise (e.g. see [7]). Further, the group $T_{\{F\}}$ is isomorphic to the group $PSp_{2n}(2)$ (see [13]) and it has no subgroup of index r from the corresponding case in Table 3. Hence $N = G$, $n = 4$, $\varepsilon = +1$ and $r = 2$. By Lemma 1 and Theorem 1 Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(32, 28), (30, 27)\}$, and $\mu \in \{64, 54\}$.

Table 3. Feasible parameters of Γ with $r = p^l$ in cases (C1)–(C3)

	n	$k+1$	$k_1,\ k_2$	θ	$-\tau$	$r\mu$	r
(C1)	5	176	135, 40	5	-35	204	2
				35	-5	144	$2, 4$
(C2), $\varepsilon = -1$:	3	36	15, 20	5	-7	36	$2, 3, 9$
				7	-5	32	$2, 4, 8, 16$
	4	136	63, 72	9	-15	140	2
				15	-9	128	$2, 4, 8, 16, 32, 64$
	5	528	255, 272	17	-31	540	$2, 3, 9, 27$
				31	-17	512	$2, 4, 8, 16, 32, 64, 128, 256$
	6	2080	1023, 1056	9	-231	2300	2
				21	-99	2156	2
				27	-77	2128	$2, 4, 8$
				33	-63	2108	2
				63	-33	2048	$r = 2^l, l \leq 10$
				77	-27	2028	$2, 13, 169$
				99	-21	2000	$2, 4, 5, 8, 25, 125$
				231	-9	1856	$2, 4, 8, 32$
(C2), $\varepsilon = +1$:	3	28	15, 12	3	-9	32	$2, 4$
				9	-3	20	2
	4	120	63, 56	7	-17	128	$2, 4, 8$
				17	-7	108	$2, 3, 9, 27$
	5	496	255, 240	15	-33	512	$2, 4, 8, 16$
				33	-15	476	2
	6	2016	1023, 992	13	-155	2156	$2, 7$
				31	-65	2048	$2, 4, 8, 16, 32$
				65	-31	1980	$2, 3, 9$
				155	-13	1872	$2, 3, 4$
(C3), $\varepsilon = -1$:	3	126	45, 80	5	-25	144	$2, 3$
				25	-5	104	$2, 4$
				$\sqrt{125}$	$-\sqrt{125}$	124	2
(C3), $\varepsilon = 1$:	$\varnothing$						

Table 4. Feasible parameters of Γ with $r = p^l$ in case (C4)

	ε, q, n	$k+1$	$k_1,\ k_2$	θ	$-\tau$	$r\mu$	r
(C4)	$-1, 3, 2$	36	20, 15	5	-7	36	$2, 3, 9$
				7	-5	32	$2, 4, 8, 16$
	$-1, 3, 3$	351	224, 126	14	-25	360	$3, 9$
				25	-14	338	$13, 169$
				35	-10	324	$3, 9, 27, 81$
	$1, 3, 2$	45	32, 12	4	-11	50	5
				11	-4	36	$3, 9$
	$1, 3, 3$	378	260, 117	13	-29	392	$2, 7$
				29	-13	360	$2, 3, 4, 9$
				$\sqrt{377}$	$-\sqrt{377}$	376	$2, 4$
	$-1, 4, 2$	120	51, 68	7	-17	128	$2, 4, 8$
				17	-7	108	$2, 3, 9, 27$
	$-1, 4, 3$	2016	975, 1040	13	-155	2156	$2, 7$
				31	-65	2048	$2, 4, 8, 16, 32$
				65	-31	1980	$2, 3, 9$
				155	-13	1872	$2, 3, 4$
	$1, 4, 2$	136	75, 60	9	-15	140	2
				15	-9	128	$2, 4, 8, 16, 32, 64$
	$1, 4, 3$	2080	1071, 1008	9	-231	2300	2
				21	-99	2156	2
				27	-77	2128	$2, 4, 8$
				33	-63	2108	2
				63	-33	2048	$r = 2^l, l \leq 10$
				77	-27	2028	$2, 13, 169$
				99	-21	2000	$2, 4, 5, 8, 25, 125$
				231	-9	1856	$2, 4, 8, 32$
	$-1, 8, 2$	2016	455, 1560	13	-155	2156	$2, 7$
				31	-65	2048	$2, 4, 8, 16, 32$
				65	-31	1980	$2, 3, 9$
				155	-13	1872	$2, 3, 4$
	$1, 8, 2$	2080	567, 1512	9	-231	2300	2
				21	-99	2156	2
				27	-77	2128	$2, 4, 8$
				33	-63	2108	2
				63	-33	2048	$r = 2^l, l \leq 10$
				77	-27	2028	$2, 13, 169$
				99	-21	2000	$2, 4, 5, 8, 25, 125$
				231	-9	1856	$2, 4, 8, 32$

A computer check in GAP [14] shows that in the case when $r = 2$, $N \simeq T$ and N is intransitive on $\mathcal{V}(\Gamma)$, Γ exists and it is unique distance-transitive $(k+1, 2, \mu)$-cover (note it can be also constructed using [17, Theorem 1] or appears in [17, Example 1]).

(C3) Let $T = P\Omega^{\pm}_{2n}(3)$ (see [3, § 3.1.3]) and the graph $\Phi_1 = \mathrm{NO}^{\varepsilon}_{2n}(3)$ have parameters

$$\Big(\frac{1}{2}3^{n-1}(3^n - \varepsilon), \frac{1}{2}3^{n-1}(3^{n-1} - \varepsilon), \frac{1}{2}3^{n-2}(3^{n-1} + \varepsilon), \frac{1}{2}3^{n-1}(3^{n-2} - \varepsilon)\Big),$$

where $\varepsilon = \pm 1$.

In view of Table 3 we have $k+1 = 126$, $\varepsilon = -1$ and $r \le 4$. Then $T \simeq U_4(3)$ and $\mathrm{d}_{\min}(T) = 112$ (see [4]). Hence $K \le Z(G)$. Enumeration of feasible parameters in GAP [14] shows that Γ does not admit H-uniform edge partitions when $\lambda = \mu$, a contradiction with Lemma 1 and Theorem 1.

If $N \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$, then $N_{\{F\}} \simeq U_4(2)$ contains a subgroup of index $r \le 4$, a contradiction. Therefore $G = N$ is a quasi-simple group and, by [7], $r = 2$. Hence, by Lemma 1 and Theorem 1, Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(24, 45), (20, 34)\}$. Since $G = N$, the groups $(G_a)^{[a]}$ and $(G_{\{F\}})^{\Sigma - \{F\}}$ are permutation isomorphic. Moreover, for the vertex $b_1 \in Q_1(a)$ the group G_{a,b_1} has exactly two non-single-point orbits on $Q_1(a)$: one orbit of length 12 and one orbit of length 32. This is impossible, since $\lambda_1 \in \{20, 24\}$.

(C4) Let $T = P\Omega_{2n+1}(q)$ (see [3, § 3.1.4]) and the graph $\Phi_1 = \mathrm{NO}_{2n+1}(q)$ have parameters

$$\Big(\frac{1}{2}q^n(q^n + \varepsilon), (q^{n-1} + \varepsilon)(q^n - \varepsilon), 2(q^{2n-2} - 1) + \varepsilon q^{n-1}(q - 1), 2q^{n-1}(q^{n-1} + \varepsilon)\Big),$$

where $\varepsilon = \pm 1$, $q = 3, 4, 8$ and $n \ge 2$. According to Table 4, the equality $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$ holds only when either $k + 1 = 36$ and $q = 3$ or $q = 4$.

For $q = 3$ we have either $n = 2$ and $\mathrm{d}_{\min}(T) = 27$, or $n = 3$ and $\mathrm{d}_{\min}(T) = 351$ (see [4]). For even q we have $P\Omega_{2n+1}(q) \simeq PSp_{2n}(q)$ and, by [12, Theorem 2], $\mathrm{d}_{\min}(T) = (q^{2n} - 1)/(q - 1)$, i.e. $\mathrm{d}_{\min}(T) = 85$ for $2n = q = 4$, $\mathrm{d}_{\min}(T) = 585$ for $4n = q = 8$ and $\mathrm{d}_{\min}(T) = 1365$ for $n = 3$ and $q = 4$. Moreover, $r = p^l \ge \mathrm{d}_{\min}(T)$ is possible only for $4n = q = 8$. Together with the fact that $PSp_4(8) \not\le GL_{10}(2)$, this implies $K \le Z(G)$.

First we consider the cases when $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) \ne k - 1$.

If $\varepsilon = +1$, $q = 3$ and $n = 2$ then $T \simeq P\Omega_5(3) \simeq PSp_4(3)$ and $\mathrm{rk}(T) = 3$. This possibility was treated in case (A5).

Let $q = n = 3$. Then $T \simeq P\Omega_7(3)$, $\mathrm{rk}(T) = 3$ and $k + 1$ is equal to 351 (for $\varepsilon = -1$) or 378 (for $\varepsilon = +1$). In any case by [4] L has no subgroup of index 3, 7 or 13.

Hence if $N \simeq T$ is transitive on $\mathcal{V}(\Gamma)$ then $r = 2$, $\varepsilon = +1$ and $N_a = N_F \simeq L_4(3)$ has two orbits on $[a]$. Moreover, for the vertex $b_2 \in Q_2(a)$ the group N_{a,b_2} has exactly two non-single-point orbits on $Q_2(a)$ (recall that $k_2 = |Q_2(a)| = 117$), and the lengths of these orbits are 80 and 36. This contradicts the fact that by Lemma 1 and Theorem 1 Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(133, 56), (126, 60)\}$.

Hence $G = N$ and, by [7], $\mathrm{M}(T) = Z_2 \times Z_3$, which together with Table 4 implies $r \le 3$ for $k+1 = 378$ and $r = 3$ for $k+1 = 351$. Then, by Lemma 1 and Theorem 1, Γ admits an H-uniform edge partition with parameters (μ_1, μ_2). More precisely, if $k + 1 = 378$, then $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(133, 56), (126, 60)\}$ for $r = 2$ and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(84, 40), (91, 36)\}$ for $r = 3$, and if $k + 1 = 351$, then $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(75, 40), (73, 45)\}$ and $r = 3$. Since the groups $(G_a)^{[a]}$ and $(G_{\{F\}})^{\Sigma - \{F\}}$ are permutation isomorphic, in the case $r = 2$ a contradiction is achieved in a similar way as above. Let $r = 3$. For $k + 1 = 351$ the group G_{a,b_1}, where $b_1 \in Q_1(a)$, has five orbits on $Q_1(a)$ (recall that $k_1 = |Q_1(a)| = 224$): two orbits of length 81, one orbit of length 60 and two single-point orbits. This is impossible, since $\lambda_1 = 73$ or 75. Let $k + 1 = 378$. Since for the vertex $b_2 \in Q_2(a)$ the group G_{a,b_2} has exactly two non-single-point orbits on $Q_2(a)$ (recall that $k_2 = |Q_2(a)| = 117$), and the lengths of these orbits are 80 and 36, then $\lambda_2 = 36$. But then $\mu_2 = 40$, which is impossible, since $G_a = G_F$ and the group G_{a,b_2} moves 36 or 80 vertices from $Q_2(a^*) \cap [b_2]$ for some vertex $a^* \in F(a)$.

Let $q = 8$. According to Table 4 $T \simeq P\Omega_5(8) \simeq PSp_4(8)$ and as noted above $\mathrm{rk}(T) = 5$. Further, the group $(\widetilde{G}_{\{F\}})^{\Sigma-\{F\}}$ has the form $L_2(64).Z_3.Z_2$ for $k+1 = 2016$ and $(L_2(8) \times L_2(8)).Z_6$ for $k+1 = 2080$. Hence, by [16, Proposition 3] and taking into account that $\mathrm{M}(T) = 1$, we obtain either $r = 4$, one of -65 or 63 is an eigenvalue of Γ and N is intransitive on $\mathcal{V}(\Gamma)$, or $N \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$. Let us consider the second case. If $k+1 = 2080$ then for a subgroup of index r in $N_{\{F\}}$ we have either $p = 3$ and r divides 3^5, or $r = p = 2$. If $k+1 = 2016$ then for a subgroup of index r in $N_{\{F\}}$ we have $r = p \leq 3$. Enumeration of the orbital graphs of N in GAP [14] shows that the case $r = 3$ is impossible, while for $r = 2$ the graph Γ exists: for $k+1 = 2016$ the parameter μ equals to 1024 or 990, and for $k+1 = 2080$ the parameter μ equals to 1024 or 1054. More precisely, for each feasible set of parameters k, μ, it turns out to be the unique (up to isomorphism) distance-transitive $(k+1, 2, \mu)$-cover.

Now let $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$.

Let us consider the case when N is transitive on $\mathcal{V}(\Gamma)$.

For transitive N, the case $\varepsilon = -1$, $q = 3$ and $n = 2$ was excluded earlier in (C2).

Let $q = 4$. Then $\mathrm{rk}(T) = 3$ and by [7] $\mathrm{M}(T) = 1$. If $n = 2$ then by [4] $N_{\{F\}} \simeq L_2(16)$ (for $k+1 = 120$) or $(\mathrm{Alt}_5 \times \mathrm{Alt}_5) : Z_2$ (for $k+1 = 136$) has no subgroup of index 3, so $r = 2$. If $n = 3$, then $N_{\{F\}} \simeq P\Omega_6^{\varepsilon}(4) : Z_2$ (see [13]) has no subgroup of index $3, 5, 7$ or 13, so $r = 2$ again. Enumeration of the orbital graphs of $PSp_{2n}(q)$ on $r(k+1)$ points in GAP [14] shows that none of these cases is realized.

A computer check in GAP [14] shows that in the case when $r = 2$, $N \simeq T$ and N is intransitive on $\mathcal{V}(\Gamma)$, Γ exists and it is unique distance-transitive $(k+1, 2, \mu)$-cover (note it can be also constructed using [17, Theorem 1]).

(D) Finally, let the pair $(\widetilde{M}, Y)$, where Y is the pre-image in $\widetilde{M}$ of a point stabilizer in T, be one of the following (up to conjugacy in $\mathrm{Aut}(\widetilde{M})$ (see [3, § 11.3.2, Theorem 11.3.2(v)–(x)])):

$$(SU_3(3), PSL_3(2)),\ (SU_3(5), 3.\mathrm{Alt}_7),\ (SU_4(3), 4.PSL_3(4)),\ (Sp_6(2), G_2(2)),\ (\Omega_7(3), G_2(3)),\ (SU_6(2), 3.PSU_4(3).2);$$

let further the graph Φ_1 have parameters

$$(36, 14, 4, 6), \quad (50, 7, 0, 1), \quad (162, 56, 10, 24), \quad (120, 56, 28, 24), \quad (1080, 351, 126, 108) \quad \text{or} \quad (1408, 567, 246, 216),$$

respectively (for their detailed description, see [3, §-§ 10.14, 10.19, 10.48, 10.39, 10.78, 10.81]). Then feasible parameters of Γ are described by Table 5, which, in particular, shows the cases $k+1 = 56$, 1080 are impossible.

Let $T \simeq SU_3(3)$. Then $\mathrm{rk}(T) = 4$, $\mathrm{M}(T) = 1$, and by [4] $\mathrm{d}_{\min}(T) = 28 > r$. Hence $K \leq Z(G)$ and $N \simeq T$. Suppose N is intransitive on $\mathcal{V}(\Gamma)$. Then by [16, Proposition 3] we have either $r = 4$ and 7 is an eigenvalue of Γ, or $r = 2$ and $\gamma = -2(\lambda(\Phi_i) + k_j\mu(\Phi_i)/k_i + 1) + k$ is an eigenvalue of Γ. In the second case $\gamma \in \{\pm 7\}$, which in view of Table 5 implies $\mu \in \{16, 18\}$. Computer check in GAP [14] shows that for $r = 2$ and each μ, Γ exists and it is the only (up to isomorphism) distance-transitive $(36, 2, \mu)$-cover.

Suppose $N \simeq T$ is transitive on $\mathcal{V}(\Gamma)$. Then $N_{\{F\}} \simeq L_3(2)$ must contain a subgroup of index r. But in view of [4] the index of a proper subgroup in $L_3(2)$ must be divisible by 7 or 8, which implies $r = 8$. Enumeration of the orbital graphs of the group $SU_3(3)$ on $36r$ points in GAP [14] shows that this is impossible.

For $r = 4$ enumeration of the orbital graphs of the group $K \times SU_3(3)$ on 144 points in GAP [14] shows this case is also impossible.

In all other cases $\mathrm{rk}(T) = 3$ and $\mathrm{d}_{\min}(T) > r$. Hence $K \le Z(G)$ and, by the remark after Proposition 1, we will assume that r is prime.

For $T \simeq PSU_3(5)$ we have $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) \ne k - 1$ and by [7] $\mathrm{M}(T) = Z_3$. In view of Table 5 $r = 2$ and hence $N = G' \simeq T$. Enumeration of the orbital graphs of the group $Z_2 \times SU_3(5)$ on 100 points in GAP [14] shows this case is impossible.

For $T \simeq Sp_6(2)$, we have $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$ and, by [7] $\mathrm{M}(T) = 1$, so $N = G' \simeq T$. Since the rank of the representation of the group $Sp_6(2)$ on cosets by its subgroup isomorphic to the group $G_2(2)'$, equals 5, we obtain that N is intransitive on $\mathcal{V}(\Gamma)$. Further, in view of Lemma 1 and Theorem 1 Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and either $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(28, 32), (27, 30)\}$ and $r = 2$, or $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(18, 20), (19, 22)\}$ and $r = 3$. Since the groups $(G_a)^{[a]}$ and $(G_{\{F\}})^{\Sigma-\{F\}}$ are permutation isomorphic, for $b_1 \in Q_1(a)$ G_{a,b_1}-orbits on $Q_1(a)$ have lengths 1, 1, 27 and 27. For $r = 3$ this is impossible, since $\lambda_1 = 18$ or 19. Hence $r = 2$. Enumeration of the orbital graphs of the group $Z_r \times Sp_6(2)$ on 240 points in GAP [14] shows that Γ exists and it is distance-transitive with $\mu = 54$ or 64.

For $T \simeq PSU_6(2)$ we have $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) \ne k - 1$ and, by [7], $\mathrm{M}(T) = Z_3 \times Z_2 \times Z_2$. Since the rank of transitive representation of $PSU_6(2)$ on its right X-cosets with $X \simeq U_4(3)$ equals 5, then G is a quasi-simple group and $r = 2$. In view of Lemma 1 and Theorem 1 Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(286, 429), (280, 410)\}$. Since the groups $(G_a)^{[a]}$ and $(G_{\{F\}})^{\Sigma-\{F\}}$ are permutation isomorphic, for $b_1 \in Q_1(a)$ G_{a,b_1}-orbits on $Q_1(a)$ have lengths 1, 320, 30, 96 and 120. This is a contradiction, since $\lambda_1 = 286$ or 280.

Table 5. Feasible parameters of Γ with $r = p^l$ in case (D)

$(\widetilde{M}, Y)$	$k+1$	k_1, k_2	θ	$-\tau$	$r\mu$	r
$(SU_3(3), PSL_3(2))$	36	14, 21	5	-7	36	$2, 3, 9$
			7	-5	32	$2, 4, 8, 16$
$(SU_3(5), 3.\mathrm{Alt}_7)$	50	7, 42	7	-7	48	$2, 4, 8$
$(Sp_6(2), G_2(2))$	120	56, 63	7	-17	128	$2, 4, 8$
			17	-7	108	$2, 3, 9, 27$
$(SU_6(2), 3.PSU_4(3).2)$	1408	567, 840	21	-67	1452	$2, 11$
			67	-21	1360	$2, 4, 8$

□

Theorem 3. *Suppose that* $\Gamma = \Gamma(\widetilde{G}, G, K)$ *is a minimal abelian* $(k+1, r, \mu)$*-cover,* $k+1 \le 2500$*,* $\mathrm{rk}(\widetilde{G}^{\Sigma}) = 3$ *and* $T = \mathrm{Soc}(\widetilde{G}^{\Sigma}) \simeq PSL_d(q)$*. Assume* $\widetilde{G} = G$ *whenever* $\mathrm{rk}(T) = 3$*. Suppose further that* $(T, k+1) \ne (\mathrm{Alt}_s, \binom{s}{2})$*. Then* $\widetilde{G}^{\Sigma} \simeq P\Gamma L_2(8)$*,* $k+1 = 36$*,* $r = 2$*,* $\mu \in \{16, 18\}$*,* $G' \simeq T$*,* G' *is transitive on* $\mathcal{V}(\Gamma)$*, and* Γ *is a unique (up to isomorphism) distance-transitive* $(36, 2, \mu)$*-cover.*

P r o o f. Let $T \simeq PSL_d(q)$. Next we consider potential combinations for T and the complementary rank 3 graphs Φ_1 and Φ_2 associated with $\widetilde{G}^{\Sigma}$, applying their description from [8] and [3, Theorem 11.3.3]. Since $k + 1 \le 2500$, we are left with the following two cases (E) and (H).

(E) Let either $T = PSL_2(4) \simeq PSL_2(5) \simeq \mathrm{Alt}_5$, $k+1 = \binom{5}{2}$, or $T = PSL_2(9) \simeq \mathrm{Alt}_6$, $k+1 = \binom{6}{2}$, or $T = PSL_4(2) \simeq \mathrm{Alt}_8$, $k+1 = \binom{8}{2}$, or $G = P\Gamma L_2(8)$, $k+1 = \binom{9}{2}$ (see [8] and also [3, Theorem 11.3.3(ii)]). Then $\Phi_1 \simeq T(m)$ and $m = 5, 6, 8, 9$, respectively. The cases $m \leq 8$ were considered in [17, Theorem 2]. Below we treat the remaining case $m = 9$.

Let $k+1 = 36$ and $\widetilde{G}^\Sigma \simeq P\Gamma L_2(8)$. Then $T \simeq L_2(8)$, $\mathrm{rk}(T) = 4$, $\mathrm{M}(T) = 1$, the graph Φ_1 has parameters $(36, 14, 7, 4)$ and $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) \neq k - 1$. If $r = p^l$, p is prime, then $p \leq 3$. Note that $L_2(8) \not\leq GL_l(3)$ for $l < 4$ and $L_2(8) \not\leq GL_l(2)$ for $l < 5$. Hence $K \leq Z(G)$. By [16, Proposition 3] $r \neq 3$ and if $r \leq 16$, then by [16, Proposition 3] $G' \simeq T$ is transitive on $\mathcal{V}(\Gamma)$, which, in view of [4], implies $r = 2$. Enumeration of the orbital graphs of the group $Z_2 \times L_2(8)$ on 72 points in GAP [14] shows that $\mu = 16$ or 18, and Γ is a unique distance-transitive $(36, 2, \mu)$-cover (see also [16, Example]).

(H) If $T = PSL_3(4)$, $T_{\{F\}} \simeq \mathrm{Alt}_6$ and Φ_1 is the Gewirtz graph (with parameters $(56, 10, 0, 2)$) or $T = PSL_4(3)$, $T_{\{F\}} \simeq PSp_4(3)$ and $\Phi_1 \simeq \mathrm{NO}_6^+(3)$ (with parameters $(117, 36, 15, 9)$), then there is no feasible set of parameters. □

Remark 1. In proofs of Theorems 2 and 3, in a computer search for distance-regular orbital graphs we used GAP packages GRAPE [15] and coco2p [10].

Remark 2. An explicit construction of covers with $r = 2$ and intransitive group G' from the conclusions of Theorem 2 can be found in [17, Theorem 1, Example 1].

Corollary 1. *Suppose that Ψ is a non-bipartite abelian (n, r', μ')-cover with a transitive group of automorphisms X that induces a primitive almost simple permutation group X^Ξ on the set Ξ of its antipodal classes such that $\mathrm{rk}(X^\Xi) = 3$ and the pair (X^Ξ, n) satisfies conditions of Theorem 2 or 3. Then Ψ has a minimal quotient $\Gamma(\widetilde{G}, G, K)$ that is an (n, r, μ)-cover from the conclusion of the respective theorem with $\mathrm{Soc}(X^\Xi) \simeq G/K$ and $r'\mu' = r\mu$.*

4. Concluding remarks

In this paper, we continued studying abelian antipodal distance-regular graphs Γ of diameter 3 with the property $(*)$: Γ *has a transitive group of automorphisms $\widetilde{G}$ that induces a primitive almost simple permutation group $\widetilde{G}^\Sigma$ on the set Σ of its antipodal classes.* As in [16], we focused on the case $\mathrm{rk}(\widetilde{G}^\Sigma) = 3$. In [16] and [17], it was shown that in the alternating and sporadic cases for $\widetilde{G}^\Sigma$ the family of non-bipartite graphs Γ with the property $(*)$ and $\mathrm{rk}(\widetilde{G}^\Sigma) = 3$ is finite and limited to a small number of potential examples with $|\Sigma| \in \{10, 28, 120, 176, 3510\}$. Here we assumed that the socle of $\widetilde{G}^\Sigma$ is a classical simple group. The case of classical simple socle seems to be both most interesting and complicated, since, on one hand, there is an infinite family of non-bipartite representatives Γ (see [17, Example 1]), and on the other hand, its study requires a profound inspection of $\widetilde{G}^\Sigma$. So we started classification of graphs Γ with "small" $|\Sigma|$. In order to describe minimal quotients of Γ, we used the technique for bounding their spectrum that is based on analysis of their local properties and the structure of $\widetilde{G}^\Sigma$, which was developed in [16] and applied in [16] and [17] for the cases of sporadic, alternating and exceptional socle (the latter was investigated under condition $|\Sigma| \leq 2500$). As a result, we significantly refined the sets of feasible parameters of Γ with $|\Sigma| \leq 2500$ in the case of classical socle, showing, in particular, that for most of these sets Γ must be a covering of a certain distance-transitive Taylor graph.

We also wish to mention two more challenging examples of graphs with the property $(*)$, namely, abelian $(n, 3, 12)$-covers with $n = 36$ or 45 and $\mathrm{rk}(\widetilde{G}^\Sigma) = 4$ or 5, respectively (for their constructions, see [9]). A computer assisted inspection shows that they are the only minimal abelian (n, r, μ)-covers $\Gamma(\widetilde{G}, G, K)$ such that $3 \leq \mathrm{rk}(\widetilde{G}^{\Sigma}) \leq 5$, $r > 2$, $n \leq 2500$ and $G = G'$ is a quasi-simple group.

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