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ON SOLVING AN ENHANCED EVASION PROBLEM FOR LINEAR DISCRETE–TIME SYSTEMS

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Abstract: We consider the problem of an enhanced evasion for linear discrete-time systems, where there are two conflicting bounded controls and the aim of one of them is to be guaranteed to avoid the trajectory hitting a given target set at a given final time and also at intermediate instants. First we outline a common solution scheme based on the construction of so called solvability tubes or repulsive tubes. Then a much more quick and simple for realization method based on the construction of the tubes with parallelepiped-valued cross-sections is presented under assumptions that the target set is a parallelepiped and parallelotope-valued constraints on controls are imposed. An example illustrating this method is considered.

Keywords: Linear control systems, Discrete-time systems, Uncertainty, Evasion problem, Parallelepipeds.

1. Introduction

We consider linear discrete-time systems under conflicting controls that may have different aims. Namely, the aim of the one control may be to guide the trajectory to a given target set, and the aim of the other control may be the opposite. This gives rise to two subproblems under conditions of uncertainty, namely, the approach problem and the evasion one.

There are well known approaches for solving the problems of such sort that are based on the construction of special tubes of trajectories known as stable bridges or solvability tubes [2, 16–19]. Since an exact construction of trajectory tubes is usually a very complicated problem, different numerical methods are being devised. Of the many works, we only indicate as examples [1–15, 17–22, 24], including those based on estimation of sets by more simple sets such as ellipsoids [1, 2, 4–6, 17–20] and parallelepipeds/parallelotopes [9–15, 21, 22]. In particular, for discrete-time systems, ellipsoidal and polyhedral solution schemes have been developed for the terminal target approach problem and the terminal evasion problem at a given final time (see, for example, [2, 11, 12, 20] and [14] for both problems respectively). But two methods presented in [14] guarantee the evasion from the given set only at the given final time. Computer simulations corroborate this.

The present paper is devoted to solving the problem of enhanced evasion for linear discrete-time systems with two bounded controls, where the aim of one of them is to avoid, regardless of the actions of the other, the trajectory hitting the given target set not only at the final time, but also at intermediate instants. First the common solution scheme is outlined. Then a much more quick and simple for realization method based on the construction of repulsive parallelepiped-valued tubes is presented. In fact, here, in contrast to [14], a pair of polyhedral tubes is constructed and explicit formulas for feedback control strategies are given on the base of both tubes. A corresponding illustrative example is included.

Note that the solutions to the evasion problems may be useful for construction of dangerous disturbances, for example, in problems of aircraft control [3, 22].

We will use the following notation. Let the symbol $\mathbb{R}^n$ denotes the n-dimensional vector space; $\|x\|_\infty = \max_{1\le i\le n} |x_i|$ be the vector norm for $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$ (we use $\top$ as the transposition symbol); $\mathbb{R}^{n\times m}$ be the space of real $n\times m$-matrices $A = \{a_i^j\} = \{a^j\}$ with elements a_i^j and columns a^j (the superscript numbers the columns of the matrix and the subscript numbers the components of vectors). Let $\operatorname{diag}\pi$ be the diagonal matrix A with $a_i^i = \pi_i$ (where π_i are the components of the vector π); $\operatorname{Abs} A = \{|a_i^j|\}$ for $A = \{a_i^j\} \in \mathbb{R}^{n\times m}$. Let the symbol I stands for the identity matrix and 0 stands for zero matrices and vectors; $\det A$ be the determinant of $A \in \mathbb{R}^{n\times n}$. The value of $\operatorname{sign} z$ is equal to -1, 0, 1 for $z < 0$, $z = 0$, $z > 0$ respectively. We use the notation $k = 1, \dots, N$ instead of $k = 1, 2, \dots, N$ for the sake of brevity.

2. Problem statements

Consider the following discrete-time system

$$x[k] = A[k]\,x[k-1] + B[k]u[k] + C[k]v[k], \quad k = 1, \dots, N, \tag{2.1}$$

where

$$u[k] \in \mathcal{R}[k], \quad v[k] \in \mathcal{Q}[k], \quad k = 1, \dots, N, \tag{2.2}$$

with the target set $\mathcal{M} \subset \mathbb{R}^n$. Here $x[k] \in \mathbb{R}^n$ are states, the matrices $A[k] \in \mathbb{R}^{n\times n}$, $B[k] \in \mathbb{R}^{n\times n_u}$, $C[k] \in \mathbb{R}^{n\times n_v}$ are given and matrices $A[k]$ are nonsingular; $u[k] \in \mathbb{R}^{n_u}$ and $v[k] \in \mathbb{R}^{n_v}$ are controls, which may have different aims; $\mathcal{M}$, $\mathcal{R}[k]$, and $\mathcal{Q}[k]$ are given compact sets. The functions $u[\cdot]$ and $v[\cdot]$ satisfying (2.2) are called *admissible.*

We consider the evasion problem, where the aim of v is to guarantee $x[N] \notin \mathcal{M}$ and moreover $x[k] \notin \mathcal{M}$ for all $k = 1, \dots, N-1$ regardless of the admissible realizations of u, and formulate it as follows.

Problem 1 (Evasion problem, enhanced evasion problem). *For system (2.1)–(2.2), find sets $\hat{\mathcal{W}}[k]$, $k = 0, 1, \dots, N$, satisfying $\hat{\mathcal{W}}[k] \supseteq \mathcal{M}$, $k = 1, \dots, N$, and find a corresponding feedback control strategy $v = v[k, x]$ satisfying $v[k, x] \in \mathcal{Q}[k]$, $k = 1, \dots, N$, such that each solution $x[\cdot]$ to the equation*

$$x[k] = A[k]\,x[k-1] + B[k]u[k] + C[k]v[k, x[k-1]], \quad k = 1, \dots, N,$$

that starts from any initial point $x[0] = x^0$ with $x^0 \notin \hat{\mathcal{W}}[0]$ would satisfy $x[k] \notin \hat{\mathcal{W}}[k]$, $k = 1, \dots, N$, for all admissible functions $u[\cdot]$.

Similarly to [3, 22] the multivalued function $\hat{\mathcal{W}}[\cdot]$ with the crossections $\hat{\mathcal{W}}[k]$, $k = 0, 1, \dots, N$, can be called a repulsive tube.

It is possible to formulate this problem in a form more close to statements and constructions from [16, Ch. 2] if we consider the solvability tube $\check{\mathcal{W}}[\cdot]$ with cross-sections $\check{\mathcal{W}}[k] = \mathbb{R}^n \setminus \hat{\mathcal{W}}[k]$, where the symbol $\mathbb{R}^n \setminus \mathcal{X}$ means the complement of $\mathcal{X}$: $\mathbb{R}^n \setminus \mathcal{X} = \{x \in \mathbb{R}^n \,|\, x \notin \mathcal{X}\}$. The term "enhanced evasion problem" is used to emphasize the difference from the following problem.

Problem 1′ (Terminal evasion problem). *It is similar to Problem 1, but we require $\hat{\mathcal{W}}[k] \supseteq \mathcal{M}$ only for $k = N$.*

As it will be presented below the solution to Problem 1 can be obtained through relations which involve rather labor-consuming operations with sets, namely, Minkowski's sum ($\mathcal{X}^1 + \mathcal{X}^2 = \{y |\ y = x^1 + x^2, x^k \in \mathcal{X}^k\}$), Minkowski's difference ($\mathcal{X}^1 \dot{-} \mathcal{X}^2 = \{y \,|\, y + \mathcal{X}^1 \subseteq \mathcal{X}^2\}$), union, and intersection.

Therefore we consider a similar polyhedral evasion problem under the assumptions which are similar to [14].

Assumption 1. *Let matrices* $A[k]$ *in* (2.1) *be nonsingular*: $\det A[k] \neq 0$, $k = 1, \ldots, N$, *the sets* $\mathcal{R}[k]$ *and* $\mathcal{Q}[k]$ *in* (2.2) *be parallelotopes, and the target set* $\mathcal{M}$ *be a parallelepiped*:

$$\mathcal{R}[k] = \mathcal{P}[r[k], \bar{R}[k]], \quad \bar{R}[k] \in \mathbb{R}^{n_u \times n_u}; \qquad \mathcal{Q}[k] = \mathcal{P}[q[k], \bar{Q}[k]], \quad \bar{Q}[k] \in \mathbb{R}^{n_v \times n_v};$$
$$\mathcal{M} = \mathcal{P}(p_{\mathrm{f}}, P_{\mathrm{f}}, \pi_{\mathrm{f}}). \tag{2.3}$$

By a *parallelepiped* $\mathcal{P}(p, P, \pi) \subset \mathbb{R}^n$ we call a set defined as

$$\mathcal{P} = \mathcal{P}(p, P, \pi) = \{x \in \mathbb{R}^n |\, x = p + \sum_{i=1}^{n} p^i \pi_i \xi_i, \ \ \|\xi\|_\infty \leq 1\}.$$

Here $p \in \mathbb{R}^n$; $P = \{p^i\} \in \mathbb{R}^{n\times n}$ is a nonsingular matrix ($\det P \neq 0$) with columns p^i such that $\|p^i\|_2 = 1$; $\pi \in \mathbb{R}^n$, $\pi \geq 0$. We call p the center of the parallelepiped and P its orientation matrix. Note that the above conditions $\|p^i\|_2 = 1$ for the Euclidean norm may be omitted to simplify formulas. We say that a parallelepiped is *nondegenerate* if all $\pi_i > 0$.

By a *parallelotope* $\mathcal{P}[p, \bar{P}] \subset \mathbb{R}^n$ we call a set defined as

$$\mathcal{P} = \mathcal{P}[p, \bar{P}] = \{x \in \mathbb{R}^n |\, x = p + \bar{P}\xi, \ \ \|\xi\|_\infty \leq 1\}.$$

Here $p \in \mathbb{R}^n$ and $\bar{P} = \{\bar{p}^i\} \in \mathbb{R}^{n\times m}$, $m \leq n$, i.e., the matrix $\bar{P}$, which determines the shape, may be singular and not square. We say that a parallelotope $\mathcal{P}$ is *nondegenerate* if $m = n$ and $\det \bar{P} \neq 0$. Note that each parallelepiped $\mathcal{P}(p, P, \pi)$ is a parallelotope $\mathcal{P}[p, \bar{P}]$, where $\bar{P} = P{\cdot}\mathrm{diag}\,\pi$, and each nondegenerate parallelotope $\mathcal{P}[p, \bar{P}]$ is a parallelepiped $\mathcal{P}(p, P, \pi)$ with $P = \bar{P}$, $\pi = (1, 1, \ldots, 1)^\top$.

Problem 2 (Polyhedral evasion problem). *Under Assumption 1, find a solution of Problem 1 in a class of polyhedral tubes* $\mathcal{P}[\cdot] = \mathcal{P}(p[\cdot], P[\cdot], \pi[\cdot])$ *with parallelepiped-valued cross-sections. Moreover introduce a family of such tubes* $\mathcal{P}[\cdot]$ (*i.e. instead of* $\hat{\mathcal{W}}[\cdot]$ *there are the tubes* $\mathcal{P}[\cdot]$).

Recall that in [11, 12] the solutions to terminal target polyhedral approach problems are given even for more general classes of systems, namely, for systems (2.1) with uncertainties / controls in the matrices $A[k]$ and with state constraints. There the families of the tubes $\mathcal{P}^-[\cdot] = \mathcal{P}[p^-[\cdot], \bar{P}^-[\cdot]]$ with parallelotope-valued cross-sections and corresponding control strategies $u[k, x]$ have been constructed to guarantee $x[N] \in \mathcal{M}$ at the given final time N.

In [14], the following problem was considered.

Problem 2′ (Polyhedral terminal evasion problem). *Under Assumption 1, find a solution of Problem 1′ in a class of polyhedral* (*parallelotope-valued*) *tubes* $\mathcal{P}^+[\cdot] = \mathcal{P}[p^+[\cdot], \bar{P}^+[\cdot]]$.

In [14], two techniques to solve Problem 2′ to ensure $x[N] \notin \mathcal{M}$ were presented using two families of the tubes $\mathcal{P}^+[\cdot] = \mathcal{P}[p^+[\cdot], \bar{P}^+[\cdot]]$ and $\mathcal{P}^{\mathrm{e}}[\cdot] = \mathcal{P}[p^{\mathrm{e}}[\cdot], \bar{P}^{\mathrm{e}}[\cdot]]$.

In the present paper, the first of these techniques is extended to solve Problem 2. Note that now we will use the tubes $\mathcal{P}[\cdot]$ with parallelepiped-valued cross-sections because this is more convenient in order to take into account the set $\mathcal{M}$ at times $k < N$.

3. Main results

To solve Problem 1 let us consider the following system of recurrence relations for calculating the tubes $\hat{\mathcal{W}}[\cdot]$ and $\hat{\mathcal{W}}^1[\cdot]$:

$$\begin{gathered}
\hat{\mathcal{W}}^0[k{-}1] = \hat{\mathcal{W}}[k] + (-B[k]\mathcal{R}[k]), \quad k = N, \ldots, 1; \\
\hat{\mathcal{W}}^1[k{-}1] = A[k]^{-1}(\hat{\mathcal{W}}^0[k{-}1] \dot{-} C[k]\mathcal{Q}[k]), \quad k = N, \ldots, 1; \\
\hat{\mathcal{W}}[k{-}1] = \hat{\mathcal{W}}^1[k{-}1] \bigcup \mathcal{M}, \quad k = N, \ldots, 2; \\
\hat{\mathcal{W}}[N] = \mathcal{M}; \quad \hat{\mathcal{W}}[0] = \hat{\mathcal{W}}^1[0].
\end{gathered} \tag{3.1}$$

Theorem 1. *Let the tubes $\hat{\mathcal{W}}[\cdot]$ and $\hat{\mathcal{W}}^1[\cdot]$ satisfy* (3.1) *and all of their cross-sections appear to be nonempty. Then the tube $\hat{\mathcal{W}}[\cdot]$ together with the control strategy $v[k,x]$ of the following form*

$$v[k,x] \in \begin{cases} \mathcal{V}[k,x] \text{ for } x \notin \hat{\mathcal{W}}^1[k-1]; \\ \mathcal{Q}[k], \text{ otherwise,} \end{cases} \qquad (3.2)$$
$$\mathcal{V}[k,x] = \mathcal{Q}[x] \bigcap \{v \mid C[k]v \in (\mathbb{R}^n \setminus \hat{\mathcal{W}}^0[k-1]) - A[k]x\}$$

give a solution to Problem 1.

P r o o f (Sketch of the proof). The lines of reasoning from [2, 10] with necessary modifications can be used. First the relations for $\check{\mathcal{W}}[k] = \mathbb{R}^n \setminus \hat{\mathcal{W}}[k]$ and $\check{\mathcal{W}}^1[k] = \mathbb{R}^n \setminus \hat{\mathcal{W}}^1[k]$ can be written using duality interconnections basing, for example, on [23, p. 137]. Let $\check{\mathcal{W}}[\cdot]$ and $\check{\mathcal{W}}^1[\cdot]$ be found. Inclusions $\check{\mathcal{W}}[k] \subseteq \mathbb{R}^n \setminus \mathcal{M}$ and $\hat{\mathcal{W}}[k] \supseteq \mathcal{M}$ follow from (3.1). Then we can verify, for any k, that if $x = x[k-1] \in \check{\mathcal{W}}^1[k-1]$, then we obtain $\mathcal{V}[k,x] \neq \emptyset$ and

$$x[k] = A[k]x + B[k]u[k] + C[k]v[k,x] \in \check{\mathcal{W}}[k]$$

for any $v[k,x] \in \mathcal{V}[k,x]$ and arbitrary $u[k] \in \mathcal{R}[k]$. □

So, to guarantee $x[k] \notin \mathcal{M}$ for all $k = 1, \ldots, N$ we first need to find the tubes $\hat{\mathcal{W}}[\cdot]$ and $\hat{\mathcal{W}}^1[\cdot]$ by solving recurrence relations (3.1) backward starting from $\hat{\mathcal{W}}[N] = \mathcal{M}$. Then starting from any $x^0 \notin \hat{\mathcal{W}}[0]$ we can apply an arbitrary control strategy v that satisfies (3.2). According to the proof, if $x^0 \notin \hat{\mathcal{W}}[0]$, then only the first line in (3.2) can be implemented.

Also note that, in general, the sets $\hat{\mathcal{W}}[k]$ satisfying (3.1) are not guaranteed to be convex even if $\mathcal{R}[k]$, $\mathcal{Q}[k]$, and $\mathcal{M}$ are convex, and the sets $\mathbb{R}^n \setminus \hat{\mathcal{W}}^0[k-1]$ are the sets with holes.

To solve Problem 2, we use elementary external polyhedral estimates for results of operations with sets. Recall that the result of a linear transformation of a parallelepiped is a parallelepiped or a parallelotope. The so called touching external estimate $\boldsymbol{P}^+_V(\mathcal{Q})$ for the set $\mathcal{Q}$ with the orientation matrix V can be found on the base of the values of the support function for $\mathcal{Q}$ [15]. It is easy to find touching estimates $\boldsymbol{P}^+_V(\mathcal{P}^1 + \mathcal{P}^2)$ and $\boldsymbol{P}^+_V(\mathcal{P}^1 \bigcup \mathcal{P}^2)$ for a sum and for a union of parallelepipeds/parallelotopes using explicit formulas [15]. Minkowski's difference $\mathcal{P}^1 \dot{-} \mathcal{P}^2$ of a parallelepiped and a parallelotope is either a parallelepiped or an empty set (concrete formulas can be found in [9]).

Notice that to check whether a point x belongs to the parallelepiped $\mathcal{P}(p, P, \pi)$ it is useful to use relative coordinates $\xi = P^{-1}(x-p)$.

Lemma 1. *Given $x \in \mathbb{R}^n$ and $\mathcal{P} = \mathcal{P}(p, P, \pi)$, let $\xi = P^{-1}(x-p)$. Then $x \notin \mathcal{P}$ iff $|\xi_{i_*}| > \pi_{i_*}$ for some $i_* \in \{1, \ldots, n\}$.*

To solve Problem 2 we introduce the system of the following recurrence relations for calculating parallelepipeds $\mathcal{P}[k] = \mathcal{P}(p[k], P[k], \pi[k])$ and $\mathcal{P}^1[k] = \mathcal{P}(p^1[k], P^1[k], \pi^1[k])$, which determine the couple of the tubes $\mathcal{P}[\cdot]$ and $\mathcal{P}^1[\cdot]$:

$$\begin{gathered} \mathcal{P}^0[k-1] = \boldsymbol{P}^+_{P[k]}(\mathcal{P}[k] + (-B[k]\mathcal{R}[k])), \quad k = N, \ldots, 1; \\ \mathcal{P}^1[k-1] = A[k]^{-1}(\mathcal{P}^0[k-1] \dot{-} C[k]\mathcal{Q}[k]), \quad k = N, \ldots, 1; \\ \mathcal{P}[k-1] = \boldsymbol{P}^+_{P[k-1]}(\mathcal{P}^1[k-1] \bigcup \mathcal{M}), \quad k = N, \ldots, 2; \\ \mathcal{P}[N] = \boldsymbol{P}^+_{P[N]}(\mathcal{M}); \quad \mathcal{P}[0] = \mathcal{P}^1[0]. \end{gathered} \qquad (3.3)$$

Given the tubes $\mathcal{P}[\cdot]$ and $\mathcal{P}^1[\cdot]$, let us introduce the notation:

$$\xi[k,x] = P^1[k{-}1]^{-1}(x - p^1[k{-}1]);$$
$$\Theta[k] = P[k]^{-1}C[k]\bar{Q}[k];$$
$$\Phi[k,x] = (\operatorname{diag}\pi^1[k{-}1])^{-1}\operatorname{Abs}\xi[k,x].$$

Here $\xi[k,x]$ stands for the relative coordinates of x with respect to the cross-section

$$\mathcal{P}^1[k{-}1] = \mathcal{P}(p^1[k{-}1], P^1[k{-}1], \pi^1[k{-}1])$$

of the tube $\mathcal{P}^1[\cdot]$; the matrix $\Theta[k]$ is determined by the parameters of system (2.1), (2.2), and (2.3), and by the cross-section $\mathcal{P}[k] = \mathcal{P}(p[k], P[k], \pi[k])$ of the tube $\mathcal{P}[\cdot]$; the vector $\Phi[k,x]$ is determined by $\xi[k,x]$ and the cross-section $\mathcal{P}^1[k{-}1]$ of the tube $\mathcal{P}^1[\cdot]$.

Then we can apply several formulas for construction of the control strategy $v[k,x]$ basing on the main formula of the form

$$\begin{gathered} v^0[k,x] = q[k] + \bar{Q}[k]\chi[k,x], \\ \chi_j[k,x] = \operatorname{sign}\Theta^j_{i_*}[k]\cdot\operatorname{sign}\xi_{i_*}[k,x], \quad j = 1,\dots,n_v. \end{gathered} \tag{3.4}$$

Let us consider three following variants of the formulas:

$$v^{(0)}[k,x] = \begin{cases} v^0[k,x] \text{ for } x \notin \mathcal{P}^1[k{-}1]; \\ \text{arbitrary } v \in \mathcal{Q}[k], \text{ otherwise,} \end{cases} \tag{3.5}$$

where i_* in (3.4) is any index $i_* = i_*[k] \in \{1,\dots,n\}$ such that $|\xi_{i_*}[k,x]| > \pi^1_{i_*}[k{-}1]$;

$$v^{(1)}[k,x] = v^0[k,x], \quad \forall x \in \mathbb{R}^n, \tag{3.6}$$

where $i_* = i_*[k] \in \operatorname{Argmax}_{1\le i\le n}\Phi_i[k,x]$;

$$v^{(2)}[k,x] = \begin{cases} v^{(1)}[k,x] \text{ for } x \notin \mathcal{P}^1[k{-}1]; \\ q[k], \text{ otherwise.} \end{cases} \tag{3.7}$$

Theorem 2. *Under Assumption 1, let $P[k]$, $k = N,\dots,1$, be arbitrary nonsingular orientation matrices (i.e., arbitrariness is allowed when choosing $P[N]$ and matrices $P[k{-}1]$ in the 3rd line in (3.3)) and system (3.3) has a solution such that all parallelepipeds $\mathcal{P}^1[k]$, $k = N,\dots,1$, turn out to be nondegenerate. Then the tube $\mathcal{P}[\cdot]$ together with each of the control strategies $v^{(l)}[k,x]$, $l \in \{0,1,2\}$, from (3.5)–(3.7), which are determined by the couple of the tubes $\mathcal{P}[\cdot]$ and $\mathcal{P}^1[\cdot]$, give a particular solution to Problem 2.*

P r o o f (Sketch of the proof). It can be verified, using Lemma 1, that if $x[0] \notin \mathcal{P}[0]$, then the control strategy $v^{(0)}[\cdot,\cdot]$ ensures that $x[k] \notin \mathcal{P}[k]$ for all $k > 0$. The control strategies $v^{(1)}$ and $v^{(2)}$ are in fact special cases when $v^{(0)}$ is concretized. □

Remark 1. Theorem 2 depicts the parametric family of the tubes $\mathcal{P}[\cdot]$. Here the matrix function $P[\cdot]$ appears as a parameter. We note two following heuristic techniques to choose $P[k]$ for $k < N$ in the 3rd line in (3.3) (then only $P[N]$ is the parameter).

(a) Given $P[k]$, put $P[k{-}1] = P^1[k{-}1] = A[k]^{-1}P[k]$.

(b) Put $P[k{-}1]$ using arguments of local volume minimization of the type:

$$V \in \operatorname{Argmin}_{V\in\{P^1,P^2\}}\operatorname{vol}\boldsymbol{P}^+_V(\mathcal{P}^1 \bigcup \mathcal{P}^2),$$

where $\mathcal{P}^k = \mathcal{P}(p^k, P^k, \pi^k)$, $k = 1,2$.

Corollary 1. *Theorem 2 is true, with an evident modification, if in the evasion problem the aim of v is to ensure $x[N] \notin \mathcal{M}$ and $x[k] \notin \mathcal{M}$ only for $k \in \mathcal{K}$, where $\mathcal{K}$ is some subset of $\{1, \dots, N-1\}$ (in particular, we have $\mathcal{K} = \emptyset$ if we require only $x[N] \notin \mathcal{M}$). Namely, it is sufficient to replace the formulas for $\mathcal{P}[k-1]$ in the 3rd line of (3.3)) by $\mathcal{P}[k-1] = \mathcal{P}^1[k-1]$ for all $k-1$ such that $k-1 \notin \mathcal{K}$. Then, for $\mathcal{K} = \emptyset$, the parallelepiped-valued tubes $\mathcal{P}[\cdot]$ turn out to coincide with the parallelotope-valued tubes $\mathcal{P}^+[\cdot]$ from [14, Theorem 1].*

Note that the solutions to Problem 2 described by Theorem 2 can be easily calculated by the explicit formulas.

So, to guarantee $x[k] \notin \mathcal{M}$ for all $k = 1, \dots, N$ we can find several pairs of the tubes $\mathcal{P}[\cdot]$ and $\mathcal{P}^1[\cdot]$, which are determined by recurrence relations (3.3). Then for a given x^0 we can choose the most suitable tube $\mathcal{P}[\cdot]$, for example, similarly to [14, Sec. IV] (we need to fulfill the condition $x^0 \notin \mathcal{P}[0]$ to meet the above claims for the trajectory) and apply any of the control strategies $v^{(l)}[k, x]$, $l \in \{0, 1, 2\}$, from (3.5)–(3.7), which are determined by the selected tube $\mathcal{P}[\cdot]$ and the corresponding tube $\mathcal{P}^1[\cdot]$. If we get $x^0 \in \mathcal{P}[0]$ for all calculated tubes, then we, generally speaking, cannot guarantee $x[k] \notin \mathcal{M}$ for all $k = 1, \dots, N$, but this can happen for some of realizations of $u[\cdot]$.

4. Example

Let us illustrate the presented constructions on the example of the same system as in [14, Sec. IV]. The system is obtained by Euler's approximations of a differential one considered on an interval $t \in [0, \theta]$:

$$A[k] \equiv I + h_N \cdot \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}, \quad B[k] \equiv h_N \cdot (0, 1)^\top, \quad \mathcal{R}[k] \equiv \mathcal{P}(0, I, 1) \subset \mathbb{R}^1,$$
$$C[k] \equiv h_N \cdot (1, 0)^\top, \quad \mathcal{Q}[k] \equiv \mathcal{P}(0, I, 0.2) \subset \mathbb{R}^1, \quad \mathcal{M} = \mathcal{P}((-0.5, 0)^\top, I, (0.5, 0.5)^\top),$$
$$h_N = \theta/N, \quad \theta = 2, \quad N = 200.$$

Given x^0, let us denote by $\boldsymbol{A}_v^1$; $\boldsymbol{A}_v^2$; $\boldsymbol{A}_u^1$ the following three aims: to ensure $x[N] \notin \mathcal{M}$; $x[k] \notin \mathcal{M}$, $k = 1, \dots, N$; $x[N] \in \mathcal{M}$ via control strategies v; v; u respectively. To construct these controls v; v; u we will use the solutions to Problem $2'$; to Problem 2; to the terminal target approach problem from [11, 12] through construction of several tubes $\mathcal{P}^{+,\alpha}[\cdot]$; $\mathcal{P}^{\beta}[\cdot]$; $\mathcal{P}^{-,\gamma}[\cdot]$ from parametric families of the tubes described in [14, Theorem 1] and also in Corollary 1; in Theorem 2; in [11, 12] respectively (see [14, the end of Sec. III] about using the families of the tubes for more details).

We consider 5 initial points

$$x^{0,1} = (-0.6, 2)^\top, \quad x^{0,2} = (0, 1.5)^\top, \quad x^{0,3} = (0.87, -1.5)^\top,$$
$$x^{0,4} = (0.88, -1.5)^\top, \quad x^{0,5} = (1, -1.5)^\top$$

and construct corresponding trajectories $x^{j,(i)}[\cdot]$, $j = 1, \dots, 5$, $i = 1, 2$, under controls $v^{(i)}$, $i = 1, 2$, from (3.6) and (3.7). We consider two Tests. In Test 1, we apply controls v, u with aims $\boldsymbol{A}_v^1$, $\boldsymbol{A}_u^1$ respectively; in Test 2, we apply controls v, u with aims $\boldsymbol{A}_v^2$, $\boldsymbol{A}_u^1$. Note that in Test 2 the aim $\boldsymbol{A}_u^1$ is not opposite to $\boldsymbol{A}_v^2$ (constructing u with the aim $\boldsymbol{A}_u^2$ opposite to $\boldsymbol{A}_v^2$ is out of the scope of this paper). It is possible to use several formulas to construct u basing on tubes $\mathcal{P}^{-,\gamma}[\cdot]$ (see, for example, [12]). Here we have applied the formulas which are similar to [13, Formula (13)]. We do not supply the trajectories $x^{j,(i)}[\cdot]$ by numbers of the Tests to simplify the notation.

The results of computer simulations are visualized in Fig. 1, where cross-sections $\mathcal{P}^{+,\alpha}[0]$, $\alpha = 1, \dots, 4$, and $\mathcal{P}^{\beta}[0]$, $\beta = 1, \dots, 4$, of several tubes calculated for solving Problem $2'$ and

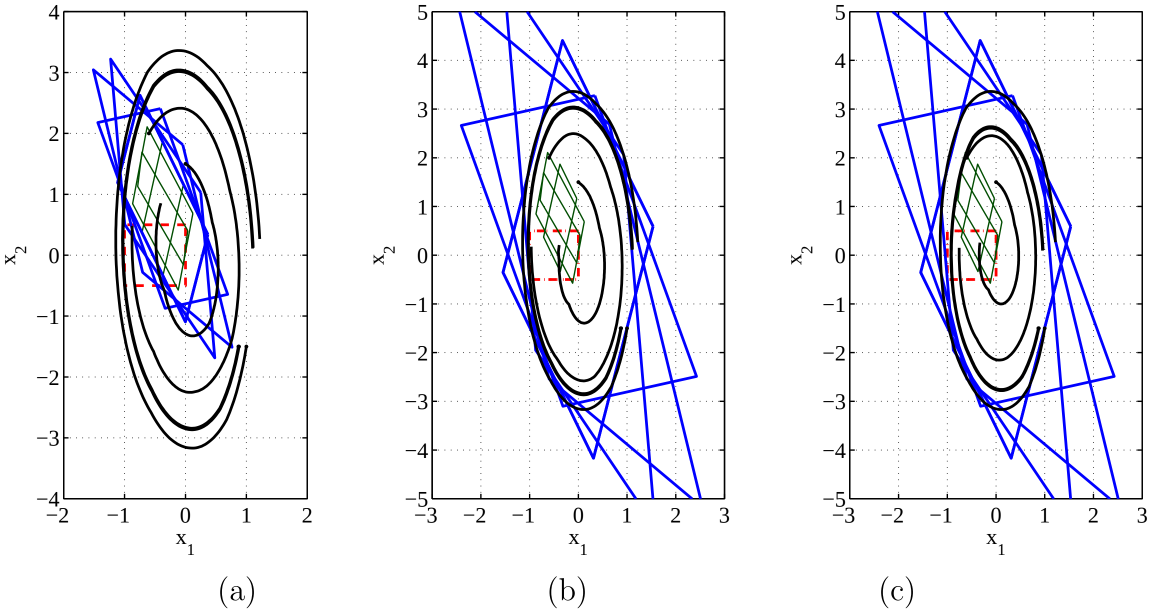

Figure 1. Used constructions and results of evasion from $\mathcal{M}$ (dashed red lines) in Example: several cross-sections $\mathcal{P}^{+,\alpha}[0]$ (left figure, blue thick lines), $\mathcal{P}^{\beta}[0]$ (two right figures, blue lines), $\mathcal{P}^{-,\gamma}[0]$ (green thin lines), and the controlled trajectories under suitable control strategies v^i, $i \in \{1,2\}$, and u. (a) Test 1: using $v^{(1)}$ based on $\mathcal{P}^{+,\alpha}[\cdot]$. (b) Test 2: using $v^{(1)}$ based on $\mathcal{P}^{\beta}[\cdot]$. (c) Test 2: using $v^{(2)}$ based on $\mathcal{P}^{\beta}[\cdot]$

Problem 2 respectively are shown by thick lines, cross-sections $\mathcal{P}^{-,\gamma}[0]$, $\gamma = 1,\dots,3$, by thin lines; the target set $\mathcal{M}$ is presented by dashed lines. The tubes $\mathcal{P}^{+,\alpha}[\cdot]$ and $\mathcal{P}^{-,\gamma}[\cdot]$ are the same as in [14, Sec. IV]; $\mathcal{P}^{\beta}[\cdot]$ are constructed as described in Theorem 2 and Remark 1(b) under the same orientation matrices $P[N]$ at the final instant as for $\mathcal{P}^{+,\alpha}[\cdot]$.

The point $x^{0,1} \in \mathcal{P}^{-,\gamma_*}[0]$ for some γ_* and we obtained $x[N] \in \mathcal{M}$ (aim $\boldsymbol{A}_u^1$ is achieved) for all trajectories started at $x^{0,1}$ in both Test 1 and Test 2 as it is theoretically guaranteed similarly to [12, Theorem 3.1].

Each of the points $x^{0,j}$, $j = 2,\dots,5$, is outside at least one of $\mathcal{P}^{+,\alpha}[0]$ (see Fig. 1(a)). In Test 1, we obtained $x[N] \notin \mathcal{M}$ (aim $\boldsymbol{A}_v^1$ is achieved) for all trajectories started at these $x^{0,j}$ as it was theoretically guaranteed by [14, Theorem 1] and also by Corollary 1. Note that in Test 1 $x^{j,(i)}[\cdot]$, $j = 2,3$, $i = 1,2$, hit $\mathcal{M}$ at some instants $k < N$. The reason is that in Test 1 we used controls v designed for solving Problem 2′ but not Problem 2.

It is also curious that in Test 2 we obtained $x^{2,(i)}[N] \in \mathcal{M}$, $i = 1,2$, in opposite to Test 1. The reason here is that we have used controls $v^{(i)}$ basing on the tubes $\mathcal{P}^{\beta}[\cdot]$ without the guarantee to achieve the aim $\boldsymbol{A}_v^2$ because we have $x^{0,2} \in \bigcap_{\beta=1}^{4} \mathcal{P}^{\beta}[0]$.

The point $x^{0,5} \notin \mathcal{P}^{\beta_*}[0]$ for some β_* (see Fig. 1(b), Fig. 1(c)). In Test 2, we obtained $x[k] \notin \mathcal{M}$, $k = 1,\dots,N$, (aim $\boldsymbol{A}_v^2$ is achieved) for the trajectories $x^{5,(i)}[\cdot]$, $i = 1,2$, started at $x^{0,5}$ as it is theoretically guaranteed by Theorem 2.

For the trajectories started at $x^{0,j}$, $j = 2,3,4$, we have no any guarantees about hitting $\mathcal{M}$ in Test 2 because we have $x^{0,j} \in \bigcap_{\beta=1}^{4} \mathcal{P}^{\beta}[0]$ for these j. And we obtained, in particular, the following results in Test 2. For very close initial points $x^{0,3}$ and $x^{0,4}$ we obtained that $x^{3,(1)}[k] \in \mathcal{M}$ and $x^{3,(2)}[k] \in \mathcal{M}$ for 1 and 13 instants k respectively; $x^{4,(1)}[k] \notin \mathcal{M}$ for all k, and $x^{4,(2)}[k] \in \mathcal{M}$ for 14 instants k. Thus, in this example, $v^{(1)}$ turned out to be more successful than $v^{(2)}$ for the initial points without guarantees.

5. Conclusion

We deal with linear discrete-time systems under two conflicting controls and given target sets. Two subproblems arrise, namely the approach problem and the evasion one. Formerly we elaborated the polyhedral control synthesis for the terminal approach problem and for the terminal evasion problem using polyhedral (parallelotope-valued) tubes. In this paper, the enhanced evasion problem is considered to avoid the trajectory hitting the given target set not only at the given final time, but also at intermediate instants. The common solution scheme is outlined. Then the solution technique is elaborated based on polyhedral (parallelepiped-valued) tubes. The recurrence relations with explicit formulas are presented for the couple of such tubes, the finding of which is much less time-consuming than the construction of the exact solutions. Control strategies, which can be calculated also by explicit formulas on the base of these tubes, are constructed. The illustrative example demonstrating the theoretical results is presented.

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