

HJB–INEQUALITIES IN ESTIMATING REACHABLE SETS OF A CONTROL SYSTEM UNDER UNCERTAINTY¹

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Abstract: Using the technique of generalized inequalities of the Hamilton–Jacobi–Bellman type, we study here the state estimation problem for a control system which operates under conditions of uncertainty and nonlinearity of a special kind, when the dynamic equations describing the studied system simultaneously contain the different forms of nonlinearity in state velocities. Namely, quadratic functions and uncertain matrices of state velocity coefficients are presented therein. The external ellipsoidal bounds for reachable sets are found, some approaches which may produce internal estimates for such sets are also mentioned. The example is included to illustrate the result.

Keywords: Control, Nonlinearity, Uncertainty, Ellipsoidal calculus, State estimation.

1. Introduction

In the paper the nonlinear dynamical control systems with unknown but bounded uncertainties with its set-membership description [20–22, 24] are studied and the main goal of the present research is to construct the outer (external) estimates for related reachable sets. Several approaches may be used for these purposes but most of them are suitable only for the case of linear dynamical systems [8, 24, 26]. However researches in nonlinear control systems theory are very important for various applications, e.g., [3–7]. The key issue in nonlinear set-membership estimation theory is to find suitable techniques, which allow to find estimates (of external and internal kind) for unknown system states and do not involve difficult and lengthy computations. Some approaches to achieve this goal may be taken and further developed using techniques of differential inclusions theory [5] but in general these ideas produce very complicated numerical schemes and hard working algorithms.

We use here the advantages of ellipsoidal calculus [8, 24, 26] and further develop the Hamilton–Jacobi–Bellman (HJB) techniques initiated in researches [12, 17, 19] to construct computationally acceptable set-valued estimates of reachable sets for a new class of nonlinear control systems under uncertainty [9–11, 13–16, 25].

The paper is organized as follows. First, in Section 2 we introduce some notations, give necessary definitions and formulate the main problem. Ellipsoidal external estimates are developed further in Section 3 where equations describing parameters of estimating ellipsoids are presented. The example is given in Section 4 to illustrate the theoretical results.

The study continues previous researches in this field and deals now with a special case, when a nonlinearity of quadratic type together with bilinear terms defined by uncertain matrix are

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presented in the dynamical equations having also an uncertainty in initial states. This case is both of theoretical and of applied importance. Note, however, that the structure of the system under consideration differs from that previously studied, including that given in earlier papers [16, 17].

The problems studied in this paper are generated both by the problems of the theory of guaranteed state estimation for nonlinear dynamical systems with uncertain dynamics and by practical problems of control-estimation under unpredictable interferences.

2. Main notations and problem statement

We introduce first a short list of main notations. Let \mathbb{R}^n be the n -dimensional Euclidean space with the inner product $x'y = \sum_{i=1}^n x_i y_i$ for $x, y \in \mathbb{R}^n$, here a prime indicates a transpose, $\|x\| = (x'x)^{1/2}$. Let $\text{comp } \mathbb{R}^n$ be the set of all compact subsets of \mathbb{R}^n , $h(A, B)$ be the Hausdorff distance between $A, B \in \text{comp } \mathbb{R}^n$. We denote also $B(a, r) = \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$, a symbol I will stand for the identity $n \times n$ -matrix.

We use the symbol $\mathbb{R}^{n \times n}$ for a set of all $n \times n$ -matrices and $E(a, Q)$ for an ellipsoid in \mathbb{R}^n , $E(a, Q) = \{x \in \mathbb{R}^n : (Q^{-1}(x-a), (x-a)) \leq 1\}$ with a center $a \in \mathbb{R}^n$ and with a symmetric positive definite $n \times n$ -matrix Q . For any $n \times n$ -matrix $M = \{m_{ij}\}$ we denote $\text{Tr}(M) = \sum_{i=1}^n m_{ii}$.

We will study the control nonlinear system

$$\dot{x} = A(t)x + f(x)d + u(t), \quad t \in [t_0, T], \quad x_0 \in X_0. \quad (2.1)$$

We assume further that $\|x\| \leq K$ ($K > 0$), $x, d \in \mathbb{R}^n$, the nonlinear function $f(x)$ is quadratic in x , $f(x) = x'Bx$, and B is a symmetric and positive definite $n \times n$ -matrix. Here the coordinates d_i of the vector d are the coefficients with which the nonlinear function $f(x)$ enters the right side of the differential control system (2.1), in particular, they can be interpreted as independent parameters of the studied model or as coefficients of approximate estimates of the state velocities of the simulated system.

The $n \times n$ -matrix function $A(t)$ in (2.1) is assumed to be of the form

$$A(t) = A^1(t) + A^0, \quad (2.2)$$

where the matrix A^0 (with its dimension $n \times n$) is given and the measurable $n \times n$ -matrix function $A^1(t)$ is unknown but bounded, $A^1(t) \in \mathcal{A}^1$ ($t \in [t_0, T]$). Namely, we have

$$A(t) \in \mathcal{A} = A^0 + \mathcal{A}^1, \quad (2.3)$$

$$\mathcal{A}^1 = \{A = \{a_{ij}\} \in \mathbb{R}^{n \times n} : |a_{ij}| \leq c_{ij}, \quad i, j = 1, \dots, n\},$$

where the numbers $c_{ij} \geq 0$ ($i, j = 1, \dots, n$) are given.

We will assume that X_0 in (2.1) is an ellipsoid,

$$X_0 = E(a_0, Q_0),$$

with a symmetric and positive definite matrix $Q_0 \in \mathbb{R}^{n \times n}$ and with a center a_0 .

It is assumed that $f(x)$ in (2.1) is a scalar function of the form $f(x) = x'Bx$, with a given symmetric positive definite $n \times n$ -matrix B . The set \mathcal{U} of admissible controls $u(\cdot)$ in (2.1) consists of all functions $u(t)$ which are measurable in Lebesgue sense on $[t_0, T]$ and such that the constraint

$$u(t) \in U \quad \text{for a.e. } t \in [t_0, T] \quad (2.4)$$

is fulfilled, where U is a given set, $U \in \text{comp } \mathbb{R}^n$.

Let the absolutely continuous function $x(t) = x(t; u(\cdot), A(\cdot), x_0)$ be a solution to dynamical system (2.1)–(2.4) with initial state $x_0 \in X_0$, with admissible control $u(\cdot)$ and with a matrix $A(\cdot)$ satisfying (2.2)–(2.3).

Definition 1. *The reachable set $X(t)$ at time t ($t_0 < t \leq T$) of system (2.1)–(2.4) is defined as follows*

$$X(t) = \{x \in \mathbb{R}^n : \exists u(\cdot) \in \mathcal{U}, \exists x_0 \in X_0, \exists A(\cdot) \in \mathcal{A}, x = x(t) = x(t; u(\cdot), A(\cdot), x_0)\}.$$

It is well known that the exact construction of reachable sets $X(t)$ of a control system is very difficult even for systems with a linear dynamics. The theory based on ideas of construction external and internal ellipsoidal estimates of reachable sets for systems with a linear dynamics have been deeply developed in [8, 24]. Recent new results devoted to construction of external (and in some special cases internal) set-valued estimates of reachable sets $X(t)$ for separate kinds of nonlinear systems may be found in [10–12, 14, 15, 25].

The approach presented here for estimating reachable sets of the system (2.1)–(2.3) is based on the techniques of developed ellipsoidal calculus and uses also approach connected with Hamilton–Jacobi–Bellman equations which is applied to nonlinear control systems of the class described above. Therefore this research establishes a connection between these two approaches to the estimation of unknown states of uncertain dynamical systems of the considered type.

We need to define also an additional trajectory tube $X(t; u(\cdot))$ ($t_0 < t \leq T$, $u(\cdot) \in \mathcal{U}$) which depends on a control $u(\cdot)$.

Definition 2. *Let $u(\cdot)$ be an admissible control. The set $X(t; u(\cdot))$ at time t ($t_0 < t \leq T$) of system (2.1) is defined as the set*

$$X(t; u(\cdot)) = \{x \in \mathbb{R}^n : \exists x_0 \in X_0, \exists A(\cdot) \in \mathcal{A}, x = x(t; u(\cdot), A(\cdot), x_0)\}.$$

Note that for each fix t ($t_0 < t \leq T$) and for a fixed control $u(\cdot)$ ($u(\cdot) \in \mathcal{U}$) the set $X(t; u(\cdot))$ represents the reachable set of system (2.1) taken with respect to $x_0 \in X_0$ only. Accordingly, the estimating ellipsoidal tubes and their cross-sections, generally speaking, also depend on admissible controls $u(\cdot)$, so in the formulations of main problems we would like to emphasize this circumstance by using a slightly modified notation $E(\hat{a}, \hat{Q}; T, u(\cdot))$ for estimating ellipsoids, adding a time moment and control here as additional arguments.

Thus, the main two problems considered here are as follows.

Problem 1. For each feasible control $u(\cdot) \in \mathcal{U}$, find the optimal (closest with respect to inclusion of sets) external ellipsoidal estimate $E(\hat{a}, \hat{Q}; T, u(\cdot))$ of the reachable set $X(T; u(\cdot))$ of the dynamical system (2.1),

$$X(T; u(\cdot)) \subset E(\hat{a}, \hat{Q}; T, u(\cdot)).$$

Problem 2. Given a vector $x^* \in \mathbb{R}^n$, find the feasible control $u^*(\cdot) \in \mathcal{U}$ and a number $\epsilon^* > 0$ such that

$$d(x^*, E(\hat{a}^*, \hat{Q}^*; T, u^*(\cdot))) = \inf_{u(\cdot) \in \mathcal{U}} d(x^*, E(\hat{a}^*, \hat{Q}^*; T, u(\cdot))) = \epsilon^*.$$

3. Main results

Here we describe the general scheme which allows to find the solutions of Problems 1–2. This scheme uses the dynamic programming ideas which are slightly modified to apply to the class of systems under study.

Let us mention first some important results [19, 21] from the optimal control theory which serve as the basis for further constructions.

Consider the control system described by the ordinary differential equation

$$\dot{x} = f(t, x, u(t)), \quad t \in [t_0, T] \tag{3.1}$$

with function $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ measurable in t and continuous in other variables. Here x stands for the state vector, t stands for time and control $u(\cdot)$ is a measurable function satisfying the constraints

$$u(\cdot) \in \mathcal{U} = \{u(\cdot) : u(t) \in U, \quad t \in [t_0, T]\} \quad (3.2)$$

where $U \in \text{comp } \mathbb{R}^m$.

Assume that the initial condition $x(t_0)$ to the system (3.1) is unknown but bounded

$$x(t_0) = x_0, \quad x_0 \in X_0 \in \text{comp } \mathbb{R}^n. \quad (3.3)$$

Assume that an absolutely continuous function $x(t) = x(t, u(\cdot), t_0, x_0)$ is a solution to (3.1) and we have $x(t_0) = x_0$ which satisfies (3.3) and a related control function $u(t)$ satisfies (3.2).

We study the control system (3.1)–(3.3) and we will assume further that $f(t, x, u)$ in (3.1) is continuous in $\{t, x, u\}$ and continuously differentiable in x . We also suppose that solutions to (3.1)–(3.3) may be extended to the whole interval $[t_0, T]$.

We use the notation $X(t) = X(t; t_0, X_0)$ for the reachable set of the system (3.1)–(3.3) at time t . It is well known that the set $X(t)$ may be interpreted as a level set of a value function $V(t, x)$ for an special auxiliary control problem [19, 24]. This value function for the new auxiliary problem satisfies the HJB equation of the following type

$$V_t(t, x) + \max_{u \in U} (V_x, f(t, x, u)) = 0.$$

Generally the value function may not be differentiable. So a solution to the HJB equation may be treated as a minmax or viscosity solution [9]. The precise solutions to such HJB-equations are difficult to find and the corresponding variational inequalities and related comparison theorems may be used to obtain approximate estimates of reachable sets [19].

3.1. Auxiliary constructions

The following auxiliary result will be needed further.

Lemma 1 [21]. *Assume that there exists a function $\mu(t)$ which is integrable on $[t_0, T]$ and such that the inequality*

$$\max_{u \in U} (V_x, f(t, x, u)) + V_t(t, x) \leq \mu(t)$$

is fulfilled. Then the following external estimate of the reachable set $X(t)$ of the system (3.1)–(3.3) is true

$$X(t) \subseteq \left\{ x : V(t, x) \leq \int_{t_0}^t \mu(s) ds + \max_{x \in X_0} V(t_0, x) \right\}, \quad t_0 \leq t \leq T.$$

Remark 1. It is known that we may take here $\mu(s) = 0$ [19].

The following more general inequality may be used also in the estimation context, namely

$$V_t(t, x) + \max_{u \in U} (V_x, f(t, x, u)) \leq g(t, V(t, x)) \quad (3.4)$$

where $g(t, V)$ is integrable in $t \in [t_0, T]$ and is continuously differentiable in V .

Due to above property the ordinary differential equation

$$\dot{U}(t) = g(t, U), \quad U(t_0) = U_0, \quad (3.5)$$

is called a comparison equation for (3.1)–(3.3). We will need further the following result.

Theorem 1 [19]. *Assume that the relations (3.4) and (3.5) are satisfied and the inequality*

$$\max_{x \in X_0} V(t_0, x) \leq U_0$$

is true. Then the upper estimate

$$X(t) \subseteq \{x : V(t, x) \leq U(t)\}, \quad t_0 \leq t \leq T.$$

is valid.

3.2. External estimates of reachable sets under uncertainty via HJB techniques

A number of approaches had been proposed recently to derive differential equations which describe the dynamics of external ellipsoidal estimates of reachable sets of uncertain control systems. In particular, the differential equations of ellipsoidal estimates for reachable sets of a nonlinear dynamical control system were derived in [12]. There the case was studied when state velocities of the contain special quadratic forms but in that case the presence of uncertainty in matrix coefficients was not investigated.

The following result continues this research and describes the dynamics of external ellipsoidal estimates of the reachable set $X(t) = X(t; t_0, X_0)$ ($t_0 \leq t \leq T$) for the special case when $U = E(\hat{a}, \hat{Q})$ with a center \hat{a} and a positive definite matrix \hat{Q} given.

First we find the smallest number $k > 0$ for which the inclusion

$$X_0 = E(a_0, Q_0) \subseteq E(a_0, k^2 B^{-1}) \quad (3.6)$$

is true, this initial step will help to get better resulting estimate for the whole trajectory tube $X(t) = X(t; t_0, X_0)$ ($t_0 \leq t \leq T$). The smallest number $k > 0$ satisfying (3.6) may be determined using the procedure described, for example, in [15].

The following main result is true.

Theorem 2. *For any $t \in [t_0, T]$ the following inclusion is true*

$$X(t; t_0, X_0) \subseteq E(a^+(t), r^+(t)B^{-1}), \quad (3.7)$$

here functions $a^+(t)$, $r^+(t)$ are the solutions of the following system

$$\begin{aligned} \dot{a}^+(t) &= A^0 a^+(t) + ((a^+(t))' B a^+(t) + r^+(t))d + \hat{a}, \quad t_0 \leq t \leq T, \\ \dot{r}^+(t) &= \max_{\|l\|=1} \left\{ l' (2r^+(t)B^{1/2}(A_0 + 2d(a^+(t))' B)B^{-1/2} \right. \\ &\quad \left. + (q(r^+(t)))^{-1} r^+(t)B^{1/2} \hat{Q}^* B^{1/2}) l \right\} + q(r^+(t))r^+(t), \\ q(r) &= ((nr)^{-1} \text{Tr}(B \hat{Q}^*))^{1/2}, \end{aligned} \quad (3.8)$$

the matrix \hat{Q}^ is positive definite and satisfies the inclusion*

$$\mathcal{A}^1 a_0 + E(0, \hat{Q}) + k_0 D^{1/2} B^{1/2} B(0, 1) \subseteq E(0, \hat{Q}^*) \quad (3.9)$$

and the initial state is the following

$$a^+(t_0) = a_0, \quad r^+(t_0) = k^2. \quad (3.10)$$

P r o o f. Using the result of Theorem 1 and basing on the scheme of reasoning discussed in [15] and [19] with necessary modifications because of the special structure of the system (2.1)–(2.3), we derive the estimates (3.7)–(3.10). Note that the above result is essentially and ideologically close to the estimates given in [13] (see Theorem 2), but it differs significantly in details and in final relations, thereby supplementing the already existing range of methods and providing a deeper theoretical basis for solving problems of estimating the states of uncertain systems of the class under study. \square

Despite the seeming cumbersomeness of the formulas describing the ellipsoid that is external in terms of inclusion for the reachable set (at the current moment of time), the calculations of the external estimates given in the Theorem 2 are fast enough (performed "in real time") and easy to implement.

The outer ellipsoids obtained by the scheme of Theorem 2 are optimal in the sense that they touch the real reachable sets at some points and cannot be reduced without violating the basic requirement to contain the estimated reachable set.

Theorem 2 solves the Problem 1 and gives the way to find an approximate solution for the Problem 2.

4. Example

We illustrate here the proposed state estimation scheme for a nonlinear uncertain system of the studied kind. The external estimates calculated on the base of Theorem 2 remain ellipsoidal-valued (and therefore convex) and contain reachable sets of the considered system.

Example 1. Consider the control system

$$\begin{cases} \dot{x}_1 = (2 + \nu)x_1 + u_1, \\ \dot{x}_2 = (2 + \nu)x_2 + u_2, \\ \dot{x}_3 = (2 + \nu)x_3 + x_1^2 + x_2^2 + x_3^2 + u_3. \end{cases} \quad (4.1)$$

Here we take $x_0 \in X_0 = B(0, 1)$, $0 \leq t \leq T = 0.4$ and $U = B(0, 0.1)$, a parameter ν is unknown but bounded, namely $\nu \in [0, 0.1]$. We emphasize that the constraint on the unknown parameter ν in the control system (4.1) has a different form than in the example (13) in [17], in accordance with a different formulation of the main problem studied here and because of a different technique used for its solution.

The reachable set $X(T)$ and its external ellipsoidal estimate found on the base of Theorem 2 $E(a^+(t), Q^+(t))$ for $t = T$ are shown in Fig. 1.

The following Fig. 2 for which different results [12, 17] were used is included to illustrate the possibilities of the approach in common, in particular of obtaining two-sided (external and internal) ellipsoidal estimates for the reachable sets of control systems with uncertainty. Fig. 2 presents two sets also, the same reachable set $X(T)$ and its internal estimating ellipsoid $E^-(T) = E(a^-(T), Q^-(T))$,

Here, in both Fig. 1 and Fig. 2, one can see a possible gap between the external and internal estimates of the reachable sets under study, this gap cannot be eliminated within the framework of the approach described here. We also note that the ellipsoidal estimates constructed above (each in its own class, of internal or external kind) are exact in the sense that these estimates are unimprovable (they cannot be reduced or increased, respectively) without violating the basic requirements for their construction. It can also be underlined that the algorithms for constructing these ellipsoidal estimates are very simple to implement (for example, through the Matlab system) and do not require much computation time.

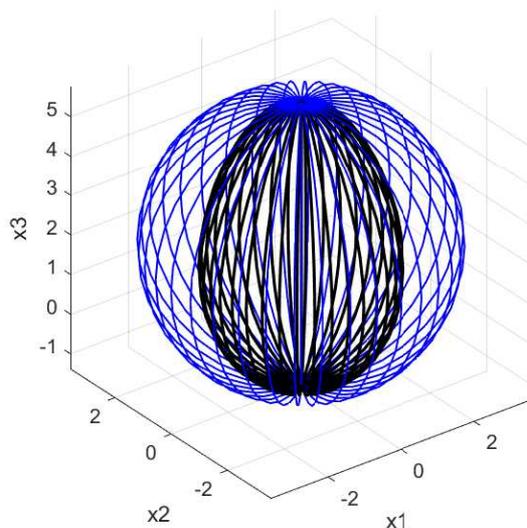


Figure 1. External estimating ellipsoid $E^+(T) = E(a^+(T), Q^+(T))$ (blue color) and the reachable set (black color) $X(T)$ in the space of $\{x_1, x_2, x_3\}$ -coordinates.

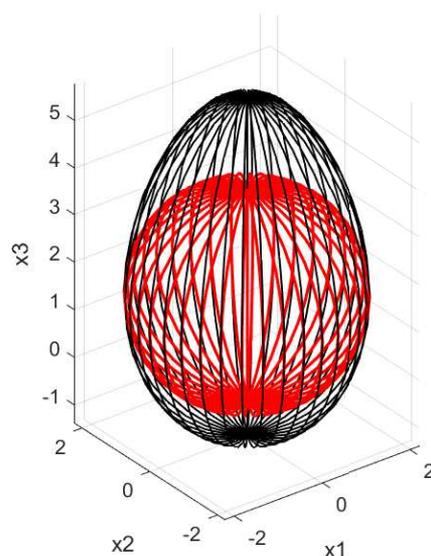


Figure 2. Internal estimating ellipsoid $E^-(T) = E(a^-(T), Q^-(T))$ (red color) and the reachable set (black color) $X(T)$ in the space of $\{x_1, x_2, x_3\}$ -coordinates.

Remark 2. The above results are also applicable to more complicated classes of problems of control under nonlinearity and uncertainty including [1, 12, 15, 17–19, 25] and to the case of presence of additional state constraints, in this case it is also possible to use basic ideas of the research [23].

Remark 3. A detailed description of somewhat different, but similar in essence, approaches to solving problems of control and state estimation and using special information sets for studying control problems under uncertainty can be found in [2].

5. Conclusion

A new method of external estimation of the states of a nonlinear control system with uncertainty is proposed, based on the ideas and results of the theory of Hamilton–Jacobi–Bellmann equations. The relationship between the new approaches proposed here and the ideas and results of earlier studies in the theory of estimating the states of dynamical systems under conditions of uncertainty and nonlinearity is established. The possibilities of computer simulation for problems of this class are discussed. The numerical simulation results for constructing upper and inner estimates of reachable sets related to the proposed techniques and illustrating the basic ideas and algorithms are included.

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