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ON CAUCHY-TYPE BOUNDS FOR THE EIGENVALUES OF A SPECIAL CLASS OF MATRIX POLYNOMIALS

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Abstract: Let $\mathbb{C}^{m\times m}$ be the set of all $m\times m$ matrices whose entries are in $\mathbb{C}$, the set of complex numbers. Then $P(z):=\sum\limits_{j=0}^{n}A_jz^j,\ A_j\in\mathbb{C}^{m\times m},\ 0\leq j\leq n$ is called a matrix polynomial. If $A_n\neq 0$, then $P(z)$ is said to be a matrix polynomial of degree n. In this paper we prove some results for the bound estimates of the eigenvalues of some lacunary type of matrix polynomials.

Keywords: Matrix polynomial, Eigenvalue, Positive-definite matrix, Cauchy's theorem, Spectral radius.

1. Introduction

Let $\mathbb{C}^{m\times m}$ be the set of all $m\times m$ matrices whose entries are in $\mathbb{C}$, the set of complex numbers. For a matrix polynomial we mean the matrix-valued function of a complex variable of the form

$$P(z):=\sum_{j=0}^{n}A_jz^j,\quad A_j\in\mathbb{C}^{m\times m},\quad 0\leq j\leq n.$$

If $A_n\neq 0$, then $P(z)$ is called a matrix polynomial of degree n.

A complex number λ is said to be an eigenvalue of the matrix polynomial $P(z)$, if there exists a nonzero vector $u\in\mathbb{C}^m$, such that $P(\lambda)u=0$. The vector u is called an eigenvector of $P(z)$ associated to the eigenvalue λ.

For matrices $A,B\in\mathbb{C}^{m\times m}$, we write $A\geq 0$ or $A>0$, if A is positive semi-definite or positive definite respectively. By $A\geq B$, we mean $A-B\geq 0$ and $A>B$, means $A-B>0$.

We denote by $\lambda_{max}(A)$ and $\lambda_{min}(A)$ the maximum and minimum eigenvalues of a Hermitian matrix A respectively. Also the spectral radius denoted by $\rho(A)$ of a matrix A is defined by

$$\rho(A)=\max\big\{|\lambda|:\ \lambda \text{ is an eigenvalue of } A\big\}.$$

The identity matrix and the conjugate transpose of a vector u are respectively denoted by I and u^*. Also for $\alpha\geq 0$, denote the open disk

$$K^*(0,\alpha):=\{z\in\mathbb{C}:\ |z|<\alpha\}$$

and the closed disk

$$K(0,\alpha):=\{z\in\mathbb{C}:\ |z|\leq\alpha\}.$$

Polynomial eigenvalue problems have vital applications in a wide range of science and engineering fields (see for example [4, 9]). It is generally challenging to compute the eigenvalues of a matrix polynomial, but bounds on such eigenvalues are relatively easy to obtain. These bounds can be used by iterative methods to calculate them and are also valuable for the computation of pseudospectra.

A simple but classical result due to Cauchy [2; 7, Theorem 27.1, p. 122] on the location of zeros of a polynomial with complex coefficients states:

Theorem 1. *Let*

$$p(z) := \sum_{j=0}^{n} a_j z^j, \quad a_n \neq 0$$

be a polynomial of degree n *with complex coefficients. Then the zeros of* $p(z)$ *lie in* $\{z : |z| < \rho\}$, *where* ρ *is the unique positive root of the equation*

$$|a_n|z^n - |a_{n-1}|z^{n-1} - \cdots - |a_1|z - |a_0| = 0.$$

An extension to matrix polynomials of Cauchy's classical result was obtained in [1, 5, 8]. It states:

Theorem 2. *Let*

$$P(z) := \sum_{j=0}^{n} A_j z^j, \quad \det(A_n) \neq 0$$

be a matrix polynomial. Then the eigenvalues of $P(z)$ *lie in* $|z| \leq \rho$, *where* ρ *is the unique positive root of the equation*

$$\|A_n^{-1}\|^{-1} z^n - \|A_{n-1}\| z^{n-1} - \cdots - \|A_1\| z - \|A_0\| = 0.$$

Throughout this paper, $\|\cdot\|$ denotes a subordinate matrix norm.

2. Main results

We call a matrix polynomial lacunary, if some of its coefficients are missing. In this paper, we obtain bounds for the eigenvalues of a class of lacunary matrix polynomials. The first result we prove in this paper states:

Theorem 3. *Let*

$$P(z) := Iz^n - Iz^{n-1} - A_1 z + A_0, \quad \|A_0\| \cdot \|A_1\| \neq 0, \quad n > 2$$

be a matrix polynomial. Then the eigenvalues of $P(z)$ *lie in* $K(0, \delta)$, *where* $\delta > 1$ *is the largest positive root of the equation*

$$z^{n+1} - 2z^n - \|A_1\|z^2 + (\|A_1\| - \|A_0\|)z + \|A_0\| = 0.$$

P r o o f. Let u be a unit vector, then we have for $|z| > 1$,

$$\begin{aligned} \|P(z)u\| &= \|uz^n - uz^{n-1} - A_1uz + A_0u\| \\ &\geq |z|^n - \left\|uz^{n-1} + A_1uz - A_0u\right\| \\ &\geq |z|^n - |z|^{n-1} - \|A_1\||z| - \|A_0\| \\ &= |z|^n\left(1 - \left(\frac{1}{|z|} + \|A_1\|\frac{1}{|z|^{n-1}} + \|A_0\|\frac{1}{|z|^n}\right)\right) \\ &> |z|^n\left(1 - \left(\frac{1}{|z|-1} + \|A_1\|\frac{1}{|z|^{n-1}} + \|A_0\|\frac{1}{|z|^n}\right)\right) \\ &= \frac{1}{|z|-1}\left(|z|^{n+1} - 2|z|^n - \|A_1\||z|^2 + (\|A_1\| - \|A_0\|)|z| + \|A_0\|\right) \\ &= \frac{1}{|z|-1} H(|z|), \end{aligned} \tag{2.1}$$

where

$$H(z) = z^{n+1} - 2z^n - \|A_1\|z^2 + (\|A_1\| - \|A_0\|)z + \|A_0\|.$$

Here $H(z)$ has two sign changes within its sequence of coefficients and $H(0) = \|A_0\| > 0$ and $H(1) = -1 < 0$, therefore by Descartes' rule of signs $H(z)$ has two positive zeros. Let δ be the largest positive zero of $H(z)$, then $H(|z|) > 0$ if $|z| > \delta$. Noting that $\delta > 1$, therefore from (2.1), we have

$$\|P(z)u\| > 0 \quad \text{if} \quad |z| > \delta.$$

Hence the eigenvalues of $P(z)$ lie in the closed disk $K(0, \delta)$, where $\delta > 1$ is the largest positive root of $H(z)$. □

The following result can be deduced from the above theorem.

Corollary 1. *Let*

$$P(z) := Iz^n - Iz^{n-1} - A_1 z + A_0, \quad \|A_0\| \cdot \|A_1\| \neq 0, \quad n > 2$$

be a matrix polynomial. Then the eigenvalues of $P(z)$ lie in $K(0, \delta')$, where $\delta' > 1$ is the largest positive root of the equation

$$z^{n+1} - 2z^n - Mz^2 + M = 0$$

and

$$M = \max(\|A_1\|, \|A_0\|).$$

P r o o f. Let

$$H(z) = z^{n+1} - 2z^n - \|A_1\|z^2 + (\|A_1\| - \|A_0\|)z + \|A_0\|.$$

Then we have for $|z| \geq 1$

$$\begin{aligned} H(|z|) &= |z|^{n+1} - 2|z|^n - \|A_1\||z|^2 + (\|A_1\| - \|A_0\|)|z| + \|A_0\| \\ &= |z|^{n+1} - 2|z|^n - \|A_1\|(|z|^2 - |z|) - \|A_0\|(|z| - 1) \\ &\geq |z|^{n+1} - 2|z|^n - M(|z|^2 - |z|) - M(|z| - 1) \\ &= |z|^{n+1} - 2|z|^n - M|z|^2 + M = G(|z|), \end{aligned} \tag{2.2}$$

where

$$G(z) = z^{n+1} - 2z^n - Mz^2 + M.$$

Since $\|A_0\| \cdot \|A_1\| \neq 0$, therefore $M \neq 0$ and hence $G(z)$ has two sign changes within its sequence of coefficients. Also $G(0) = M > 0$ and $G(1) = -1 < 0$, thus by Descartes rule of signs $G(z)$ has two positive zeros. Let $\delta' > 1$ be the largest positive zero of $G(z)$. Therefore from (2.2), we have

$$H(|z|) \geq G(|z|) > 0 \quad \text{if} \quad |z| > \delta'.$$

Thus

$$H(|z|) > 0 \quad \text{if} \quad |z| > \delta'.$$

Thus $\delta \leq \delta'$, where δ is the largest positive zero of $H(z)$. However by Theorem 3 all eigenvalues of $P(z)$ lie in $K(0, \delta)$, therefore

$$K(0, \delta) \subseteq K(0, \delta').$$

This proves the corollary. □

We next prove the following results which give bounds on the eigenvalues of a matrix polynomial in terms of the norms of coefficient matrices.

Theorem 4. *Let*

$$P(z) := Iz^n - Iz^{n-1} - A_1z + A_0, \quad \|A_0\| \cdot \|A_1\| \neq 0, \quad n > 2$$

be a matrix polynomial. Then the eigenvalues of $P(z)$ lie in the closed disk

$$K\left(0, (1 + \sqrt{1 + 4\|A_0\| + 4\|A_1\|})/2\right).$$

P r o o f. Let u be a unit vector. Then just as in the proof of the Theorem 3, we have for $|z| > 1$

$$\begin{aligned}\|P(z)u\| &\geq |z|^n - \left(|z|^{n-1} + \|A_1\||z| + \|A_0\|\right) \\ &> |z|^n - \left(|z|^{n-1} + \|A_1\||z|^{n-2} + \|A_0\||z|^{n-2}\right) \\ &= |z|^{n-2}(|z|^2 - |z| - \|A_1\| - \|A_0\|) = |z|^{n-2}H(|z|),\end{aligned} \tag{2.3}$$

where

$$H(z) = z^2 - z - \|A_1\| - \|A_0\|.$$

Now $H(z) = 0$ implies

$$z = \frac{1 \pm \sqrt{1 + 4\|A_1\| + 4\|A_0\|}}{2}.$$

Also $\lim_{z\to\infty} H(z) = \infty$, thus if

$$|z| > \frac{1 + \sqrt{1 + 4\|A_1\| + 4\|A_0\|}}{2},$$

then $H(|z|) > 0$. This implies from (2.3) that for $|z| > 1$

$$\|P(z)u\| > 0 \quad \text{if} \quad |z| > \frac{1 + \sqrt{1 + 4\|A_1\| + 4\|A_0\|}}{2}.$$

We note that

$$\frac{1 + \sqrt{1 + 4\|A_1\| + 4\|A_0\|}}{2} > 1,$$

therefore the eigenvalues of $P(z)$ lie in the closed disk $K\left(0, (1 + \sqrt{1 + 4\|A_1\| + 4\|A_0\|})/2\right)$. □

It is clear that if $\rho > 0$ then

$$1 + \rho > \frac{1}{2}(1 + \sqrt{1 + 4\rho})$$

and therefore we have the following:

Corollary 2. *Let*

$$P(z) := Iz^n - Iz^{n-1} - A_1z + A_0, \quad \|A_0\| \cdot \|A_1\| \neq 0, \quad n > 2$$

be a matrix polynomial. Then the eigenvalues of $P(z)$ lie in the open disk $K^*(0, 1 + \|A_0\| + \|A_1\|)$.

The next result which we prove gives an upper bound for the positive eigenvalues. For the proof we need the following lemmas.

Lemma 1 [3]. *If the real polynomial*

$$p(z) = z^n - z^{n-1} - a_1z + a_0, \quad a_1a_0 > 0, \quad n > 2,$$

has two positive zeros, its largest positive zero δ satisfies $\delta < 1 + \sqrt{a_1}$.

The above lemma is due to Dehmer and Mowshowitz [3]. We also need the following lemma.

Lemma 2 [6, p. 235]. *Let $M \in \mathbb{C}^{m\times m}$ be a Hermitian matrix, then*

$$\lambda_{\min}(M) = \min_{u\in\mathbb{C}^m,\, u^*u=1}\{u^*Mu\}$$

and

$$\lambda_{\max}(M) = \max_{u\in C^m,\, u^*u=1}\{u^*Mu\}.$$

Theorem 5. *Let*

$$P(z) =: Iz^n - Iz^{n-1} - A_1z + A_0, \quad A_1 \geq A_0 > 0, \quad n > 2$$

be a matrix polynomial. If λ is a positive eigenvalue of $P(z)$, then

$$\lambda < 1 + \sqrt{\|A_1\|}.$$

P r o o f. Let u be a unit vector. Define

$$P_u(z) = u^*P(z)u = z^n - z^{n-1} - u^*A_1uz + u^*A_0u.$$

Then $P_u(z)$ is a polynomial with complex coefficients. Also since $A_0 > 0$, therefore $P_u(z)$ has two sign changes within its sequence of coefficients. Moreover $P_u(0) = u^*A_0u > 0$ and

$$P_u(1) = u^*A_0u - u^*A_1u = u^*(A_0 - A_1)u \leq 0,$$

therefore by Descartes' rule of signs $P_u(z)$ has two positive roots. Hence by Lemma 1, the largest positive zero δ_u of $P_u(z)$ satisfies

$$\delta_u < 1 + \sqrt{u^*A_1u}.$$

Thus by Lemma 2, we have

$$\delta_u < 1 + \sqrt{\lambda_{\max}(A_1)} \leq 1 + \sqrt{\|A_1\|}. \tag{2.4}$$

Let λ be a positive eigenvalue of $P(z)$, then λ is a zero of $P_u(z)$ for some unit vector u and therefore by (2.4), we have

$$\lambda < 1 + \sqrt{\|A_1\|}.$$

This proves the theorem. □

Taking $A_1 = I$ in Theorem 5, we get the following:

Corollary 3. *Let*

$$P(z) =: Iz^n - Iz^{n-1} - Iz + A_0, \quad I \geq A_0 > 0, \quad n > 2$$

be a matrix polynomial. If λ is a positive eigenvalue of $P(z)$, then $\lambda < 2$.

The next theorem gives a bound on the eigenvalues of another class of lacunary matrix polynomials.

Theorem 6. *Let*

$$P(z) =: Iz^n - A_1z + A_0, \quad \|A_0\|\cdot\|A_1\| \neq 0, \quad n > 2$$

be a matrix polynomial. Then the eigenvalues of $P(z)$ lie in the closed disk

$$K\left(0, (\|A_1\| + \sqrt{\|A_1\|^2 + 4\|A_0\| + 4})/2\right).$$

P r o o f. Let u be a unit vector. Then just as in the proof of the Theorem 3, we have for $|z|>1$,

$$\begin{aligned}\|P(z)u\| &\geq |z|^n - \|A_1\||z| - \|A_0\| \\ &> |z|^n - \left(\|A_1\||z|^{n-1} + \|A_0\||z|^{n-2} + |z|^{n-2}\right) \\ &= |z|^{n-2}(|z|^2 - \|A_1\||z| - \|A_0\| - 1) = |z|^{n-2}H(|z|),\end{aligned} \tag{2.5}$$

where

$$H(z) = z^2 - \|A_1\|z - \|A_0\| - 1.$$

Now $H(z)=0$ implies

$$z = \frac{\|A_1\| \pm \sqrt{\|A_1\|^2 + 4\|A_0\| + 4}}{2}.$$

Thus $H(|z|)>0$ if

$$|z| > \frac{\|A_1\| + \sqrt{\|A_1\|^2 + 4\|A_0\| + 4}}{2}.$$

Also noting that

$$\frac{\|A_1\| + \sqrt{\|A_1\|^2 + 4\|A_0\| + 4}}{2} > 1.$$

therefore from (2.5), we have

$$\|P(z)u\| > 0 \quad \text{if} \quad |z| > \frac{\|A_1\| + \sqrt{\|A_1\|^2 + 4\|A_0\| + 4}}{2}.$$

Therefore the eigenvalues of $P(z)$ lie in the closed disk

$$K\left(0, (\|A_1\| + \sqrt{\|A_1\|^2 + 4\|A_0\| + 4})/2\right).$$

□

The next result is obtained on restricting the coefficient matrices. For the proof we need the following lemma due to Dehmer and Mowshowitz [3].

Lemma 3 [3]. *If the real polynomial*

$$p(z) = z^n - a_1 z + a_0, \quad a_1 a_0 > 0, \quad n > 2,$$

has two positive zeros, its largest positive zero satisfies

$$\delta < \frac{1 + \sqrt{4a_1 + 1}}{2}.$$

Theorem 7. *Let*

$$P(z) =: Iz^n - A_1 z + A_0, \quad A_1 \geq I + A_0, \quad A_0 > 0, \quad n > 2$$

be a matrix polynomial. If λ is a positive eigenvalue of $P(z)$, then

$$\lambda < \frac{1 + \sqrt{4\|A_1\| + 1}}{2}.$$

P r o o f. Let u be a unit vector and $P_u(z) = u^*P(z)u$. Then since

$$P(z) = Iz^n - A_1z + A_0,$$

we have

$$P_u(z) = z^n - u^*A_1uz + u^*A_0u.$$

Now by hypothesis

$$P_u(1) = 1 - u^*A_1u + u^*A_0u = u^*(I - A_1 + A_0)u \leq 0$$

and

$$P_u(0) = u^*A_0u > 0.$$

Also $P_u(z)$ has two sign changes within its sequence of coefficients, therefore by Descartes' rule of signs $P_u(z)$ has two positive zeros. Hence by Lemma 3, the largest positive zero δ_u of $P_u(z)$ satisfies

$$\delta_u < \frac{1 + \sqrt{4u^*A_1u + 1}}{2}.$$

This gives on using Lemma 2

$$\delta_u < \frac{1 + \sqrt{4\lambda_{\max}(A_1) + 1}}{2} \leq \frac{1 + \sqrt{4\|A_1\| + 1}}{2}.$$

In the same way as in Theorem 5, we conclude that any positive eigenvalue λ of $P(z)$ satisfies

$$\lambda < \frac{1 + \sqrt{4\|A_1\| + 1}}{2}.$$

□

For $m = 1$, the matrices A_j reduce to $a_j \in \mathbb{C}$ and therefore in this case the above results reduce to various theorems proved by Dehmer and Mowshowitz [3].

The bounds obtained in Theorem 3–5 are incomparable. We consider the following examples:

Example 1. Let $n = 3$, $\|A_0\| = \|A_1\| = \|A\|$ for some matrix A. Then if $\|A\| = 1$, we have

$$\delta = 2.3485 > 2 = \frac{1 + \sqrt{1 + 4\|A_0\| + 4\|A_1\|}}{2}.$$

However, if $\|A\| = 6$, then

$$\delta = 3.5544 < 4 = \frac{1 + \sqrt{1 + 4\|A_0\| + 4\|A_1\|}}{2}.$$

Example 2. Let $\|A_0\| = \|A_1\| = 3$, then

$$\frac{1 + \sqrt{1 + 4\|A_0\| + 4\|A_1\|}}{2} = 3 > 1 + \sqrt{3} = 1 + \sqrt{\|A_1\|}.$$

However, if $\|A_0\| = \|A_1\| = 1/2$, then

$$\frac{1 + \sqrt{1 + 4\|A_0\| + 4\|A_1\|}}{2} = 1.618 < 1.7071 = 1 + \sqrt{\|A_1\|}.$$

Example 3. Assume $\|A_0\| = \|A_1\| = \|A\|$ for some matrix A. Then if $n = \|A\| = 4$ we have

$$\delta = 2.5279 < 3 = 1 + \sqrt{\|A_1\|}.$$

However, if $n = 3$ and $\|A\| = 6$ then

$$\delta = 3.5544 > 3.44 = 1 + \sqrt{\|A_1\|}.$$

Note, we used Desmos, an online graphing calculator and mathematical tool, for the calculations.

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REFERENCES

1. Bini D. A., Noferini V., Sharify M. Locating the eigenvalues of matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 2013. Vol. 34, No. 4. P. 1708–1727. DOI: 10.1137/120886741
2. Cauchy A.-L. *Mémoire sur la Résolution des Équations numériques et sur la Théorie de l'Élimination.* Paris: de Bure Frères, 1829. 64 p. (in French)
3. Dehmer M., Mowshowitz A. Bounds on the moduli of polynomial zeros. *Appl. Math. Comp.*, 2011. Vol. 218, No. 8. P. 4128–4137. DOI: 10.1016/j.amc.2011.09.043
4. Gohberg I., Lancaster P., Rodman L. *Matrix Polynomials.* New York: Academic Press, 1982. 433 p.
5. Higham N. J., Tisseur F. Bounds for eigenvalues of matrix polynomials. *Linear Algebra Appl.*, 2003. Vol. 358, No. 1–3. P. 5–22. DOI: 10.1016/S0024-3795(01)00316-0
6. Horn R. A., Johnson C. R. *Matrix Analysis.* Cambridge: Cambridge University Press, 2013. 643 p. DOI: 10.1017/CBO9780511810817
7. Marden M. *Geometry of Polynomials.* Math. Surveys Monogr., vol. 3. Providence, RI: American Mathematical Society, 1966. 243 p.
8. Melman A. Generalization and variations of Pellet's theorem for matrix polynomials. *Linear Algebra Appl.*, 2013. Vol. 439, No. 5. P. 1550–1567. DOI: 10.1016/j.laa.2013.05.003
9. Tisseur F., Meerbergen K. The quadratic eigenvalue problem. *SIAM Review*, 2001. Vol. 43, No. 2. P. 235–286.