URAL MATHEMATICAL JOURNAL, Vol. 9, No. 1, 2023, pp. 49–63
DOI: 10.15826/umj.2023.1.004

STATISTICAL CONVERGENCE IN A BICOMPLEX VALUED METRIC SPACE

Subhajit Bera[†], **Binod Chandra Tripathy**[††]

Tripura University,
Suryamaninagar-799022, Agartala, India

[†]berasubhajit0@gmail.com [††]tripathybc@gmail.com

Abstract: In this paper, we study some basic properties of bicomplex numbers. We introduce two different types of partial order relations on bicomplex numbers, discuss bicomplex valued metric spaces with respect to two different partial orders, and compare them. We also define a hyperbolic valued metric space, the density of natural numbers, the statistical convergence, and the statistical Cauchy property of a sequence of bicomplex numbers and investigate some properties in a bicomplex metric space and prove that a bicomplex metric space is complete if and only if two complex metric spaces are complete.

Keywords: Partial order, Bicomplex valued metric space, Statistical convergence.

1. Introduction

The concept of statistical convergence for real numbers was introduced by Fast [6], Buck [2], and Schoenberg [12] independently. Later, the concept was studied and linked with summability theory by Salat [11], Fridy [7], Tripathy [17, 19], Rath and Tripathy [8], Tripathy and Sen [18], Tripathy and Nath [16], and many others.

The concept of bicomplex numbers has been investigated from different aspects by Segre [13], Wagh [20], Srivastava and Srivastava [15], Sager and Sager [10], Rochon and Shapiro [9], Beg et al. [3], and Singh [14]. In this paper, we study different types of partial order relations on bicomplex numbers and discuss the concept of statistical convergence in bicomplex valued metric spaces.

Das et al. [5] and many other researchers discussed the statistical convergence in a metric space. In this paper, we investigate statistically convergent and statistically Cauchysequences in a bicomplex valued metric space.

In what follows, C_0, C_1, and C_2 denote the set of real, complex, and bicomplex numbers, respectively.

2. Definitions and preliminaries

2.1. Bicomplex numbers

The concept of bicomplex numbers was introduced by Segre [13]. A bicomplex number is defined as

$$\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4,$$

where $x_1, x_2, x_3, x_4 \in C_0$ and the independent units i_1 and i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1i_2 = i_2i_1$. The set of bicomplex numbers C_2 is defined as

$$C_2 = \big\{\xi : \xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4, \quad x_1, x_2, x_3, x_4 \in C_0\big\},$$

i.e.,

$$C_2 = \{\xi : \xi = z_1 + i_2 z_2, \quad z_1, z_2 \in C_1\}.$$

There are four idempotent elements in C_2, they are $0, 1$, $e_1 = (1 + i_1 i_2)/2$, and $e_2 = (1 - i_1 i_2)/2$, two of which, e_1 and e_2, are nontrivial such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.

Every bicomplex number $\xi = z_1 + i_2 z_2$ can be uniquely expressed as the combination of e_1 and e_2; namely,

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 = \mu_1 e_1 + \mu_2 e_2,$$

where $\mu_1 = (z_1 - i_1 z_2)$ and $\mu_2 = (z_1 + i_1 z_2)$.

A bicomplex number $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$ is said to be a hyperbolic number if $x_2 = 0$ and $x_3 = 0$. The set of hyperbolic numbers is denoted by $\mathcal{H}$.

The Euclidean norm $\|.\|$ on C_2 is defined as

$$\|\xi\|_{C_2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|\mu_1|^2 + |\mu_2|^2}{2}},$$

where

$$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2 = \mu_1 e_1 + \mu_2 e_2$$

and

$$\mu_1 = z_1 - i_1 z_2, \quad \mu_2 = z_1 + i_1 z_2, \quad e_1 = \frac{1 + i_1 i_2}{2}, \quad e_2 = \frac{1 - i_1 i_2}{2}.$$

With this norm, C_2 is a Banach space, also C_2 is a commutative algebra.

The product of two bicomplex numbers satisfies the inequality

$$\|\xi \cdot \eta\|_{C_2} \leq \sqrt{2}\|\xi\|_{C_2} \cdot \|\eta\|_{C_2}.$$

Definition 1.

(*i*) *The i_1-conjugate of a bicomplex number $\xi = z_1 + i_2 z_2$ is denoted by ξ^* and is defined as $\xi^* = \bar{z_1} + i_2 \bar{z_2}$ for all $z_1, z_2 \in C_1$; here $\bar{z_1}$ and $\bar{z_2}$ are the complex conjugates of z_1 and z_2, respectively, and $i_1^2 = i_2^2 = -1$.*

(*ii*) *The i_2-conjugate of a bicomplex number $\xi = z_1 + i_2 z_2$ is denoted by $\bar{\xi}$ and is defined as $\bar{\xi} = z_1 - i_2 z_2$ for all $z_1, z_2 \in C_1$, where $i_1^2 = i_2^2 = -1$.*

(*iii*) *The i_3-conjugate of a bicomplex number $\xi = z_1 + i_2 z_2$ is denoted by $\xi^{'}$ and is defined as $\xi^{'} = \bar{z_1} - i_2 \bar{z_2}$ for all $z_1, z_2 \in C_1$; here $\bar{z_1}$ and $\bar{z_2}$ are the complex conjugates of z_1 and z_2, respectively, and $i_1^2 = i_2^2 = -1$.*

2.2. Partial order relation

Definition 2 [1]. *The i_1-partial order relation $\preceq_{i_1}$ on C_1 is defined as follows: for $z_1, z_2 \in C_1$, $z_1 \preceq_{i_1} z_2$ if and only if $\operatorname{Re}(z_1) \leq \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) \leq \operatorname{Im}(z_2)$.*

Definition 3. *Let $\xi_1, \xi_2 \in C_2$, $\xi_1 = z_1 + i_2 z_2$ and $\xi_2 = z_1^* + i_2 z_2^*$. The i_2-partial order relation $\preceq_{i_2}$ on C_2 is defined as follows: $\xi_1 \preceq_{i_2} \xi_2$ if and only if $z_1 \preceq_{i_1} z_1^*$ and $z_2 \preceq_{i_1} z_2^*$, i.e., $\xi \preceq_{i_2} \eta$ if one of the following conditions is satisfied:*

(*i*) $z_1 = z_1^*$ *and* $z_2 = z_2^*$;

(*ii*) $z_1 \prec_{i_1} z_1^*$ *and* $z_2 = z_2^*$;

(iii) $z_1 = z_1^*$ *and* $z_2 \prec_{i_1} z_2^*$;
(iv) $z_1 \prec_{i_1} z_1^*$ *and* $z_2 \prec_{i_1} z_2^*$.

In particular, we write $\xi \precnsim_{i_2} \eta$ *if* $\xi \preceq_{i_2} \eta$ *and* $\xi \neq \eta$, *i.e., if one of* (*ii*), (*iii*), *and* (*iv*) *is satisfied, and we write* $\xi \prec_{i_2} \eta$ *if only* (*iv*) *is satisfied.*

For every two bicomplex numbers $\xi, \eta \in C_2$, *we can verify the following*:
(1) $\xi \preceq_{i_2} \eta \implies \|\xi\|_{C_2} \leq \|\eta\|_{C_2}$,
(2) $\|\xi + \eta\|_{C_2} \leq \|\xi\|_{C_2} + \|\eta\|_{C_2}$.

Definition 4. *Let* $\xi_1, \xi_2 \in C_2$, *where*

$$\xi_1 = z_1 + i_2 z_2 = \mu_1 e_1 + \mu_2 e_2 \quad \textit{and} \quad \xi_2 = z_1^* + i_2 z_2^* = \mu_1^* e_1 + \mu_2^* e_2.$$

The Id*-partial order relation* $\preceq_{i_{Id}}$ *on* C_2 *is defined as follows*: $\xi_1 \preceq_{i_{Id}} \xi_2$ *if and only if* $\mu_1 \preceq_{i_1} \mu_1^*$ *and* $\mu_2 \preceq_{i_1} \mu_2^*$ *on* C_1, *i.e.,* $\xi \preceq_{i_{Id}} \eta$ *if one of the following conditions is satisfied*:
(i) $\mu_1 = \mu_1^*$ *and* $\mu_2 = \mu_2^*$;
(ii) $\mu_1 \prec_{i_1} \mu_1^*$ *and* $\mu_2 = \mu_2^*$;
(iii) $\mu_1 = \mu_1^*$ *and* $z_2 \prec_{i_1} z_2^*$;
(iv) $\mu_1 \prec_{i_1} z_1^*$ *and* $\mu_2 \prec_{i_1} \mu_2^*$.

In particular, we write $\xi \precnsim_{i_{Id}} \eta$ *if* $\xi \preceq_{i_{Id}} \eta$ *and* $\xi \neq \eta$, *i.e., one of* (*ii*), (*iii*), *and* (*iv*) *is satisfied, and we write* $\xi \prec_{i_{Id}} \eta$ *if only* (*iv*) *is satisfied.*

For every two bicomplex numbers $\xi, \eta \in C_2$, *we can verify the following*:

$$\xi \preceq_{i_{Id}} \eta \implies \|\xi\|_{C_2} \leq \|\eta\|_{C_2}.$$

Remark 1. For $\xi, \eta \in C_2$, the relation $\xi \prec_{i_2} \eta$ does not guarantee that $\xi \prec_{i_{Id}} \eta$. Similarly, $\xi \prec_{i_{Id}} \eta$ does not guarantee that $\xi \prec_{i_2} \eta$.

2.3. Bicomplex valued metric space

Choi et al. [4] defined a bicomplex valued metric space as follows.

Definition 5 [4]. *A function* $d : X \times X \to C_2$ *is a bicomplex valued metric on* $X \subseteq C_2$ *with respect to the* i_2*-partial order if it has the following properties for all* $x, y, z \in X$:
(i) $0 \preceq_{i_2} d(x, y)$;
(ii) $d(x, y) = 0$ *if and only if* $x = y$;
(iii) $d(x, y) = d(y, x)$;
(iv) $d(x, y) \preceq_{i_2} d(x, z) + d(z, y)$.

The pair (X, d) *is called a bicomplex valued metric space with respect to the* i_2*-partial order. It is denoted by* (X, d_{i_2}).

Definition 6. *A function* $d : X \times X \to C_2$ *is a bicomplex valued metric on* $X \subseteq C_2$ *with respect to the* i_{Id}*-partial order if it has the following properties for all* $x, y, z \in X$:
(i) $0 \preceq_{i_{Id}} d(x, y)$;
(ii) $d(x, y) = 0$;
(iii) $d(x, y) = d(y, x)$;
(iv) $d(x, y) \preceq_{i_{Id}} d(x, z) + d(z, y)$.

The pair (X, d) *is called a bicomplex valued metric space with respect to the* i_{Id}*-partial order. It is denoted by* $(X, d_{i_{Id}})$.

Definition 7. *A function $d^{\mathcal{H}} : X \times X \to \mathcal{H}$ is a hyperbolic valued (D-valued) metric on $X \subseteq C_2$ with respect to the i_2-partial order (the i_{Id}-partial order) if it has the following properties for all $x, y, z \in X$, respectively:*
(i) $0 \preceq_{i_2} (\preceq_{i_{Id}})\, d^{\mathcal{H}}(x, y)$;
(ii) $d^{\mathcal{H}}(x, y) = 0$ *if and only if* $x = y$;
(iii) $d^{\mathcal{H}}(x, y) = d^{\mathcal{H}}(y, x)$;
(iv) $d^{\mathcal{H}}(x, y) \preceq_{i_2} (\preceq_{i_{Id}})\, d^{\mathcal{H}}(x, z) + d^{\mathcal{H}}(z, y)$.

The pair $(X, d^{\mathcal{H}})$ is called a bicomplex valued metric space with respect to the i_2-partial order (the i_{Id}-partial order). The metric space $(X, d^{\mathcal{H}})$ with respect to the i_2-partial order is denoted by $(X, d^{\mathcal{H}}_{i_2})$, and the metric space $(X, d^{\mathcal{H}})$ with respect to the i_{Id}-partial order is denoted by $(X, d^{\mathcal{H}}_{i_{Id}})$.

2.4. Statistical convergence of a sequences of bicomplex numbers

The concept of statistical convergence depends on the notion of the natural density of the set of natural numbers.

Definition 8. *A subset E of $\mathbb{N}$ is said to have natural density $\delta(E)$ if*

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k),$$

where χ_E is the characteristic function on E.

Definition 9. *For two sequences (x_k) and (y_k), we say that $x_k = y_k$ for almost all k if*

$$\delta\big(\{k \in \mathbb{N} : x_k \neq y_k\}\big) = 0.$$

Definition 10. *A sequence of bicomplex number (ξ_k) is said to be statistically convergent to $\xi \in C_2$ with respect to the Euclidean norm on C_2 if, for all $\varepsilon > 0$,*

$$\delta\big(\{k \in N : \|\xi_k - \xi\|_{C_2} \geq \varepsilon\}\big) = 0.$$

We use the notation $\text{stat-lim}\, \xi_k = \xi$.

3. Statistically convergent and statistically Cauchy sequences in a bicomplex valued metric space with respect to the i_2-partial order

Definition 11. *Let (X, d_{i_2}) be a bicomplex valued metric space, and let (ξ_k) be a sequence in (X, d_{i_2}). The sequence (ξ_k) is said to be statistically convergent to $\xi \in X$ if, for all $0 \prec_{i_2} \varepsilon \in C_2$,*

$$\delta\big(\{k : d(\xi_k, \xi) \succeq_{i_2} \varepsilon\}\big) = 0.$$

We use the notation $\text{stat-lim}\, \xi_k = \xi$.

Definition 12. *Let (X, d_{i_2}) be a bicomplex valued metric space, and let (ξ_k) be a sequence in (X, d_{i_2}). We say that (ξ_k) is a statistically Cauchy sequence if, for all $0 \prec_{i_2} \varepsilon \in X$,*

$$\delta\big(\{k : d(\xi_k, \xi_m) \succeq_{i_2} \varepsilon\}\big) = 0.$$

Definition 13. *Let (X, d_h) be a D-valued metric space, and let (ξ_k) be a sequence in (X, d_h). The sequence (ξ_k) is said to be statistically convergent to $\xi \in X$ if, for all $0 \prec_{i_2} \varepsilon \in D$,*

$$\delta\big(\{k : d_h(\xi_k, \xi) \succeq_{i_2} \varepsilon\}\big) = 0.$$

4. Main results

Lemma 1. *If a sequence (ξ_k) is statistically convergent in a bicomplex valued metric space (X, d_{i_2}), then $(d(\xi_k, \xi))$ is statistically convergent to 0 with respect to Euclidean norm on C_2.*

P r o o f. Since (ξ_k) is statistically convergent in a bicomplex valued metric space (X, d_{i_2}), for all $\varepsilon \succ_{i_2} 0$, we have

$$\begin{aligned} \delta\big(\{k : d(\xi_k, \xi) \succeq_{i_2} \varepsilon\}\big) = 0 &\implies \delta\big(\{k : \|d(\xi_k, \xi)\|_{C_2} \geq \|\varepsilon\|_{C_2}\}\big) = 0 \\ &\implies \delta\big(\{k : \|d(\xi_k, \xi)\|_{C_2} \geq \varepsilon'\}\big) = 0, \end{aligned}$$

where $\varepsilon' = \|\varepsilon\|_{C_2} > 0$. Thus, the sequences of bicomplex numbers $(d(\xi_k, \xi))$ is statistically convergent to 0 with respect to the Euclidean norm on C_2, and hence stat-lim $d(\xi_k, \xi) = 0$. □

Lemma 2. *Let (X, d_{i_2}) be a bicomplex valued metric space, then the inequality*

$$d(\xi_1, \eta_1) - d(\xi_2, \eta_2) \preceq_{i_2} d(\xi_1, \xi_2) + d(\eta_1, \eta_2)$$

holds for all $\xi_1, \xi_2, \eta_1, \eta_2 \in C_2$.

P r o o f. By the triangle inequality, we have

$$\begin{aligned} d(\xi_1, \eta_1) \preceq_{i_2} d(\xi_1, \xi_2) + d(\xi_2, \eta_2) + d(\eta_2, \eta_1) &\implies d(\xi_1, \eta_1) - d(\xi_2, \eta_2) \preceq_{i_2} d(\xi_1, \xi_2) + d(\eta_2, \eta_1) \\ &\implies d(\xi_1, \eta_1) - d(\xi_2, \eta_2) \preceq_{i_2} d(\xi_1, \xi_2) + d(\eta_1, \eta_2). \end{aligned}$$

□

Theorem 1. *Let (X, d_{i_2}) be a bi-complex valued metric space, and if the sequences (ξ_k) and (η_k) are statistically convergent to ξ and η, respectively, in (X, d_{i_2}). Then the sequence $(d(\xi_k, \eta_k))$ is statistically convergent to $d(\xi, \eta)$ with respect to Euclidean norm in C_2.*

P r o o f.

$$\begin{aligned} &\{k : d(\xi_k, \eta_k) - d(\xi, \eta) \succeq_{i_2} \varepsilon\} \subseteq \{k : d(\xi_k, \xi) \succeq_{i_2} \varepsilon\} \cup \{k : d(\eta_k, y) \succeq_{i_2} \varepsilon\} \\ &\implies \delta(\{k : d(\xi_k, \eta_k) - d(\xi, \eta) \succeq_{i_2} \varepsilon\}) \leq \delta(\{k : d(\xi_k, \xi) \succeq_{i_2} \varepsilon\}) + \delta(\{k : d(\eta_k, \eta) \succeq_{i_2} \varepsilon\}) \\ &\implies \delta(\{k : d(\xi_k, \eta_k) - d(\xi, \eta) \succeq_{i_2} \varepsilon\}) = 0 \\ &\implies \delta(\{k : \|d(\xi_k, \eta_k) - d(\xi, \eta)\|_{C_2} \geq \|\varepsilon\|_{C_2}\}) = 0. \end{aligned}$$

□

Let us formulate the following theorem without proof.

Theorem 2. *Let (X, d_{i_2}) be a bicomplex valued metric space, and let $\xi, \eta \in X$. If (ξ_k) is a sequence in X statistically convergent to ξ and statistically convergent to η, then $\xi = \eta$.*

Lemma 3. *Consider a bicomplex valued metric space (X, d_{i_2}) on C_2. Suppose that*

$$d(\xi_k, \xi) = d_1(\xi_k, \xi) + i_1 d_2(\xi_k, \xi) + i_2 d_3(\xi_k, \xi) + i_1 i_2 d_4(\xi_k, \xi).$$

The sequence (ξ_k) is statistically convergent (statistically Cauchy) in (X, d_{i_2}) if and only if (x_k) is statistically convergent (statistically Cauchy) in the real valued metric spaces (X, d_j), $j = 1, 2, 3, 4$.

Lemma 4. *Consider a bicomplex valued metric space (X, d_{i_2}) on X. Suppose that*

$$d(\xi_k, \xi) = d_1(\xi_k, \xi) + i_1 d_2(\xi_k, \xi) + i_2 d_3(\xi_k, \xi) + i_1 i_2 d_4(\xi_k, \xi).$$

Then A sequence (ξ_k), where

$$\xi_k = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4, \quad x_1, x_2, x_3, x_4 \in C_0,$$

is statistically convergent (statistically Cauchy) in (X, d_{i_2}) if and only if (x_j) is statistically convergent (statistically Cauchy) in the real valued metric spaces (C_2, d_j), $j = 1, 2, 3, 4$.

Lemma 5. *Consider a D-valued metric space $(X, d^{\mathcal{H}}_{i_2})$ on X. Suppose that*

$$d^{\mathcal{H}}(\xi_k, \xi) = d^{\mathcal{H}}_1(\xi_k, \xi) e_1 + d^{\mathcal{H}}_2(\xi_k, \xi) e_2.$$

Then $(X, d^{\mathcal{H}}_1)$ and $(X, d^{\mathcal{H}}_2)$ are real valued metric spaces. A sequence (ξ_k) is statistically convergent (statistically Cauchy) in $(X, d^{\mathcal{H}})$ with respect to the i_2-partial order if and only if (ξ_k) is statistically convergent (statistically Cauchy) in the real valued metric spaces $(X, d^{\mathcal{H}}_j)$, $j = 1, 2$.

Lemma 6. *Consider a D-valued metric space $(\mathcal{H}, d^{\mathcal{H}}_{i_2})$ on $\mathcal{H}$. Suppose that*

$$d^{\mathcal{H}}(\xi_k, \xi) = d^{\mathcal{H}}_1(\xi_k, \xi) e_1 + d^{\mathcal{H}}_2(\xi_k, \xi) e_2.$$

Then $(C_2, d^{\mathcal{H}}_1)$ and $(C_2, d^{\mathcal{H}}_2)$ are real valued metric spaces. A sequence $(\xi_k) \in \mathcal{H}$, where

$$\xi_k = \mu_{1k} e_1 + \mu_{2k} e_2,$$

is statistically convergent (statistically Cauchy) in $(\mathcal{H}, d^{\mathcal{H}})$ with respect to the i_2-partial order if and only if (μ_{jk}) is statistically convergent (statistically Cauchy) in the real valued metric spaces $(C_2, d^{\mathcal{H}}_j)$, $j = 1, 2$.

Theorem 3. *If (ξ_k) and (η_k) are statistically convergent in a bicomplex valued metric space (X, d_{i_2}) and if*

$$\|d_1(\xi_k, \eta_k)\| \le \|d(\xi_k, \eta_k)\|$$

for all $k \in \mathbb{N}$, then $(d_1(\xi_k, \eta_k))$ is also statistically convergent with respect to the Euclidean norm in C_2.

P r o o f. Using Lemma 2, for all $\varepsilon \succ_{i_2} 0$ and $k, m \ge n_0$, we obtain

$$\begin{gathered}
\{k : d_1(\xi_k, \eta_k) - d_1(\xi_m, \eta_m) \succeq_{i_2} \varepsilon\} \subseteq \{k : d_1(\xi_k, \xi_m) \succeq_{i_2} \varepsilon\} \cup \{k : d_1(\eta_k, \eta_m) \succeq_{i_2} \varepsilon\} \\
\Longrightarrow \{k : \|d_1(\xi_k, \eta_k) - d_1(\xi_m, \eta_m)\|_{C_2} \ge \|\varepsilon\|_{C_2}\} \\
\le \{k : \|d_1(\xi_k, \xi_m)\|_{C_2} \ge \|\varepsilon\|_{C_2}\} \cup \{k : \|d_1(\eta_k, \eta_m)\|_{C_2} \ge \|\varepsilon\|_{C_2}\} \\
\Longrightarrow \delta(\{k : \|d_1(\xi_k, \eta_k) - d_1(\xi_m, \eta_m)\|_{C_2} \ge \|\varepsilon\|_{C_2}\}) \\
\le \{k : \|d(\xi_k, \xi_m)\|_{C_2} \ge \|\varepsilon\|_{C_2}\} \cup \{k : \|d(\eta_k, \eta_m)\|_{C_2} \ge \|\varepsilon\|_{C_2}\} \\
\Longrightarrow \delta(\{k : \|d_1(\xi_k, \eta_k) - d_1(\xi_m, \eta_m)\|_{C_2} \ge \|\varepsilon\|_{C_2}\}) = 0.
\end{gathered}$$

Thus, $(d_1(\xi_k, \eta_k))$ is a Cauchy sequence of bicomplex numbers and, hence, $(d_1(\xi_k, \eta_k))$ is statistically convergent with respect to the Euclidean norm. □

Lemma 7. $\delta\big(\{k : d(\xi_k, \xi) \succeq_{i_2} \varepsilon\}\big) = 0$ *implies that* $\delta\big(\{k : d(\xi_k, \xi_m) \succeq_{i_2} \varepsilon\}\big) = 0$.

P r o o f.

$$\delta\big(\{k : d(\xi_k, \xi) \succeq_{i_2} \varepsilon\}\big) = 0 \implies \delta\big(\{k : d(\xi_k, \xi) \prec_{i_2} \varepsilon/2\}\big) = 1 \implies \delta\big(\{k : d(\xi_m, \xi) \prec_{i_2} \varepsilon/2\}\big) = 1.$$

We have

$$\begin{aligned}\{k : d(\xi_k, \xi) \prec_{i_2} \varepsilon/2\} \subseteq \{k : d(\xi_k, \xi_m) \prec_{i_2} \varepsilon\} &\implies \delta\big(\{k : d(\xi_k, \xi_m) \prec_{i_2} \varepsilon\}\big) = 1 \\ &\implies \delta\big(\{k : d(\xi_k, \xi_m) \succeq_{i_2} \varepsilon\}\big) = 0.\end{aligned}$$

□

Remark 2. The converse is generally not true. To justify this, consider the following example.

Example 1. Let

$$X = (0, 1 + i_1 + i_2 + i_1 i_2]$$

with the metric

$$d(\xi, \eta) = (1 + i_1 + i_2)\|\xi - \eta\|_{C_2}$$

for all $\xi, \eta \in X$.

Consider a sequence (ξ_k) in X defined as

$$\xi_k = \begin{cases} \dfrac{(1 + i_1 + i_2 + i_1 i_2)}{k} & \text{for} \quad k = i^2, \quad i \in \mathbb{N}; \\ \dfrac{(1 + i_1 + i_2 + i_1 i_2)}{k^2} & \text{otherwise.} \end{cases}$$

Then, we observe that (ξ_k) is a statistically Cauchy sequence but is not statistically convergent in X.

Lemma 8. *Let (X, d_{i_2}) be a complete bicomplex valued metric space, and let (ξ_k) be a sequence in X. Then the following properties are equivalent:*
(i) (ξ_k) is statistically convergent;
(ii) (ξ_k) is a statistically Cauchy sequence.

Theorem 4. *Assume that (ξ_k) is a sequence in a bicomplex valued metric space (X, d_{i_2}) and*

$$\delta\big(\{k : \sum_{i=1}^{k} d(\xi_i, \xi_{i+1}) \succeq_{i_2} \varepsilon\}\big) = 0.$$

Then (ξ_k) is a statistically Cauchy sequence in (X, d_{i_2}).

P r o o f. We have

$$\begin{aligned} &\delta\big(\{k : \sum_{i=1}^{k} d(\xi_i, \xi_{i+1}) \succeq_{i_2} \varepsilon\}\big) = 0 \\ \implies &\delta\big(\{k : \sum_{i=1}^{k} d_j(\xi_i, \xi_{i+1}) \geq \varepsilon_j\}\big) = 0, \quad j = 1, 2, 3, 4 \\ \implies &\delta\big(\{k : d_j(\xi_k, \xi_{k+1}) \geq \varepsilon_j\}\big) = 0, \quad j = 1, 2, 3, 4. \end{aligned}$$

Thus, (ξ_k) is a statistically Cauchy sequence in the real valued metric spaces (X, d_j), $j = 1, 2, 3, 4$. Hence, (ξ_k) is a statistically Cauchy sequence in the bicomplex valued metric space (X, d_{i_2}). □

Theorem 5. *Let* (ξ_k), *where*

$$\xi_k = z_{1k} + i_2 z_{2k},$$

be a sequence of bicomplex numbers in the bicomplex valued metric space (X, d_{i_2}). *Then the following properties are equivalent*:

(*i*) (ξ_k) *statistically converges to a point* $\xi = z_1 + i_2 z_2 \in X$;

(*ii*) (z_{1k}) *and* (z_{2k}) *statistically converge to* z_1 *and* z_2, *respectively*;

(*iii*) *there are sequences* (z_{1k}) *and* (z_{2k}) *such that* $z_{1k} = z'_{1k}$ *and* $z_{2k} = z'_{2k}$ *for almost all* k *and* (z'_{1k}) *and* (z'_{2k}) *converge to* z_1 *and* z_2, *respectively*;

(*iv*) *there is a bicomplex sequence convergent to* $\xi + i_2\bar{\xi}$, *where* $\bar{\xi}$ *is the* i_2*-conjugate of* ξ;

(*v*) *there are a statistically dense subsequence* (z_{1k_i}) *of* (z_{1k}) *and a statistically dense subsequence* (z_{2k_i}) *of* (z_{2k}) *such that* (z_{1k_i}) *and* (z_{2k_i}) *are convergent*;

(*vi*) *there are a statistically dense subsequence* (z_{1k_i}) *of* (z_{1k}) *and a statistically dense subsequence* (z_{2k_i}) *of* (z_{2k}) *such that* (z_{1k_i}) *and* (z_{2k_i}) *are statistically convergent.*

P r o o f. $(i) \implies (ii)$ The sequence (ξ_k) is statistically convergent to ξ. Then for every

$$0 \prec \varepsilon = \varepsilon_1 + i_2\varepsilon_2 \in C_2,$$

we have

$$\delta\big(\{k : d_{i_2}(\xi_k, \xi) \succeq_{i_2} \varepsilon\}\big) = \lim_{n\to\infty} \frac{1}{n} \big|\, \{k : d_{i_2}(\xi_k, \xi) \succeq_{i_2} \varepsilon\} \,\big| = 0.$$

There are two following cases.

Case 1. Consider

$$d_{i_2}(\xi_k, \xi) = |z_{1k} - z_1| + i_2|z_{2k} - z_2|$$

or

$$d_{i_2}(\xi_k, \xi) = d_1(z_{1k}, z_1) + i_2 d_1(z_{2k}, z_1),$$

where

$$d_1(z_k, z) = |z_k - z|,$$

corresponds to a real valued metric space on C_1 with the property

$$\big\{k : d_{i_2}(\xi_k, \xi) \succeq_{i_2} \varepsilon\big\} = \big\{k : |z_{1k} - z_1| + i_2|z_{2k} - z_2| \succeq_{i_2} (\varepsilon_1 + i_2\varepsilon_2)\big\}.$$

We have

$$\delta\big(\{k : |z_{1k} - z_1| \geq |\varepsilon_1|\}\big) \leq \delta\big(\{k : d_{i_2}(\xi_k, \xi) \succeq_{i_2} \varepsilon\}\big) = 0,$$

which implies

$$\delta\big(\{k : |z_{1k} - z_1| \geq |\varepsilon|\}\big) = 0.$$

Similarly,

$$\delta\big(\{k : |z_{2k} - z_2| \geq |\varepsilon|\}\big) = 0.$$

Hence, (z_{1n}) and (z_{2n}) are statistically convergent in real valued metric spaces on C_1.

Case 2. Consider

$$d_{i_2}(\xi_k, \xi) = (a_1 + i_2 a_2)\|\xi_k - \xi\|_{C_2},$$

where

$$0 \prec a_1, a_2 \in C_1(i_1),$$

or

$$d_{i_2}(\xi_k, \xi) = a_1 \|\xi_k - \xi\|_{C_2} + i_2 a_2 \|\xi_k - \xi\|_{C_2}$$

or

$$d_{i_2}(\xi_k, \xi) = a_1 d_1(\xi_k, \xi) + i_2 a_2 d_1(\xi_k, \xi),$$

where

$$d_1(\xi_k, \xi) = \|\xi_k - \xi\|_{C_2},$$

defines a real valued metric space on C_2. Then, (ξ_k) is statistically convergent in real valued metric space on C_2.

We have

$$\|\xi_k - \xi\|_{C_2} = \sqrt{(z_{1k} - z_1)^2 + (z_{2k} - z_2)^2} = d_2^2(z_{1k}, z_1) + d_2^2(z_{2k}, z_2),$$

and

$$|z_{1k} - z_1| \leq \sqrt{(z_{1k} - z_1)^2 + (z_{2k} - z_2)^2},$$

which implies that

$$\{k : |z_{1k} - z_1| \geq \varepsilon\} \subseteq \{k : d_1(\xi_k, \xi) \geq \varepsilon\}.$$

Hence, (z_{1k}) is statistically convergent in a related real valued metric space. Similarly, (z_{2k}) is statistically convergent in a real valued metric space.

(ii) $\implies$ *(iii)* The sequences (z_{1k}) and (z_{2k}) statistically converge to z_1 and z_2, respectively. Then, for every $0 < \varepsilon \in C_0$, we have

$$\delta\big(\{k : d(z_{1k}, z_1) \geq \varepsilon\}\big) = \lim_{n\to\infty} \frac{1}{n} \big|\{k : d(z_{1k}, z_1) \geq \varepsilon\}\big| = 0$$

and

$$\delta\big(\{k : d(z_{2k}, z_2) \geq \varepsilon\}\big) = \lim_{n\to\infty} \frac{1}{n} \big|\{k : d(z_{2k}, z_2) \geq \varepsilon\}\big| = 0.$$

Choose an increasing sequence of natural numbers (n_k) such that, for all $n > n_k$,

$$\frac{1}{n} \left|\left\{k : d(z_{1k}, z) \geq \frac{1}{2^k}\right\}\right| < \frac{1}{2^k}.$$

Define a sequence of complex numbers (w_{1k}) such that

$$w_{1k} = \begin{cases} z_{1k} & \text{if} \quad k \leq n_1; \\ z_{1k} & \text{if} \quad d(z_{1k}, z) \geq \dfrac{1}{2^k}; \\ z_1 & \text{otherwise.} \end{cases}$$

The sequence (w_{1k}) is convergent.

Now we have

$$\{k : z_{1k} = w_{1k}\} \supseteq \{k : d_{i_1}(z_{1k}, z_1) \prec_{i_1} \varepsilon\}.$$

Therefore, $z_{1k} = w_{1k}$ for almost all k. Similarly, $z_{2k} = w_{2k}$ for almost all k.

(iii) $\implies$ *(iv)* The sequences (z_{1k}) and (z_{2k}) converge to z_1 and z_2, respectively. Then the bicomplex sequence $(\xi_k) = (z_{1k} + i_2 z_{2k})$ converges to $\xi = z_1 + i_2 z_2$ and the bicomplex sequence $(\zeta_k) = (z_{2k} + i_2 z_{1k})$ converges to $z_2 + i_2 z_1$, i.e., to $i_2 \bar{\xi}$. Hence, there exists a bicomplex sequence $(\eta_k) = (\xi_k + \zeta_k)$ converging to $\xi + i_2 \bar{\xi}$.

(iv) $\implies$ *(v)* Consider a bicomplex sequence (η_k) converging to

$$\xi + i_2 \bar{\xi} = (z_1 + z_2) + i_2(z_1 + z_2).$$

Let
$$\eta_k = z'_{1k} + i_2 z'_{2k}.$$
There exist $\left(z''_{1k}\right)$ and $\left(z''_{2k}\right)$ such that
$$z'_{1k} = z_{1k} + z''_{1k} \quad \text{and} \quad z'_{2k} = z_{2k} + z''_{2k},$$
and as (z_{1k}) and (z_{2k}) are convergent we have
$$\lim_{k\to\infty} z''_{1k} = z_2 \quad \text{and} \quad \lim_{k\to\infty} z''_{2k} = z_1.$$
Let
$$M_1 = \left\{k : d_{i_1}\left(z''_{1k}, z_2\right) \succeq_{i_1} \varepsilon\right\} \quad \text{and} \quad M_2 = \left\{k : d_{i_1}\left(z''_{2k}, z_1\right) \succeq_{i_1} \varepsilon\right\}.$$
Let
$$K_1 = \mathbb{N} - M_1 = \left\{k_i : k_i < k_{i+1}\right\} \quad \text{and} \quad K_2 = \mathbb{N} - M_2 = \left\{k'_i : k'_i < k'_{i+1}\right\}.$$
Then $\delta(K_1) = 1$ and $\delta(K_2) = 1$. Thus, we have
$$\lim_{i\to\infty} z_{1k_i} = z_1 \quad \text{and} \quad \lim_{i\to\infty} z_{2k_i} = z_2.$$

$(v) \implies (vi)$ A subsequence (z_{1k_i}) of the sequence (z_{1k}) is convergent, hence, it is statistically convergent. Similarly, (z_{2k_i}) is statistically convergent.

$(vi) \implies (i)$ Let there exist
$$K_1 = \left\{k_i : k_i < k_{i+1}\right\} \subset \mathbb{N} \quad \text{and} \quad K_2 = \left\{k'_i : k'_i < k'_{i+1}\right\} \subset \mathbb{N}$$
such that
$$\lim_{i\to\infty} z_{1k_i} = z_1 \quad \text{and} \quad \lim_{i\to\infty} z_{2k_i} = z_2.$$
Then, for all
$$0 \prec_{i_2} \varepsilon = \varepsilon_1 + i_2\varepsilon_2 \in C_2,$$
we have
$$\begin{aligned} &\left\{k : d_{i_2}(\xi_k, \xi) \succeq_{i_2} \varepsilon\right\} \subseteq \left\{k : d_{i_1}(z_{1k}) \succeq_{i_1} \varepsilon_1\right\} \cup \left\{k : d_{i_1}(z_{2k}) \succeq_{i_1} \varepsilon_2\right\} \\ &\subseteq K_1^c \cup \left\{k \in K_1 : d_{i_1}(z_{1k}, z_1) \succeq_{i_1} \varepsilon_1\right\} \cup K_2^c \cup \left\{k \in K_2 : d_{i_1}(z_{2k}, z_2) \succeq_{i_1} \varepsilon_2\right\}. \end{aligned}$$
Therefore, (ξ_k) is statistically convergent. □

5. Statistically convergent and statistically Cauchy sequences in a bicomplex valued metric space with respect to the i_{Id}-partial order

Definition 14. *Let $(X, d_{i_{Id}})$ be a bicomplex valued metric space, and let (ξ_k) be a sequence in (X, d). The sequence (ξ_k) is said to be statistically convergent to $\xi \in X$ if, for all $0 \prec_{i_{Id}} \varepsilon \in C_2$,*
$$\delta\left(\{k : d(\xi_k, \xi) \succeq_{i_{Id}} \varepsilon\}\right) = 0.$$
We use the notation stat-lim $\xi_k = \xi$.

Definition 15. *Let $(X, d_{i_{Id}})$ be a bicomplex valued metric space, and let (ξ_k) be a sequence in $(X, d_{i_{Id}})$. We say that (ξ_k) is a statistically Cauchy sequence if, for all $0 \prec_{i_{Id}} \varepsilon \in C_2$,*
$$\delta\left(\{k : d(\xi_k, \xi_m) \succeq_{i_{Id}} \varepsilon\}\right) = 0.$$

Example 2. Consider a metric $d : C_2 \times C_2 \to C_2$ on C_2 defined as

$$d(\xi,\eta) = \big[(5+8i_1)e_1 + (7+2i_1)e_2\big]\|\xi-\eta\|_{C_2} \quad \forall\, \xi,\eta \in C_2.$$

Consider a sequence (ξ_k) in C_2 defined as

$$\xi_k = \begin{cases} 1+i_1+i_2+i_1i_2 & \text{for} \quad k=i^2, \quad i\in\mathbb{N}; \\ 1/2022 & \text{otherwise.} \end{cases}$$

Then we observe that (ξ_k) is statistically convergent in the metric space $(C_2, d_{i_{Id}})$.

Lemma 9. *Consider a bicomplex valued metric space $(X, d_{i_{Id}})$ on X. Suppose that*

$$d(\xi_k,\xi) = d'_1(\xi_k,\xi)e_1 + d'_2(\xi_k,\xi)e_2.$$

Then (X, d'_1) and (X, d'_1) are complex valued metric spaces. A sequence (ξ_k) is statistically convergent (statistically Cauchy) in $(X, d_{i_{Id}})$ if and only if (ξ_k) is a statistically convergent (statistically Cauchy) sequence in the complex valued metric spaces (X, d'_j), $j = 1, 2$.

Lemma 10. *Consider a bicomplex valued metric space $(X, d_{i_{Id}})$ on X. Suppose that*

$$d(\xi_k,\xi) = d'_1(\xi_k,\xi)e_1 + d'_2(\xi_k,\xi)e_2.$$

Then (X, d'_1) and (X, d'_1) are complex valued metric spaces. A sequence (ξ_k), where

$$\xi_k = \mu_{1k}e_1 + \mu_{2k}e_2,$$

is a statistically convergent (statistically Cauchy) sequence in $(X, d_{i_{Id}})$ if and only if (μ_{jk}) are statistically convergent (statistically Cauchy) sequences in the complex valued metric spaces (X, d'_j), $j = 1, 2$.

We formulate the following theorem without proof.

Theorem 6. *Let (ξ_k), where*

$$\xi_k = \mu_{1k}e_1 + \mu_{2k}e_2,$$

be a sequence of bicomplex numbers in the bicomplex valued metric space $(X, d_{i_{Id}})$. Then the following statements are equivalent:

(i) (ξ_k) *statistically converges to a point* $\xi = \mu_1e_1 + \mu_2e_2 \in X$;

(ii) (μ_{1k}) *and* (μ_{2k}) *statistically converge to* μ_1 *and* μ_2*, respectively;*

(iii) there are sequences (μ_{1k}) *and* (μ_{2k}) *such that* $\mu_{1k} = \mu'_{1k}$ *and* $\mu_{2k} = \mu'_{2k}$ *for almost all* k*, and* (μ'_{kn}) *and* (μ'_{2k}) *converge to* μ_1 *and* μ_2*, respectively;*

(iv) there is a bicomplex sequence converging to $\mu_1 + \mu_2 - (i_2-1)(\mu_1e_2 + \mu_2e_1)$;

(v) there are a statistically dense subsequence (μ_{1k_i}) *of* (μ_{1k}) *and a statistically dense subsequence* (μ_{2k_i}) *of* (μ_{2k}) *such that* (μ_{1k_i}) *and* (μ_{2k_i}) *are convergent;*

(vi) there are a statistically dense subsequence (μ_{1k_i}) *of* (μ_{1k}) *and a statistically dense subsequence* (μ_{2k_i}) *of* (μ_{2k}) *such that* (μ_{1k_i}) *and* (μ_{2k_i}) *are statistically convergent.*

Theorem 7. *$(X, d_{i_{Id}})$ is complete if and only if (X, d') and (X, d'') are complete metric spaces in C_1, where*

$$d(\xi, \eta) = d'(\xi, \eta)e_1 + d''(\xi, \eta))e_2.$$

P r o o f. Let $(X, d_{i_{Id}})$ be a complete metric space, and let $\xi = (\xi_k)$ be a Cauchy sequence in (X, d'). Therefore, for all $0 \prec_{i_2} \varepsilon' \in C_1$, there exists $k_0 \in \mathbb{N}$ such that

$$d'(\xi_k, \xi_m) \prec_{i_1} \varepsilon' \quad \forall k, m \geq k_0.$$

Consider

$$d(\xi_k, \xi_m) = d'(\xi_k, \xi_m)e_1 + 0 \cdot e_2 \in C_2 \quad \text{and} \quad \varepsilon = \varepsilon' e_1 + 0 \cdot e_2 \in C_2.$$

Then

$$d(\xi_k, \xi_m) = d'(\xi_k, \xi_m)e_1 + 0 \cdot e_2 \prec_{i_{Id}} e_1\varepsilon + 0 \cdot e_2.$$

This implies that (ξ_k) is a Cauchy sequence in $(X, d_{i_{Id}})$. Therefore, by the completeness of $(X, d_{i_{Id}})$, there exists ξ in $(X, d_{i_{Id}})$ such that $\xi_k \to \xi$ as $n \to \infty$ in $(X, d_{i_{Id}})$. We need to show that $\xi_k \to \xi'$ as $n \to \infty$ in (X, d') and $x'' = 0$.

Now, $\xi_k \to \xi$ as $n \to \infty$ in $(X, d_{i_{Id}})$, therefore, there exists a natural number k such that

$$\begin{aligned} d(\xi_k, \xi) \prec_{i_{Id}} \varepsilon \quad &\text{for all} \quad n > k \\ \Longrightarrow d'(\xi_k, \xi)e_1 + 0 \cdot e_2 \prec_{i_{Id}} \varepsilon' e_1 + 0 \cdot e_2 \quad &\text{for all} \quad n > k \\ \Longrightarrow d'(\xi_k, \xi) \prec_{i_1} \varepsilon' \quad &\text{for all} \quad n > k. \end{aligned}$$

Similarly, $d''(\xi_k, \xi) \prec_{i_1} \varepsilon'$ for all $n > k$. Hence, (X, d') and (X, d'') are complete metric spaces in C_1.

Conversely, let (X, d') and (X, d'') be complete metric spaces in $C(i_1)$.

Let (ξ_k) be a Cauchy sequence in $(X, d_{i_{Id}})$. Therefore, for $\varepsilon \succ_{i_{Id}} 0$, there exists $k_0 \in \mathbb{N}$ such that $\forall m, k \geq k_0$

$$\begin{aligned} d_{i_{Id}}(\xi_k, \xi_m) \prec_{i_{Id}} \varepsilon &\Longrightarrow d'(\xi_k, \xi_m)e_1 + d''(\xi_k, \xi_m)e_2 \prec_{i_{Id}} \varepsilon' e_1 + \varepsilon'' e_2 \\ &\Longrightarrow d'(\xi_k, \xi_m) \prec_{i_1} \varepsilon' \quad \text{and} \quad d''(\xi_k, \xi_m) \prec_{i_1} \varepsilon''. \end{aligned}$$

Therefore, (ξ_k) is a Cauchy sequence in (X, d') and (X, d'').

Since (X, d') and (X, d'') are complete, there exist $k_0', k_0'' \in \mathbb{N}$ such that

$$d'(\xi_k, \xi) \prec_{i_1} \varepsilon' \quad \text{for all} \quad k > k_0' \quad \text{and} \quad d''(\xi_k, \xi) \prec_{i_1} \varepsilon'' \quad \text{for all} \quad k > k_0''.$$

Now, for all $k > k_1 = \max\{k_0', k_0''\}$,

$$\begin{aligned} &d(\xi_k, \xi) = d'(\xi_k, \xi)e_1 + d''(\xi_k, \xi)e_2 \prec_{i_{Id}} \varepsilon' e_1 + \varepsilon'' e_2 \\ &\Longrightarrow d(\xi_k, \xi) \prec_{i_{Id}} \varepsilon, \quad \text{where} \quad \varepsilon = \varepsilon' e_1 + \varepsilon'' e_2 \in C_2. \end{aligned}$$

Hence, $(X, d_{i_{Id}})$ is a complete metric space. □

We formulate the following theorem without proof.

Theorem 8. *Let $(C_2, d_{i_{Id}})$ be a bicomplex valued metric space. Then the class b_∞^* of all bounded statistically convergent sequences of bicomplex numbers over C_2 is complete.*

Theorem 9. *The metric spaces (X, d_{i_2}) and $(X, d_{i_{Id}})$ are not comparable.*

P r o o f. Consider a metric $d : X \times X \to C_2$ on X defined as

$$d(\xi, \eta) = (5 + 6i_1 + 7i_2 + i_1 i_2)\|\xi - \eta\|_{C_2} \quad \forall\, \xi, \eta \in X.$$

Then, all properties of metric space with respect to the i_2-partial order holds and hence (X, d_{i_2}) is a metric space. Now we have

$$\begin{aligned} d(\xi, \eta) &= \big(5 + 6i_1 + 7i_2 + i_1 i_2\big)\|\xi - \eta\|_{C_2} \\ &= \big[(-2 + 5i_1)e_1 + (12 + 7i_1)e_2\big]\|\xi - \eta\|_{C_2} \quad \forall\, \xi, \eta \in X. \end{aligned}$$

Then the property $d(\xi, \eta) \succ 0$ with respect to the Id-partial order does not hold. Therefore, $(X, d_{i_{Id}})$ is not a metric space.

Next, consider a metric $d : X \times X \to C_2$ on X defined as

$$d(\xi, \eta) = \big[(5 + 8i_1)e_1 + (7 + 2i_1)e_2\big]\|\xi - \eta\|_{C_2} \quad \forall\, \xi, \eta \in X.$$

Then all properties of metric space with respect to the Id-partial order hold and hence $(X, d_{i_{Id}})$ is a metric space. Now we have

$$\begin{aligned} d(\xi, \eta) &= \big[(5 + 8i_1)e_1 + (7 + 2i_1)e_2\big]\|\xi - \eta\|_{C_2} \\ &= (6 + 5i_1 - 3i_2 - i_1 i_2)\|\xi - \eta\|_{C_2} \quad \forall\, \xi, \eta \in X. \end{aligned}$$

Then the property $d(\xi, \eta) \succ 0$ with respect to the i_2-partial order does not hold. Therefore, (X, d_{i_2}) is not a metric space. $\square$

6. Complete bicomplex metric space

Definition 16. *A bicomplex valued metric space on C_2 is said to be a complete bicomplex metric space if every Cauchy sequence of bicomplex numbers in C_2 converges to a point in C_2.*

Theorem 10. *Let (C_2, d_{i_2}) be a bicomplex valued metric space. Then the class b^*_∞ of all bounded statistically convergent sequences of bicomplex numbers over C_2 is complete.*

P r o o f. Let (ξ_k) be a Cauchy sequence of bicomplex numbers in b^*_∞. For a given $0 \prec_{i_2} \varepsilon \in C_2$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_k d(\xi_k^m, \eta_k^n) \prec_{i_2} \varepsilon \quad \forall\, m, n \geq n_0.$$

Then, for every fixed value of k,

$$d(\xi_k^m, \eta_k^n) \prec_{i_2} \frac{\varepsilon}{3} \quad \text{for all } m, n \geq n_0. \tag{6.1}$$

Then (ξ_k^j) is a bicomplex Cauchy sequence in (C_2, d_{i_2}). Since (C_2, d_{i_2}) is a complete bicomplex metric space, (ξ_k^i) converges to $\xi \in C_2$ for all $k \in \mathbb{N}$.

Let

$$\lim_{k\to\infty} \xi_k^m = \xi.$$

Let (ξ_k^j) statistically converge to $\eta^m \in X$ for all j. Then

$$\delta\Big(\Big\{k \in \mathbb{N} : d(\xi_k^j, \eta^j) \prec_{i_2} \frac{\varepsilon}{3}\Big\}\Big) = 1.$$

Let

$$A_j = \left\{ k \in \mathbb{N} : d(\xi_k^j, \eta^j) \prec_{i_2} \frac{\varepsilon}{3} \right\}. \tag{6.2}$$

Let n_0 be chosen such that for $k \in A_j \cap A_r$ for all $j, r \geq n_0$. Now,

$$\begin{aligned} d(\eta^j, \eta^r) &\prec_{i_2} d(\xi_k^j, \xi_k^r) + d(\xi_k^r, \eta^r) + d(\xi_k^j, \eta^j) \\ \implies d(\eta^j, \eta^r) &\prec_{i_2} \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \text{[using (6.1) and (6.2)]} \\ \implies d(\eta^j, \eta^r) &\prec_{i_2} \varepsilon. \end{aligned}$$

Hence, (η^j) is a Cauchy sequence in (C_2, d_{i_2}), which is complete. Let

$$\lim_{j \to \infty} \eta^j = \eta.$$

Now,

$$d(\xi_k, \eta) \prec_{i_2} d(\xi_k^j, \xi_k) + d(\eta^j, \eta) + d(\xi_k^j, \eta^j) \prec_{i_2} \varepsilon,$$

as $\delta(A_j) = 1$ implies that,

$$\delta\big(\{k : d(\xi_k, \eta) \prec_{i_2} \varepsilon\} \big) = 1.$$

Hence, b_∞^* is a complete bicomplex metric space. This completes the proof. □

7. Conclusion

In this paper, we have studied the statistical convergence in bicomplex valued metric spaces. This is the first paper on this topic and is expected to attract researchers for further investigations and applications.

REFERENCES

1. Azam A., Fisher B. and Khan M. Common fixed point theorems in complex valued metric spaces. *Numer. Funct. Anal. Optim.*, 2011. Vol. 32, No. 3. P. 243–253. DOI: 10.1080/01630563.2011.533046
2. Buck R. C. Generalized asymptotic density. *Amer. J. Math.*, 1953. Vol. 75, No. 2. P. 335–346. DOI: 10.2307/2372456
3. Beg I., Datta S. K., Pal D. Fixed point in bicomplex valued metric spaces. *Int. J. Nonlinear Anal. Appl.*, 2021. Vol. 12, No. 2. P. 717–727. DOI: 10.22075/IJNAA.2019.19003.2049
4. Choi J., Datta S. K., Biswas T., Islam Md N. Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. *Honam Mathematical J.*, 2017. Vol. 39, No. 1. P. 115–126. DOI: 10.5831/HMJ.2017.39.1.115
5. Das N. R., Dey R., Tripathy B. C. Statistically convergent and statistically Cauchy sequence in a cone metric space. *TWMS J. Pure Appl. Math.*, 2014. Vol. 5, No. 1. P. 59–65.
6. Fast H. Sur la convergence statistique. *Colloq. Math.*, 1951. Vol. 2, No. 3–4. P. 241–244. (in French)
7. Fridy J. A. On statistical convergence. *Analysis*, 1985. Vol. 5, No. 4. P. 301–313. DOI: 10.1524/anly.1985.5.4.301
8. Rath D., Tripathy B. C. Matrix maps on sequence spaces associated with sets of integers. *Indian J. Pure Appl. Math.*, 1996. Vol. 27, No. 2. P. 197–206.
9. Rochon D., Shapiro M. On algebraic properties of bicomplex and hyperbolic numbers. *An. Univ. Oradea Fasc. Mat.*, 2004. Vol. 11. P. 71–110.
10. Sager N., Sağir B. On completeness of some bicomplex sequence spaces. *Palestine J. Math.*, 2020. Vol. 9, No. 2. P. 891–902.
11. Šalát T. On statistically convergent sequences of real numbers. *Math. Slovaca*, 1980. Vol. 30, No. 2. P. 139–150.

12. Schoenberg I. J. The integrability of certain functions and related summability methods. *Amer. Math. Monthly*, 1959. Vol. 66, No. 5. P. 361–375. DOI: 10.2307/2308747
13. Segre C. Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici. *Math. Ann.*, 1892. Vol. 40. P. 413–467. DOI: 10.1007/BF01443559 (in Italiano)
14. Singh S. *A Study of Bicomplex Space with a Topological View Point.* Thesis. Punjab: Lovely Professional University, 2018. 120 p.
15. Srivastava R. K., Srivastava N. K. On a class of entire bicomplex sequences. *South East Asian J. Math. Math. Sci.*, 2007. Vol. 5, No. 3. p. 47–68.
16. Tripathy B. C., Nath P. K. Statistical convergence of complex uncertain sequences. *New Math. Nat. Comput.*, 2017. Vol. 13, No. 3. P. 359–374. DOI: 10.1142/S1793005717500090
17. Tripathy B. C. Matrix transformations between some classes of sequences. *J. Math. Anal. Appl.*, 1997. Vol. 206, No. 2. P. 448–450. DOI: 10.1006/jmaa.1997.5236
18. Tripathy B. C., Sen M. On generalized statistically convergent sequences. *Indian J. Pure Appl. Math.*, 2001. Vol. 32, No. 11. P. 1689–1694.
19. Tripathy B. C. On generalized difference paranormed statistically convergent sequences. *Indian J. Pure Appl. Math.*, 2004. Vol. 35, No. 5. P. 655–663.
20. Wagh M. A. On certain spaces of bicomplex sequences. *Inter. J. Phy. Chem. Math. Fundam.*, 2014. Vol. 7, No. 1. P. 1–6.