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COMBINED ALGORITHMS FOR CONSTRUCTING A SOLUTION TO THE TIME-OPTIMAL PROBLEM IN THREE-DIMENSIONAL SPACE BASED ON THE SELECTION OF EXTREME POINTS OF THE SCATTERING SURFACE[1]

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Abstract: A class of time-optimal control problems in three-dimensional space with a spherical velocity vector is considered. A smooth regular curve Γ is chosen as the target set. We distinguish pseudo-vertices that are characteristic points on Γ and responsible for the appearance of a singularity in the function of the optimal result. We reveal analytical relationships between pseudo-vertices and extreme points of a singular set belonging to the family of bisectors. The found analytical representation for the extreme points of the bisector is taken as the basis for numerical algorithms for constructing a singular set. The effectiveness of the developed approach for solving non-smooth dynamic problems is illustrated by an example of numerical-analytical construction of resolving structures for the time-optimal control problem.

Keywords: Time-optimal problem, Dispersing surface, Bisector, Pseudo-vertex, Extreme point, Curvature, Singular set, Frenet–Serret frame (TNB frame).

1. Introduction

This study continues the series of works by the authors on the development of methods and algorithms for constructing solutions to time-optimal control problems with a constant velocity vector and various geometry of target sets [6, 19]. Previously accumulated experience in solving plane problems [13] was transferred to three-dimensional space [3, 14], expanded, and supplemented with new methods and constructions. In this paper, the authors consider a time-optimal control problem in which a sufficiently smooth regular spatial curve is chosen as the target set. The optimal result function is not differentiable over the entire domain of consideration [2]. A combined approach is applied in its construction, which combines analytical methods for identifying the features of the solution of the problem and numerical algorithms for constructing the solution as a whole. To find

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singularities of the optimal result function, elements of differential geometry are used, in particular, the moving Frenet frame and the main invariants of space curves [9, 10]. Also, a significant role in the constructions is played by the angular characteristic of the point's nonconvexity with respect to the target set and its measure of nonconvexity [16]. The measure of the non-convexity of a set determines the nature of the breaks in the wave fronts generated by this set [1]. The wave fronts in the problem under consideration coincide with the level surfaces of the optimal result function [4]. The key element in constructing a solution is the selection of a singular set — the bisector of the target set [7]. In the problem under consideration, the bisector generally consists of the union of two-dimensional, one-dimensional, and zero-dimensional manifolds [11, 12]. The simulation of the non-smooth solution of the problem was carried out with the help of modernized computational procedures, previously created for solving flat problems of time-optimal control [18]. The developed procedures can be used in constructing generalized solutions of first-order partial differential equations [15], as well as in theoretical mechanics, geometric optics, seismology, and economics [5].

2. Problem statement

The paper is devoted to the study of a time-optimal problem for a 3D system consisting of a single point with the speed limited as follows:

$$\dot{\mathbf{x}} \in U(\mathbf{0}, 1) \subset \mathbb{R}^3, \tag{2.1}$$

where $U(\mathbf{c}, r)$ is a ball in $\mathbb{R}^3$ centered at a point $\mathbf{c}$ of radius $r > 0$,

$$\mathbf{x} = \mathbf{x}(\tau) \triangleq (x(\tau), y(\tau), z(\tau)), \quad \dot{\mathbf{x}} = \frac{d\mathbf{x}}{d\tau},$$

and τ is a scalar interpreted as time. For an arbitrary point $\mathbf{x}$, the optimal trajectory is a line segment connecting it to the nearest point in the Euclidean metric of the target closed set $A \subset \mathbb{R}^3$. The optimal result function [17] is

$$u(\mathbf{x}) = \rho(\mathbf{x}, A) \triangleq \min_{\mathbf{a} \in A} \|\mathbf{x} - \mathbf{a}\|.$$

The time-optional problem under consideration is tightly connected with the Hamilton–Jacobi differential equations

$$\min_{(v_1, v_2, v_3) \in U(\mathbf{0}, 1)} \left(v_1 \frac{\partial u}{\partial x} + v_2 \frac{\partial u}{\partial y} + v_3 \frac{\partial u}{\partial z} \right) + 1 = 0 \tag{2.2}$$

and Eikonal equations

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1 \tag{2.3}$$

with a boundary condition

$$u|_{\partial A} = 0, \tag{2.4}$$

where ∂A is the boundary of A.

The restriction of the optimal result function $u = u(x, y, z)$ to the closure $\mathrm{cl}(\mathbb{R}^3 \setminus A)$ of the set $\mathbb{R}^3 \setminus A$ coincides with the generalized (minimax) solution of the Dirichlet problem [15] for equation (2.2) with a boundary condition (2.4). A more detailed proof for an arbitrary finite-dimensional Euclidean space is given in [6]. The fundamental (generalized) solution $u_k(\mathbf{x})$ of the Dirichlet problem for equation (2.3) with boundary condition (2.4) (introduced by S.N. Kruzhkov [5]) is equal to the function $u(x, y, z)$ on $\mathbb{R}^3 \setminus A$ in absolute value but has the opposite sign:

$$u_k(x, y) = -\rho\big((x, y, z), A\big).$$

It should be noted that equation (2.3) is used to describe light propagation in a homogeneous medium, provided that the speed is normalized and reduced to 1. The wave front at the time point $\tau > 0$ coincides with the level surface

$$\Phi(\tau) \triangleq \{\mathbf{x} \in \mathbb{R}^3 \colon u(\mathbf{x}) = \tau\}$$

of the optimal result function $u(\mathbf{x})$. In the whole space $\mathbb{R}^3$, the function $u(\mathbf{x})$ satisfies the Lipschitz condition with the constant $L = 1$.

3. Basic notation and definitions

Let $A \subset \mathbb{R}^3$ be a closed set in $\mathbb{R}^3$. We denote by $\Omega_A(\mathbf{x})$ the union of all points closest to $\mathbf{x}$ in the set A.

Definition 1 [19]. *A set*

$$L(A) \triangleq \left\{\mathbf{x} \in \mathbb{R}^3 \colon \operatorname{card} \Omega_A(\mathbf{x}) > 1\right\}$$

is called a bisector of a closed non-empty set A.

Here, $\operatorname{card} \Omega_A(\mathbf{x})$ is the cardinality of the set $\Omega_A(\mathbf{x})$.

The bisector is a specific case of a symmetric set on which the wave front loses its smoothness [1]. In English academic sources, similar sets are termed as “conflict set” [13], “symmetry set”, and “medial axe” [3]. Their geometric properties in 3D space were studied, for example, in [14]. Some topological properties of non-smooth wave front sets in Euclidean spaces of small dimensions (2 to 6) were investigated by V. D. Sedykh in [11, 12].

According to control theory, $L(A)$ is classified as a dispersing surface [4, ex. 6.10.1] in the time-optimal problem for dynamic systems (2.1). More than one optimal trajectory directed differently to the surface, e.g., line segments $[\mathbf{x}, \mathbf{y}_i]$, $i = \overline{1, k}$, where $\mathbf{y}_i \in \Omega_A(\mathbf{x})$, $k = \operatorname{card} \Omega_A(\mathbf{x})$, originates from each of its points. This determines that the optimal result function $u(\mathbf{x})$ is non-differentiable on the set $L(A)$. It should be mentioned that, for $u(\mathbf{x})$ as a function of the Euclidean distance, the superdifferential $D^+u(\mathbf{x})$ is defined at points $\mathbf{x} \in L(A)$, for more details see [2, Ch. II, Sect. S 8]. The value $D^+u(\mathbf{x})$ is used in [6] to prove that the function restriction to the set $\mathbb{R}^3 \setminus A$ is a generalized solution of the Hamilton–Jacobi equation (2.2).

Definition 2 [7]. *Non-coinciding points* $\mathbf{y}_i^- \in A$ *and* $\mathbf{y}_i^+ \in A$ *are called quasi-symmetric if*

$$\exists \mathbf{x} \in L(A) \colon \left\{\mathbf{y}_i^-, \mathbf{y}_i^+\right\} \subseteq \Omega_A(\mathbf{x}).$$

In this case, the point $\mathbf{x}$ *is called generated by the pair of points* $\mathbf{y}_i^-$ *and* $\mathbf{y}_i^+$.

Definition 3 [19]. *The point* $\mathbf{y}_0$ *is called a pseudo-vertex of the set A if there exists a sequence of pairs of quasi-symmetric points* $\left\{\mathbf{y}_i^-, \mathbf{y}_i^+\right\}_{i=1}^{\infty} \subset A$ *and a sequence of points* $\mathbf{x}_i \subset L(A)$, *for which the following conditions hold:*

$$\forall i \in \mathbb{N} \ \left\{\mathbf{y}_i^-,\ \mathbf{y}_i^+\right\} \subseteq \Omega_A(\mathbf{x}_i)$$

and

$$\lim_{i \to \infty} \left\{\mathbf{y}_i^-, \mathbf{y}_i^+\right\} = \{\mathbf{y}_0, \mathbf{y}_0\}.$$

If there is an additional limit

$$\lim_{i \to \infty} \mathbf{x}_i = \mathbf{x}_0,$$

then, $\mathbf{x}_0$ *is an extreme point of the bisector corresponding to the pseudo-vertex* $\mathbf{y}_0$.

Remark 1. The union of the bisector’s extreme points forms the edge of the surface coinciding with the closure of $L(A)$. In general, the dispersing surface is not a closed set, and the extreme points do not belong to it, but they determine its geometry.

4. Singular set characteristics

Hereinafter, we consider the case of a set A whose boundary Γ is a curve defined by the parametric equation:

$$\Gamma = \left\{ \mathbf{r}(t) \in \mathbb{R}^3 \colon t \in T \right\}, \tag{4.1}$$

where $T \subseteq \mathbb{R}$ is a closed connected interval.

Condition 1. We assume that the vector-valued function $\mathbf{r}(t)$ is three times differentiable on T, and the following biregularity condition is satisfied:

$$\forall t \in T \quad \left[\mathbf{r}'(t), \mathbf{r}''(t)\right] \neq \mathbf{0}, \tag{4.2}$$

where $[\cdot,\cdot]$ is the vector product, whereas the function $\mathbf{r}(t)$ satisfies a Lipschitz condition.

Condition (4.2) ensures that, for any $t \in T$, a TNB frame [9] consisting of three unit vectors is defined:

$$\mathbf{e}_1(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \tag{4.3}$$

$$\mathbf{e}_2(t) = \frac{\left[\left[\mathbf{r}'(t), \mathbf{r}''(t)\right], \mathbf{r}'(t)\right]}{\left\|\left[\left[\mathbf{r}'(t), \mathbf{r}''(t)\right], \mathbf{r}'(t)\right]\right\|}, \tag{4.4}$$

$$\mathbf{e}_3(t) = \frac{\left[\mathbf{r}'(t), \mathbf{r}''(t)\right]}{\left\|\left[\mathbf{r}'(t), \mathbf{r}''(t)\right]\right\|}. \tag{4.5}$$

According to the classification used in differential geometry, $\mathbf{e}_1(t)$ is a tangent unit vector, $\mathbf{e}_2(t)$ is a normal unit vector, and $\mathbf{e}_3(t)$ is a binormal unit vector. The curve Γ is characterized by two parameters at a point. They are its curvature

$$k(t) = \frac{\left\|\left[\mathbf{r}'(t), \mathbf{r}''(t)\right]\right\|}{\|\mathbf{r}'(t)\|^3} \tag{4.6}$$

and torsion

$$\varkappa(t) = \frac{\left(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)\right)}{\left\|\left[\mathbf{r}'(t), \mathbf{r}''(t)\right]\right\|^2}. \tag{4.7}$$

Here $(\cdot,\cdot,\cdot)$ is a triple scalar product.

Definition 4. *The line*

$$V(t) = \left\{ \mathbf{r}(t) + k^{-1}(t)\mathbf{e}_2(t) + \xi \mathbf{e}_3(t) \in \mathbb{R}^3 \colon \xi \in \mathbb{R} \right\} \tag{4.8}$$

is called conjugate to the curve (4.1) *at the point* $\mathbf{r}(t)$.

The biregularity condition (4.2) ensures that, for any t, the curvature is defined and has a non-zero solution. Hence, the conjugate line (4.8) is also defined. It should be noted that a normal at the point $\mathbf{r}(t)$ is to be constructed as

$$P(t) = \left\{ \mathbf{z} \in \mathbb{R}^3 \colon \left\langle \mathbf{z} - \mathbf{r}(t), \mathbf{r}'(t) \right\rangle = 0 \right\}, \tag{4.9}$$

whereas the wave front $\Phi(\tau)$ generated by the point $\mathbf{r}(t)$ falls partly inside the circle:

$$\Theta(t,\tau) = \partial U(\mathbf{r}(t), \tau) \cap P(t), \tag{4.10}$$

where $\tau > 0$.

Lemma 1. *If a sequence of pairs of quasi-symmetric parameters* $\{t_i^-, t_i^+\}_{i=1}^{\infty} \subset T$, *a sequence of points* $\{\mathbf{x}_i\}_{i=1}^{\infty} \subset L(\Gamma)$, *a parameter* $t_0 \in T$, *and a point* $\mathbf{x}_0 \in L(\Gamma)$ *satisfy the conditions*

$$\forall i \in \mathbb{N}\ \{\mathbf{r}(t_i^-), \mathbf{r}(t_i^+)\} \subseteq \Omega_\Gamma(\mathbf{x}_i), \tag{4.11}$$

$$\lim_{i\to\infty} \{t_i^-, t_i^+\} = \{t_0, t_0\}, \tag{4.12}$$

$$\lim_{i\to\infty} \mathbf{x}_i = \mathbf{x}_0, \tag{4.13}$$

then the following relation is true:

$$\lim_{i\to\infty} \big(\langle \mathbf{x}_i - \mathbf{r}(t_i^-), \mathbf{e}_2(t_i^-)\rangle - k^{-1}(t_i^-)\big) = 0. \tag{4.14}$$

P r o o f. Consider a Frenet–Serret frame (trihedron). We should note that if $\overline{\mathbf{r}}(s) = (\overline{x}(s), \overline{y}(s), \overline{z}(s))$ is a vector-valued function that is three times differentiable on the interval $S \subset \mathbb{R}$ and defined by a natural parameter (arc length) $s \geqslant 0$, then the following Taylor expansion is true for any $s \in S$ and sufficiently small increments of Δs:

$$\overline{\mathbf{r}}(s+\Delta s) = \overline{\mathbf{r}}(s) + \overline{\mathbf{r}}'(s)\Delta s + \frac{1}{2}\overline{\mathbf{r}}''(s)\Delta s^2 + \frac{1}{6}\overline{\mathbf{r}}'''(s)\Delta s^3 + \mathbf{o}(\Delta s^3). \tag{4.15}$$

Here, $\mathbf{o}(\delta)$ is a vector-valued function with $\|\mathbf{o}(\delta)\| = o(\delta)$; $o(\delta)$ being an infinitesimal with a higher order of smallness with respect to $\delta \in \mathbb{R}$.

Consider a classical orthonormal Frenet–Serret frame $\{\overline{\mathbf{e}}_1(s), \overline{\mathbf{e}}_2(s), \overline{\mathbf{e}}_3(s)\}$ and specify the coordinates of the vector $\overline{\mathbf{r}}(s+\Delta s) = \big(\overline{x}(s+\Delta s),\ \overline{y}(s+\Delta s), \overline{z}(s+\Delta s)\big)$ based on (4.15) (for more details, see [10, Ch. 5]):

$$\overline{x}(s+\Delta s) = \overline{x}(s) + \Delta s - \frac{1}{6}\overline{k}^2(s)\Delta s^3 + o(\Delta s^3),$$
$$\overline{y}(s+\Delta s) = \overline{y}(s) + \frac{1}{2}\overline{k}(s)\Delta s^2 + \frac{1}{6}k'(s)\Delta s^3 + o(\Delta s^3),$$
$$\overline{z}(s+\Delta s) = \overline{z}(s) + \frac{1}{6}\overline{k}(s)\overline{\varkappa}(s)\Delta s^3 + o(\Delta s^3),$$

where $\overline{k}(s)$ and $\overline{\varkappa}(s)$ are the curvature and torsion of the curve at the point $\overline{\mathbf{r}}(s)$.

In what follows, to achieve the result stated, it is sufficient to use only the lower terms of the above expansions:

$$\overline{x}(s+\Delta s) = \overline{x}(s) + \Delta s + o(\Delta s),$$
$$\overline{y}(s+\Delta s) = \overline{y}(s) + \frac{1}{2}\overline{k}(s)\Delta s^2 + o(\Delta s^2),$$
$$\overline{z}(s+\Delta s) = \overline{z}(s) + \frac{1}{6}\overline{k}(s)\overline{\varkappa}(s)\Delta s^3 + o(\Delta s^3).$$

Let us turn to the original curve described by means of the parameter $t \in \mathbb{R}$. We have

$$\mathbf{r}(t) = \overline{\mathbf{r}}\big(s(t)\big),$$

where

$$s'(t) = \|\mathbf{r}'(t)\|.$$

The coordinates of the vector

$$\mathbf{r}(t+\Delta t) = \big(x(t+\Delta t), y(t+\Delta t), z(t+\Delta t)\big),$$

where

$$\mathbf{r}(t+\Delta t) \triangleq \overline{\mathbf{r}}\big(s(t+\Delta t)\big) = \big(\overline{x}\big(s(t+\Delta t)\big), \overline{y}\big(s(t+\Delta t)\big), \overline{z}\big(s(t+\Delta t)\big)\big)\,,$$

are calculated in the orthonormal basis

$$\big\{\overline{\mathbf{e}}_1\big(s(t)\big), \overline{\mathbf{e}}_2\big(s(t)\big), \overline{\mathbf{e}}_3\big(s(t)\big)\big\} = \{\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)\}$$

as follows:

$$x(t+\Delta t) = x(t) + \big\|\mathbf{r}'(t)\big\|\,\Delta t + o(\Delta t), \tag{4.16}$$

$$y(t+\Delta t) = y(t) + \frac{\|\mathbf{r}'(t)\|^2\,\Delta t^2 k(t)}{2} + o(\Delta t^2), \tag{4.17}$$

$$z(t+\Delta t) = z(t) + \frac{k(t)\varkappa(t)\,\|\mathbf{r}'(t)\|^3\,\Delta t^3}{6} + o(\Delta t^3). \tag{4.18}$$

Here, $k(t) = \overline{k}\big(s(t)\big)$ and $\varkappa(t) = \overline{\varkappa}\big(s(t)\big)$. When deriving formulas (4.16)–(4.18), it is taken into account that $\Delta s = s'(t)\Delta t + o(\Delta t)$ with $\Delta t \to 0$ as $\Delta s \to 0$.

Let us relate the moving coordinate system to the point $t = t_i^-$. Provided that $\Delta t = t_i^+ - t_i^-$, we obtain the following equalities by (4.16)–(4.18):

$$x(t_i^+) = x(\widehat{t}_i) + \|\mathbf{r}'(\widehat{t}_i)\|\Delta t + o(\Delta t), \tag{4.19}$$

$$y(t_i^+) = y(t_i^-) + \frac{1}{2}k(t_i^-)\|\mathbf{r}'(t_i^-)\|^2\Delta t^2 + o(\Delta t^2), \tag{4.20}$$

$$z(t_i^+) = z(t_i^-) + \frac{1}{6}k(t_i^-)\varkappa(t_i^-)\|\mathbf{r}'(t_i^-)\|^3\Delta t^3 + o(\Delta t^3). \tag{4.21}$$

Let us calculate the derivatives of the coordinates at the point $t = t_i^+$ up to infinitesimals:

$$x'(t_i^+) = \|\mathbf{r}'(t_i^-)\| + \varepsilon(\Delta t), \tag{4.22}$$

$$y'(t_i^+) = k(t_i^-)\|\mathbf{r}'(t_i^-)\|^2\Delta t + o(\Delta t), \tag{4.23}$$

$$z'(t_i^+) = o(\Delta t). \tag{4.24}$$

Here, $\varepsilon(t)$ is an infinitesimal.

Denote the coordinates of the point $\mathbf{x}_i$ in the proposed coordinate system by (x_i^*, y_i^*, z_i^*). By the conditions, the sequences $\{t_i^-, t_i^+\}_{i=1}^\infty$ and $\{\mathbf{x}_i\}_{i=1}^\infty$ are bounded, and the function $\mathbf{r}(t)$ is Lipschitz, hence, the sequence $\{(x_i^*, y_i^*, z_i^*)\}_{i=1}^\infty$ is bounded. Therefore,

$$\exists \mu > 0\colon\ \forall i \in \mathbb{N} \quad |x_i^*| + |y_i^*| + |z_i^*| \leqslant \mu. \tag{4.25}$$

Since, by construction, $\mathbf{x}_i \in P(t_i^-)$, we have

$$x_i^* = 0. \tag{4.26}$$

On the other hand, if $\mathbf{x}_i \in P(t_i^+)$, then

$$\langle \mathbf{x}_i - \mathbf{r}(t_i^+), \mathbf{r}'(t_i^+)\rangle = 0. \tag{4.27}$$

Based on the equality $t_i^+ = t_i^- + \Delta t$ and representations (4.19)–(4.21) as well as (4.22)–(4.24), we can write equality (4.27) in the form

$$\begin{aligned} &\big(x_i^* - \big(\big\|\mathbf{r}'(t_i^-)\big\|\,\Delta t + o(\Delta t)\big)\,\big)\,\big(\big\|\mathbf{r}'(t_i^-)\big\| + \varepsilon(\Delta t)\big) + \\ &+ (y_i^* - o(\Delta t))\left(\big\|\mathbf{r}'(t_i^-)\big\|^2 k(t_i^-)\Delta t + o(\Delta t)\right) + (z_i^* - o(\Delta t))o(\Delta t) = 0. \end{aligned} \tag{4.28}$$

From (4.25), it follows that $|z_i^*| \leqslant \mu$; hence $z_i^* o(\Delta t) = o(\Delta t)$. Therefore, grouping all infinitesimals of a higher order than Δt and substituting the value x_i^* from (4.26) into equality (4.28), we can transform (4.28) to the following form:

$$-\|\mathbf{r}'(t_i^-)\|^2 \Delta t + y_i^* \left\|\mathbf{r}'(t_i^-)\right\|^2 k(t_i^-)\Delta t + o(\Delta t) = 0. \tag{4.29}$$

Let us express $y_i^* k(t_i^-)$ from (4.29). Thus, we get the limit relation

$$\lim_{i\to\infty} y_i^* k(t_i^-) = \lim_{i\to\infty} \frac{\left\|\mathbf{r}'(t_i^-)\right\|^2 \Delta t - o(\Delta t)}{\left\|\mathbf{r}'(t_i^-)\right\|^2 \Delta t} = \lim_{\Delta t\to 0} \frac{\left\|\mathbf{r}'(t_i^-)\right\|^2 \Delta t - o(\Delta t)}{\left\|\mathbf{r}'(t_i^-)\right\|^2 \Delta t} = 1. \tag{4.30}$$

Since, in the adopted coordinate system, the positive direction of the ordinate axis coincides with the unit vector $\mathbf{e}_2(t_i^-)$, we have

$$y_i^* = \left\langle \mathbf{e}_2(t_i^-), \mathbf{x}_i - \mathbf{r}(t_i^-) \right\rangle. \tag{4.31}$$

From (4.30) and (4.31), it follows that

$$\lim_{i\to\infty} \left\langle \mathbf{e}_2(t_i^-), \mathbf{x}_i - \mathbf{r}(t_i^-) \right\rangle k(t_i^-) = 1. \tag{4.32}$$

We move the number 1 to the left side of (4.32) under the limit sign and divide the expression obtained under the limit sign by $k(t_i^-) \neq 0$. As a result, we get (4.14). □

Lemma 1 enables formulating a statement about the coordinates of the extreme points generated by the pseudo-vertex of the spatial curve.

Theorem 1. *Let there be a pseudo-vertex* $\mathbf{r}(t_0)$ *on the curve* (4.1). *If the extreme point of the bisector* $\mathbf{x}_0$ *corresponds to the pseudo-vertex* $\mathbf{r}(t_0)$, *then the following inclusion holds*:

$$\mathbf{x}_0 \in V(t_0). \tag{4.33}$$

P r o o f. If $\mathbf{x}_0$ is the extreme point of the bisector corresponding to the pseudo-vertex $\mathbf{r}(t_0)$ of the set Γ, then Definition 3 implies the existence of a sequence of pairs of non-coinciding numbers $\{t_i^-, t_i^+\}_{i=1}^{\infty} \subset T$ and a sequence of points $\{\mathbf{x}_i\}_{i=1}^{\infty} \subset L(\Gamma)$ satisfying the conditions (4.11)–(4.13). Since it follows from (4.11) that the point $\mathbf{x}_i$ for any i lies in the normal plane (4.9) (constructed at the point $\mathbf{r}(t_i^-)$), we have

$$\forall i \in \mathbb{N} \ \left\langle \mathbf{x}_i - \mathbf{r}(t_i^-), \mathbf{r}'(t_i^-) \right\rangle = 0. \tag{4.34}$$

The vector product is a continuous function of two vector variables, and $\mathbf{r}(t)$ is a three times differentiable function. Therefore, it is possible to calculate the value of the limit as follows:

$$\lim_{i\to\infty} \left\langle \mathbf{x}_i - \mathbf{r}(t_i^-), \mathbf{r}'(t_i^-) \right\rangle = \left\langle \lim_{i\to\infty} \mathbf{x}_i - \lim_{i\to\infty} \mathbf{r}(t_i^-), \lim_{i\to\infty} \mathbf{r}'(t_i^-) \right\rangle = \left\langle \mathbf{x}_0 - \mathbf{r}(t_0), \mathbf{r}'(t_0) \right\rangle. \tag{4.35}$$

According to (4.34) and (4.35), the following equality is true:

$$\left\langle \mathbf{x}_0 - \mathbf{r}(t_0), \mathbf{r}'(t_0) \right\rangle = 0. \tag{4.36}$$

The biregularity condition ensures that the curvature (4.6) at any point of the curve is continuous and strictly positive; hence, the inverse function $k^{-1}(t)$ is continuous in some neighborhood of t_0. Consider the function (4.4). Its numerator represents a composition of vector products of continuous vector-valued functions, and the denominator is equal to the norm of the numerator. In this case, the numerator is different from 0 according to the condition (4.2). Therefore,

$$\begin{gathered}\lim_{i\to\infty}\left(\left\langle \mathbf{x}_i-\mathbf{r}(t_i^-),\mathbf{e}_2(t_i^-)\right\rangle - k^{-1}(t_i^-)\right)=\\ =\langle \lim_{i\to\infty}\mathbf{x}_i-\lim_{i\to\infty}\mathbf{r}(t_i^-),\lim_{i\to\infty}\mathbf{e}_2(t_i^-)\rangle-\lim_{i\to\infty}k^{-1}(t_i^-)=\langle \mathbf{x}_0-\mathbf{r}(t_0),\mathbf{e}_2(t_0)\rangle-k^{-1}(t_0).\end{gathered}\tag{4.37}$$

From (4.14) and (4.37), it follows that

$$\langle \mathbf{x}_0-\mathbf{r}(t_0),\mathbf{e}_2(t_0)\rangle-k^{-1}(t_0)=0.\tag{4.38}$$

It should be noted that (4.8) for $t=t_0$ can be represented as a set of points, for which the following conditions hold:

$$\mathbf{z}\in P(t_0)\tag{4.39}$$

and

$$\langle \mathbf{z}-\mathbf{r}(t_0),\mathbf{e}_2(t_0)\rangle=k^{-1}(t_0).\tag{4.40}$$

Equality (4.36) is equivalent to the condition (4.39), and equality (4.38) is equivalent to the condition (4.40). Hence, (4.33) holds. □

Remark 2. Equations (4.8) and (4.33) for the extreme points of a singular set are generalizations to three-dimensional equations for the extreme points of a singular set for solving the corresponding planar time-optimal control problem (see (4.1) and (4.2) from [18]).

Remark 3. Strictly speaking, a Frenet–Serret frame is not unique. Depending on the parameters, the vectors (4.3) and (4.5) can be directed differently. However, the vector (4.4) is always coincides with the direction, in which the curve (4.1) is locally convex in the neighborhood of the point $\mathbf{r}(t)$. Therefore, the equation of the conjugate line (4.8) is an invariant and is determined solely by certain characteristics of the curve Γ.

5. Example of solving the time-optimal problem (2.1)

To construct singular sets in 3D space, the authors have upgraded a software package [8], previously used to solve flat tome-optimal problems. It is based on algorithms for calculating the parameters t^- and t^+, which define pairs of quasi-symmetric points $\mathbf{r}(t^-)$ and $\mathbf{r}(t^+)$ and the points $\mathbf{x}\in L(\Gamma)$ generated by them. A key element is searching for pseudo-vertices of the target set. Finding a pseudo-vertex makes it possible, using the results of Section 4, to construct sets of extreme points of the bisector. These sets help to numerically construct the singular set itself. The level surface $\Phi(\tau)$ of the optimal result function $u(\mathbf{x})$ corresponding to the time point $\tau>0$ is constructed as a union of circles (4.10), from which the parts cut off by the bisector $L(\Gamma)$ are removed. For each circle $\Theta(t,\tau)$, $t\in T$, it is required to find out, which arcs on it get into $\Phi(\tau)$.

Example 1. Consider an example of a time-optimal problem with a target set represented by the curve (4.1), where the function

$$\mathbf{r}(t)=\left(\cos t,\sin t,\frac{\cos 3t}{3}\right)\tag{5.1}$$

is defined on $T=[0,2\pi]$. The function (5.1) satisfies Condition 1 and the Lipschitz condition with constant $L=3$. An analysis of its first-order derivatives

$$\mathbf{r}'(t)=(-\sin t,\cos t,-\sin 3t)\tag{5.2}$$

and its second-order derivatives

$$\mathbf{r}''(t) = (-\cos t, -\sin t, -3\cos 3t) \tag{5.3}$$

allows us to prove that the biregularity condition (4.2) holds. We should note that the torsion (4.7) is not identically zero; hence, the curve Γ is not flat. Although, $\varkappa(t) = 0$ is possible at some points $t \in T$.

Modeling the wave front propagation makes it possible to define that the set (4.1) has six pseudo-vertices corresponding to the values of the parameter

$$t_1 = 0, \quad t_2 = \pi/3, \quad t_3 = 2\pi/3, \quad t_4 = \pi, \quad t_5 = 4\pi/3, \quad t_6 = 5\pi/3.$$

According to Theorem 1, the extreme points of the bisector lie on the lines conjugate to Γ at the pseudo-vertices. Fig. 1 shows the curve Γ as a purple line, its pseudo-vertices $\mathbf{r}(t_i)$, $i = \overline{1,6}$, as bubbles, and the dispersing surface $L(\Gamma)$ as the translucent blue surface. The sets of extreme points W_i corresponding to the pseudo-vertices $\mathbf{r}(t_i)$, $i = \overline{1,6}$, are found by means of the derivatives of the vector-valued function of the first-order (5.2) and second-order (5.3):

$$W_1 = \Big\{\Big(\xi, 0, \frac{1-\xi}{3}\Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty)\Big\},$$
$$W_2 = \Big\{\Big(\frac{\sqrt{3}}{2}\xi, \frac{\xi}{2}, \frac{\xi-1}{3}\Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty)\Big\},$$
$$W_3 = \Big\{\Big(-\frac{\xi}{2}, \frac{\sqrt{3}}{2}\xi, \frac{1-\xi}{3}\Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty)\Big\},$$
$$W_4 = \Big\{\Big(-\xi, 0, \frac{\xi-1}{3}\Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty\Big\},$$
$$W_5 = \Big\{\Big(-\frac{\sqrt{3}}{2}\xi, -\frac{\xi}{2}, \frac{1-\xi}{3}\Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty)\Big\},$$
$$W_6 = \Big\{\Big(\frac{\xi}{2}, -\frac{\sqrt{3}}{2}\xi, \frac{\xi-1}{3}\Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty)\Big\}.$$

The sets W_i, $i = \overline{1,6}$, are shown by red lines in Fig. 1. The embedding $W_i \subset V(t_i)$ is valid for all $i = \overline{1,6}$.

The wave front $\Phi(\tau)$ corresponding to the time point $\tau = 0.5$ (that is, the set of points for which the optimal result function is equal to τ) is shown in Fig. 2 as a surface with colors changing from blue to red as they grow along the Z axis. The wave front $\Phi(\tau)$ corresponding to the time point $\tau = 1$ is shown in Fig. 3.

The dispersing surface is characterized by 6 sheets:

$$L_1 = \Big\{(x, y, z) \in \mathbb{R}^3 \colon x = \xi,\ y = 0,\ z < \frac{1-\xi}{3},\ \xi \in [0, \infty)\Big\},$$
$$L_2 = \Big\{(x, y, z) \in \mathbb{R}^3 \colon x = \frac{\sqrt{3}}{2}\xi,\ y = \frac{\xi}{2},\ z > \frac{\xi-1}{3},\ \xi \in [0, \infty)\Big\},$$
$$L_3 = \Big\{(x, y, z) \in \mathbb{R}^3 \colon x = -\frac{\xi}{2},\ y = \frac{\sqrt{3}}{2}\xi,\ z < \frac{1-\xi}{3},\ \xi \in [0, \infty)\Big\},$$
$$L_4 = \Big\{(x, y, z) \in \mathbb{R}^3 \colon x = 0,\ y = \xi,\ z > \frac{\xi-1}{3},\ \xi \in [0, \infty)\Big\},$$
$$L_5 = \Big\{(x, y, z) \in \mathbb{R}^3 \colon x = \frac{\xi}{2},\ y = -\frac{\sqrt{3}}{2}\xi,\ z < \frac{1-\xi}{3},\ \xi \in [0, \infty)\Big\},$$
$$L_6 = \Big\{(x, y, z) \in \mathbb{R}^3 \colon x = \frac{\sqrt{3}}{2}\xi,\ y = -\frac{\xi}{2}, z > \frac{\xi-1}{3},\ \xi \in [0, \infty)\Big\},$$

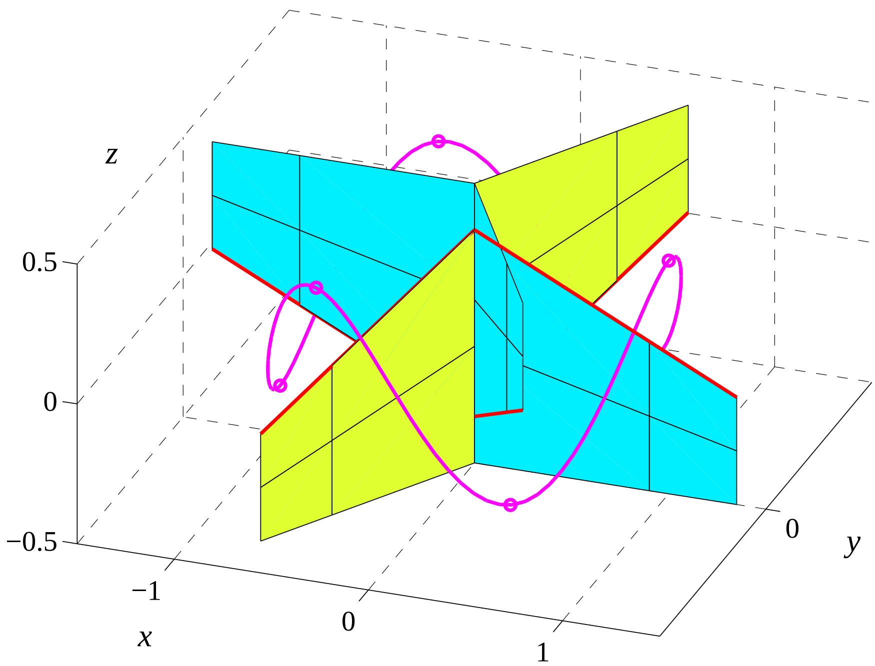

Figure 1. The curve Γ, the pseudo-vertices, and the dispersing surface $L(\Gamma)$.

All sheets have a non-empty intersection

$$L^* = \bigcap_{i=\overline{1,6}} L_i = \Big\{(x, y, z) \in \mathbb{R}^3 \colon x = 0,\ y = 0,\ z \in \big(-\frac{1}{3}, \frac{1}{3}\big)\Big\}.$$

We have $\mathrm{card}(\Omega_\Gamma(\mathbf{x})) = 6$ for all points $\mathbf{x} \in L^*$, and $\mathrm{card}(\Omega_\Gamma(\mathbf{x})) = 3$ for all other points

$$\big\{(x, y, z) \in \mathbb{R}^3 \colon\ x = 0,\ y = 0,\ |z| \geqslant 1/3\big\}$$

on the applicate axis.

Remark 4. The resolving constructions in Example 1 can be considered as a problem solution for an Eikonal equation with the boundary condition given on the graph of the vector-valued function (5.1). In this case, wave fronts represent light propagation surfaces in a homogeneous medium with the source distributed uniformly along the curve Γ. The bisector $L(\Gamma)$ is the union of non-smoothness points of the wave fronts due to the fact that the radiation comes from different points on the curve Γ.

6. Conclusion

One class of time-optimal problems in 3D space with a spherical velocity vectogram is investigated in the case of the target set coinciding with a curve Γ defined by the parametric equation. Characteristic points, such as pseudo-vertices responsible for the origin of the singular set $L(\Gamma)$, are identified. The optimal result function $u(\mathbf{x})$ loses its smoothness on the surface $L(\Gamma)$. Analytical expressions are obtained for the coordinates of the extreme points of the bisector corresponding to a pseudo-vertex. The equations are written in terms of the curvature, principal normal, and

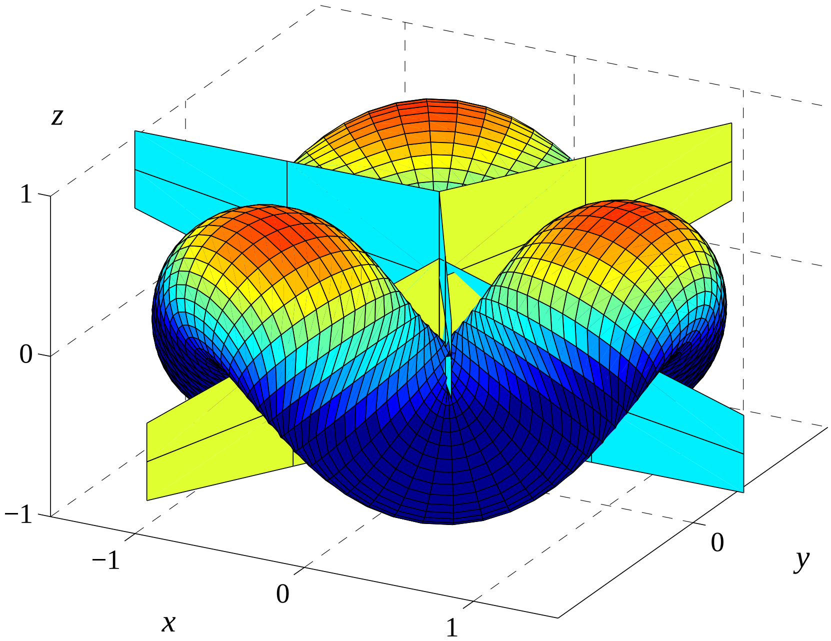

Figure 2. The level surface $\Phi(0.5)$ of the optimal result function and the dispersing surface $L(\Gamma)$.

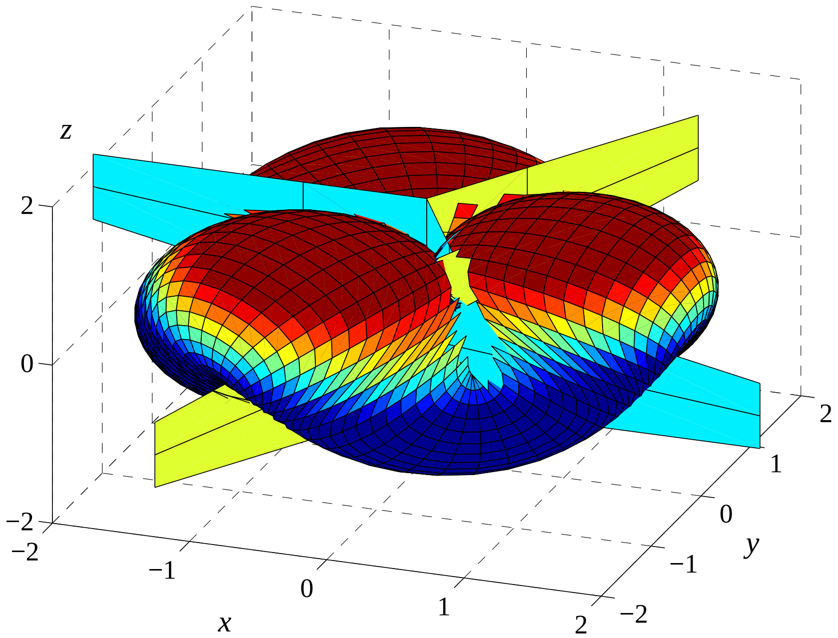

Figure 3. The level surface $\Phi(1)$ of the optimal result function and the dispersing surface $L(\Gamma)$.

binormal of the curve Γ. An example of modeling the construction of a solution to a time-optimal problem with a closed curve taken as the target set is given. Four pseudo-vertices and the sets W_i, $i = \overline{1,6}$, of extreme points corresponding to them, which are rays on lines conjugate to $L(\Gamma)$, are found. Based on the sets W_i, $i = \overline{1,6}$, a bisector is constructed, which is the union of two plane sets lying in orthogonal planes and having a common line segment. The level surfaces $\Phi(\tau)$ are constructed at various time points τ. We should note that, in the previously studied problems on the plane, only one bisector point can correspond to each pseudo-vertex (or two in a very special case, e.g., in [7]). In 3D space, an infinite set of extreme points corresponding to one pseudo-vertex can exist. In the future, it is planned to extend the developed algorithms to solve problems with more complex geometry.

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