

# IMPULSE–SLIDING REGIMES IN SYSTEMS WITH DELAY<sup>1</sup>

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**Abstract:** The paper is devoted to the formalization of a concept of impulse-sliding regimes generated by positional impulse controls for systems with delay. We define the notion of impulse-sliding trajectory as a limit of a sequence of Euler polygonal lines generated by a discrete approximation of the impulse position control. The equations describing the trajectory of impulse-sliding regime are received.

**Key words:** Impulse position control, Systems with delay, Impulse-sliding regime, Euler polygonal lines.

## Introduction

Usually, the positional control algorithms are introduced by substitution in the program control models the initial time and the initial model position to an arbitrary time moment and to an arbitrary state. Such replacement may result in that we will need to realize an impulse control action at each time moment. This fact leads to the appearance of a moving or a so called sliding impulse. Such phenomenon from the point of view of the theory of differential equations requires an appropriate formalization. In addition, this motion type in the space of positions creates the motion sliding on some functional manifold. Impulse-sliding regimes in systems without delay were considered in [1–3]. Impulse-sliding regimes for linear systems with delay were studied in [4]. The reaction of nonlinear systems with delay to impulse actions is understood here as in the paper [5]. The definition of a solution of nonlinear systems with delay given in [5] is a generalization for the notion of solution for systems without delay in [6, 7].

### 1. Formalization of impulse-sliding regime

Consider a dynamic system with impulse control

$$\dot{x}(t) = f(t, x(t), x(t - \tau)) + B(t, x(t))u, \quad t \in [t_0, \vartheta], \quad (1.1)$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0], \quad (1.2)$$

where  $f(\cdot, \cdot, \cdot)$  is a function with value in  $R^n$ ,  $B(\cdot, \cdot)$  is a  $m \times n$ -matrix function. Elements of  $f$  and  $B$  are continuous functions and satisfy the conditions, which guarantee the existence and

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uniqueness of a solution for any summable function  $u(t)$ . Let  $x_t(\cdot)$  be a function-prehistory  $x_t(\cdot) = \{x(t+s); -\tau \leq s < 0\}$ . The function  $\varphi(t)$  here is a function of bounded variation for  $t \in [t_0 - \tau, t_0]$ .

We will assume that the function  $B(t, x)$  satisfies the well-known Frobenius condition [8],

$$\sum_{\nu=1}^n \frac{\partial b_{ij}(t, x)}{\partial x_\nu} b_{\nu l}(t, x) = \sum_{\nu=1}^n \frac{\partial b_{il}(t, x)}{\partial x_\nu} b_{\nu j}(t, x). \quad (1.3)$$

According to [5, 6] this condition ensures the uniqueness of the system response to the impact of a generalized control  $u(t)$  (the generalized derivative of a bounded variation function). We note that there are various ways of defining a solution for the equation (1.1) which lead generally to various implementations of the trajectories [6]. We will use the definition that is based on the closure of the set of smooth trajectories in the space of functions of bounded variation [6]. This definition is the most natural from the point of view of control theory. This is due to the fact that impulse controls are often some control idealizations operating in short time intervals and with great intensity.

By an impulse positional control we will mean an operator  $t, x_t(\cdot) \rightarrow U(t, x(t))$  mapping the space of extended states  $\{t, x(t)\}$  into the space of  $m$ -vector-valued distributions

$$U(t, x(t)) = r(t, x(t)) \delta_t. \quad (1.4)$$

In this paper we assume that a delay is only in  $f(t, x(t), x(t - \tau))$  and a control function does not contain a delay.

Here  $r(t, x(t))$  is  $m$ -dimensional vector function,  $\delta_t$  is the Dirac impulse function concentrated at  $t$ . The system reaction to the impulse position control  $U(t, x(t))$  (which we call an impulse-sliding regime) is defined as the set of Euler polygonal functions  $x^h(\cdot)$ ,  $h = \max(t_{k+1} - t_k)$  corresponding to all decompositions  $t_0 < t_1 < \dots < t_p = \vartheta$  of the interval  $[t_0, \vartheta]$ . The Euler polygonal function (Euler line)  $x^h(\cdot)$  is constructed as a left continuous function of bounded variation such that the equation holds

$$\dot{x}^h(t) = f(t, x^h(t), x^h(t - \tau)) + \sum_{i=1}^p B(t, x^h(t)) r(t_i, x(t_i)) \delta_{t_i} \quad (1.5)$$

with the initial condition  $x(t) = \varphi(t)$ ,  $t \in [t_0 - \tau, t_0]$ .

The Euler line satisfies the equation

$$x^h(t) = \varphi(t_0) + \int_{t_0}^t f(\xi, x^h(\xi), x^h(\xi - \tau)) d\xi + \sum_{t_i < t} S(t_i, x^h(t_i), r(t_i, x(t_i))) \quad (1.6)$$

and the jump functions are defined by the equations

$$S(t_i, x^h(t_i), r(t_i, x^h(t_i))) = z(1) - z(0), \quad (1.7)$$

$$\dot{z}(\xi) = B(t, z(\xi)) r(t_i, x^h(t_i)), \quad z(0) = x^h(t_i). \quad (1.8)$$

Here the jump function  $S(t, x, \mu)$  is the solution of the equation

$$\frac{\partial y}{\partial \mu} = B(t, y). \quad (1.9)$$

We will assume that the equality

$$r\left(t, x(t) + S(t, x(t), r(t, x(t)))\right) = 0. \quad (1.10)$$

is true.

This equality means that after the action of impulse at the system at time  $t$ , the state  $\{t, x(t)\}$  will belong to the manifold  $r(t, x(t)) = 0$ .

## 2. Properties of the impulse-sliding regime

**Lemma 1.** *Assume that for all admissible values  $t_1, t_2, x_1, x_2, y_1$  and  $y_2$  the following inequalities are true*

$$\|f(t, x, y)\| \leq C(1 + \sup_{[t_0-\tau]} \|x(\cdot)\|), \quad (2.1)$$

$$\|S(t_1, x_1, r(t_1, x_1)) - S(t_2, x_2, r(t_2, x_2))\| \leq L(|t_1 - t_2| + \|x_1 - x_2\|). \quad (2.2)$$

Then for all decompositions  $h$  and all  $t \in [t_0, \vartheta]$  the set of Euler polygonal functions  $x^h(\cdot)$  is bounded, what means that there exists a constant  $M$  such that

$$\|x^h(t)\| \leq M. \quad (2.3)$$

*P r o o f.* From (1.6) and (2.1) the following inequality follows

$$\|x^h(t)\| \leq \|\varphi(t_0)\| + C \int_{t_0}^t (1 + \sup_{[t_0-\tau, \xi]} \|x^h(\cdot)\|) d\xi + \sum_{t_i < t} \|S(t_i, x^h(t_i), r(t_i, x^h(t_i)))\|. \quad (2.4)$$

Due to the fact that

$$S(t_{i-1}, x^h(t_{i-1} + 0), r(t_{i-1}, x^h(t_{i-1} + 0))) = 0,$$

in view of (2.2), we have the inequalities

$$\begin{aligned} \|S(t_i, x^h(t_i), r(t_i, x^h(t_i)))\| &= \|S(t_i, x^h(t_i), r(t_i, x^h(t_i)))\| - \\ &- S(t_{i-1}, x^h(t_{i-1} + 0), r(t_{i-1}, x^h(t_{i-1} + 0))) \leq L(t_i - t_{i-1} + \|x^h(t_i) - x^h(t_{i-1} + 0)\|). \end{aligned} \quad (2.5)$$

At the same time, in view of (2.1),

$$\begin{aligned} \|x^h(t_i) - x^h(t_{i-1} + 0)\| &\leq \int_{t_{i-1}}^{t_i} \|f(\xi, x^h(\xi), x^h(\xi - \tau))\| d\xi \\ &\leq C(t_i - t_{i-1} + L \int_{t_{i-1}}^{t_i} (1 + \sup_{[t_0-\tau, \xi]} \|x^h(\cdot)\|) d\xi). \end{aligned} \quad (2.6)$$

In consequence, from (2.4) in view of (2.5) and (2.6) we get the following inequality

$$\|x^h(t)\| \leq \|\varphi(t_0)\| + (L + C)(t - t_0) + L(1 + C) \int_{t_0}^t \sup_{[t_0-\tau, \xi]} \|x^h(\cdot)\| d\xi. \quad (2.7)$$

As in [9], from the last inequality we get

$$\sup_{[t_0-\tau, t]} \|x^h(\cdot)\| \leq R + (L + C)(t - t_0) + L(1 + C) \int_{t_0}^t \sup_{[t_0-\tau, \xi]} \|x^h(\cdot)\| d\xi, \quad (2.8)$$

where

$$R = \sup_{[t_0-\tau, t_0]} \|\varphi(\cdot)\|.$$

Applying the result of [10] we get from (2.8) the estimate

$$\sup \|x^h(\cdot)\| \leq (R + (L + C)(\vartheta - t_0)) e^{L(1+C)(\vartheta-t_0)},$$

which completes the proof of the lemma. □

Note that as a constant  $M$  we can take the following number

$$M = (R + (L + C)(\vartheta - t_0))e^{L(1+C)(\vartheta-t_0)}.$$

Let  $D$  be a bounded closed set which contains all  $x^h(\cdot)$ . By continuity we may assume that all functions  $f(t, x, y)$ ,  $B(t, x)$  and  $r(t, x)$  are bounded.

Denote

$$M_1 = \max_{[t_0, \vartheta] \times D \times D} \|f(t, x, y)\|, \quad M_2 = \max_{[t_0, \vartheta] \times D} \|B(t, x)\|, \quad M_3 = \max_{[t_0, \vartheta] \times D} \|r(t, x)\|. \quad (2.9)$$

**Lemma 2.** *Under the above assumptions from each confinal sequence of Euler functions  $\{x^h(\cdot)\}$  we can select a subsequence  $\{x^{h_p}(\cdot)\}$  uniformly at  $(t_0, \vartheta]$  converging to absolutely continuous function  $x(\cdot)$ . Moreover for all  $t \in (t_0, \vartheta]$  we have  $r(t, x(t)) = 0$  ( $x(t) = \varphi(t)$  for  $t \in [t_0 - \tau, t_0]$ ), in other words the limit element of the impulse-sliding regime moves over the manifold which is described by the equation  $r(t, x(t)) = 0$ .*

**P r o o f.** The proof of convergence of  $x^h(\cdot)$  uses the generalization of Arzela's lemma from [11]. Let  $x^{h_i}(\cdot)$  be a confinal sequence. Then according to (1.6) we have

$$\|x^{h_i}(t'') - x^{h_i}(t')\| \leq \int_{t'}^{t''} \|f(t, x^h(t), x^h(t - \tau))\| ds + \sum_{k=m(t')+1}^{m(t'')} \|S(t_k, x^{h_i}(t_k), r(t_k, x^{h_i}(t_k)))\|, \quad (2.10)$$

where  $m(t)$  is the nearest point on the left in the decomposition which generates the polygonal line  $x^{h_i}(\cdot)$ . In accordance with (1.6) we have

$$\begin{aligned} \|S(t_k, x^{h_i}(t_k), r(t_k, x^h(t_k)))\| &= \|S(t_k, x^h(t_k), r(t_k, x^h(t_k)))\| - \\ &- \|S(t_{k-1}, x^{h_i}(t_{k-1} + 0), r(t_{k-1}, x^{h_i}(t_{k-1} + 0)))\|. \end{aligned}$$

Considering (2.2) we get

$$\|S(t_k, x^{h_i}(t_k), r(t_k, x^h(t_k)))\| \leq L(t_k - t_{k-1} + \|x^{h_i}(t_k) - x^{h_i}(t_{k-1} + 0)\|).$$

At the same time

$$x^{h_i}(t_k) - x^{h_i}(t_{k-1} + 0) = \int_{t_{k-1}}^{t_k} f(\xi, x^h(\xi)) d\xi.$$

By taking into account (2.8), we obtain

$$\|S(t_k, x^{h_i}(t_k), r(t_k, x^h(t_k)))\| \leq L(t_k - t_{k-1} + M_1(t_k - t_{k-1})) = L(1 + M_1)(t_k - t_{k-1}). \quad (2.11)$$

From (2.10) and (2.11) it follows that

$$\|x^{h_i}(t'') - x^{h_i}(t')\| \leq (M_1 + L(1 + M_1))(t'' - t') + L(2 + M)(t' - t_{t_i h_i}), \quad (2.12)$$

where  $t_{t_i h_i}$  is the nearest point at the left in partition  $h_i$  to the point  $t'$ . The last inequality allows to apply the generalization of Arzela's lemma from [11] and ensures the existence of a subsequence  $x^{h_i}(\cdot)$  which uniformly converges to the function  $x(\cdot)$ .

Now we pass to the limit in the inequality (2.12) as  $i \rightarrow \infty$ . As a result we have

$$\|x(t'') - x(t')\| \leq (M_1 + L(1 + M_1))(t'' - t').$$

This means that  $x(t)$  is an absolutely continuous function at  $(t_0, \vartheta]$ .

Now let us show that the limit element  $x^h(\cdot)$  belongs to the manifold  $r(t, x) = 0$ . Let  $t_{m_i h_i}$  be the nearest point from the left in partition  $h_i$  by the time  $t$ . The following inequality holds

$$\begin{aligned} \|r(t, x(t))\| &\leq \|r(t, x(t)) - r(t, x^{h_i}(t)) + r(t, x^{h_i}(t))\| \\ &\leq \|r(t, x(t)) - r(t, x^{h_i}(t))\| + \|r(t_{m_i h_i}, x^{h_i}(t_{m_i h_i} + 0)) - r(t, x^{h_i}(t))\| \\ &\leq L[\|x(t) - x^{h_i}(t)\| + (t - t_{m_i h_i})] + \|x^{h_i}(t_{m_i h_i} + 0) - x^{h_i}(t)\| \\ &\leq L[\|x(t) - x^{h_i}(t)\| + (L + M)(t - t_{m_i h_i})]. \end{aligned}$$

By the uniform convergence of the sequence  $x^{h_i}(\cdot)$  the first term at the right hand part at the last inequality tends to zero. The second one tends to zero because  $h_i \rightarrow 0$  when  $i \rightarrow \infty$ . Therefore  $r(t, x(t)) \equiv 0$  when  $t \in (t_0, \vartheta]$ , this completes the proof of lemma.  $\square$

**Lemma 3.** *Let  $r(t, x)$  be a vector function continuously differentiable in all variables. Then the following equality holds*

$$\begin{aligned} &S(t_k, x^h(t_k), r(t_k, x^h(t_k))) - S(t_{k-1}, x^h(t_{k-1} + 0), r(t_{k-1}, x^h(t_{k-1} + 0))) \\ &= \int_{t_{k-1}}^{t_k} \left[ \frac{\partial S(\xi, x^h(\xi), r(t, x^h(\xi)))}{\partial \xi} + \frac{\partial S(\xi, x^h(\xi), r(\xi, x^h(\xi)))}{\partial x} f(\xi, x^h(\xi), x^h(\xi - \tau)) + \right. \\ &\quad \left. + \frac{\partial S(\xi, x^h(\xi), r(\xi, x^h(\xi)))}{\partial r} \left( \frac{\partial r(\xi, x^h(\xi))}{\partial \xi} + \frac{\partial r(\xi, x^h(\xi))}{\partial x} \dot{f}(\xi, x^h(\xi), x^h(\xi - \tau)) \right) \right] d\xi. \end{aligned} \quad (2.13)$$

The lemma follows from the the formula for differentiating a composite function.

**Theorem 1.** *Let all assumptions given above hold. Then an impulse-sliding regime on  $(t_0, \vartheta]$  is described by the equation*

$$\begin{aligned} \dot{x}(t) &= \frac{\partial S(t, x(t), r(t, x(t)))}{\partial t} + \frac{\partial S(t, x(t), r(t, x(t)))}{\partial r} \frac{\partial r(t, x(t))}{\partial t} + \\ &+ \left[ E + \frac{\partial S(t, x(t), r(t, x(t)))}{\partial x} + \frac{\partial S(t, x(t), r(t, x(t)))}{\partial r} \times \frac{\partial r(t, x(t))}{\partial x} \right] f(t, x(t), x(t - \tau)), \\ x(t_0 + 0) &= x(t_0) + S(t_0, x(t_0), r(t_0, x(t_0))). \end{aligned} \quad (2.14)$$

**P r o o f.** According to (1.6) and Lemma 3  $x^h(t)$  satisfies the equation

$$\begin{aligned} x^{h_i}(t) &= \varphi(t_0) + \int_{t_0}^t f(\xi, x^{h_i}(\xi), x^{h_i}(\xi - \tau)) d\xi + \int_{t_0}^{t_{m_i h_i}} \left[ \frac{\partial S(\xi, x^{h_i}(\xi), r(t, x^{h_i}(\xi)))}{\partial \xi} + \right. \\ &+ \left( \frac{\partial S(\xi, x^{h_i}(\xi), r(\xi, x^{h_i}(\xi)))}{\partial x} + \frac{\partial S(\xi, x^{h_i}(\xi), r(\xi, x^{h_i}(\xi)))}{\partial r} \times \frac{\partial r(\xi, x^{h_i}(\xi))}{\partial x} \right) f(\xi, x^{h_i}(\xi), x^{h_i}(\xi - \tau)) + \\ &\quad \left. + \frac{\partial S(\xi, x^{h_i}(\xi), r(\xi, x^{h_i}(\xi)))}{\partial r} \cdot \frac{\partial r(\xi, x^{h_i}(\xi))}{\partial \xi} \right] d\xi. \end{aligned}$$

Passing to the limit at the last equation and bearing in mind that  $x(t)$  is an absolutely continuous function, we can see that the theorem is true.  $\square$

### 3. Conclusion

The formalization of the impulse-sliding regime for a nonlinear system with time delay is made. The equation to describe the limiting element of impulse-sliding regime is obtained.

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