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ON THE PROPERTIES OF THE SET OF TRAJECTORIES OF THE NONLINEAR CONTROL SYSTEM WITH QUADRATIC INTEGRAL CONSTRAINT ON THE CONTROL FUNCTIONS

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Abstract: In this paper the control system described by a nonlinear differential equation is studied. It is assumed that the control functions have a quadratic integral constraint, more precisely, the admissible control functions are chosen from the ellipsoid of the space $L_2([t_0,\theta];\mathbb{R}^m)$. Different properties of the set of trajectories are investigated. It is proved that a small perturbation of the set of control functions causes also appropriate small perturbation of the set of trajectories. It is also shown that the set of trajectories has a small change if along with the integral constraint on the control functions, a sufficiently large norm type geometric constraint on the control functions is introduced. It is established that every trajectory is robust with respect to the fast consumption of the remaining control resource, and hence every trajectory of the system can be approximated by a trajectory generated by full consumption of the total control resource.

Keywords: Nonlinear control system, Quadratic integral constraint, Set of trajectories, Robustness.

1. Introduction

The control systems described by nonlinear differential equations are investigated in a vast number of papers. Depending on the character of the control efforts the control systems are classified as a) the control systems with geometric constraint on the control functions; b) the control systems with integral constraint on the control functions; and c) the control systems with mixed constraints on the control functions which include both the geometric and the integral constraints on the control functions. The geometric constraints on the control functions appear in the case when the control resource is not exhausted by consumption. But, if the control resource is exhausted by consumption, say as energy, food, fuel, finance, etc., then the integral constraints on the control functions is inevitable (see, e.g., [1, 2, 9, 12, 15, 16]). For example, the behaviour of the flying objects with rapidly changing mass is described as a control system with integral constraint on the control functions (see, e.g., [2, 12]).

One of the important notions of the control systems theory is the set of trajectories and attainable set concepts. Attainable set of the system at the given instant of time consists of points to which arrive the trajectories of the system and can be defined as a section of the set of trajectories at the given instant of time. Different topological properties and approximate construction methods of the set of trajectories described by various types of the integral and differential equations, where the control functions have integral constraints, are considered in papers [4–8, 11, 13, 14]. In papers [4, 5, 11, 14] the compactness, closedness, path-connectedness properties and approximate construction methods of the set of trajectories and attainable sets of the control systems which are affine with respect to the control vector are discussed. In papers [6–8, 13] the same problems are investigated for nonlinear control systems. In presented paper the properties of the set of trajectories of the nonlinear control systems are studied where the admissible control functions are chosen from the ellipsoid of the space L_2.

The paper is organized as follows. In Section 2, the basic conditions which have to satisfy the system's equation are formulated and preliminary properties of the system's trajectories are given. In Section 3 it is shown that introduction of the sufficiently large norm type constraint along with integral constraint and a small perturbation of the given ellipsoid, which characterizes the integral constraint, induce a small change of the set of trajectories (Theorem 2). A perturbation evaluation for the set of trajectories is presented. In Section 4 it is proved that every trajectory is robust with respect to the fast and full consumption of remaining control resource (Proposition 7). Applying this result it is proved that every trajectory can be approximated by the trajectory generated by full consumption of the total control resource (Theorem 3).

2. The system's dynamics

Consider control system described by nonlinear ordinary differential equation

$$\dot{x}(t) = f\left(t, x(t), u(t)\right), \quad x(t_0) = x_0 \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the phase state vector, $u(t) \in \mathbb{R}^m$ is the control vector, $t \in [t_0, \theta]$ is the time.

Let $B(\cdot) : [t_0, \theta] \to \mathbb{R}^{m \times m}$ be a continuous matrix function and $B(t)$ be a positive definite $m \times m$ matrix for every $t \in [t_0, \theta]$. For given $\varepsilon \in [0, 1]$ and $\alpha > 0$ we denote

$$U_\varepsilon = \left\{ u(\cdot) \in L_2\big([t_0, \theta]; \mathbb{R}^m\big) : \int_{t_0}^{\theta} \langle B(t)u(t), u(t)\rangle dt \le 1 + \varepsilon \right\},$$

$$U_\varepsilon^\alpha = \left\{ u(\cdot) \in U_\varepsilon : \|u(t)\| \le \alpha \text{ for almost all } t \in [t_0, \theta] \right\},$$

$$U_0^* = \left\{ u(\cdot) \in L_2\big([t_0, \theta]; \mathbb{R}^m\big) : \int_{t_0}^{\theta} \langle B(t)u(t), u(t)\rangle dt = 1 \right\},$$

where $L_2\left([t_0, \theta]; \mathbb{R}^m\right)$ is the space of Lebesgue measurable functions $u(\cdot) : [t_0, \theta] \to \mathbb{R}^m$ such that

$$\left\|u(\cdot)\right\|_2 < +\infty, \quad \left\|u(\cdot)\right\|_2 = \left(\int_{t_0}^{\theta} \left\|u(t)\right\|^2 ds \right)^{1/2},$$

$\|\cdot\|$ denotes the Euclidean norm, $\langle \cdot, \cdot \rangle$ stands for inner product.

Proposition 1. *The sets U_ε and U_ε^α are bounded, closed and convex subsets of the space $L_2\left([t_0, \theta]; \mathbb{R}^m\right)$. The set U_0^* is bounded and closed subset of the space $L_2\left([t_0, \theta]; \mathbb{R}^m\right)$.*

It is not difficult to show that there exists $c_* > 0$ such that the inequality

$$\left\|u(\cdot)\right\|_2 \le c_* \tag{2.2}$$

is satisfied for every $u(\cdot) \in U_\varepsilon$ and $\varepsilon \in [0, 1]$.

It is assumed that the function $f(\cdot, \cdot, \cdot) : [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies the following conditions:

2.A. The function $f(\cdot, \cdot, \cdot) : [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous.

2.B. For every bounded set $D \subset [t_0, \theta] \times \mathbb{R}^n$ there exist $\gamma_1 = \gamma_1(D) > 0$, $\gamma_2 = \gamma_2(D) > 0$ and $\gamma_3 = \gamma_3(D) > 0$ such that the inequality

$$\|f(t, x_1, u_1) - f(t, x_2, u_2)\| \le \left[\gamma_1 + \gamma_2(\|u_1\| + \|u_2\|)\right] \|x_1 - x_2\| + \gamma_3 \|u_1 - u_2\|$$

is satisfied for every $(t, x_1, u_1) \in D \times \mathbb{R}^m$ and $(t, x_2, u_2) \in D \times \mathbb{R}^m$.

2.C. There exists $\kappa > 0$ such that the inequality

$$\|f(t,x,u)\| \leq \kappa\left(\|x\|+1\right)\left(\|u\|+1\right)$$

is held for every $(t,x,u) \in [t_0,\theta] \times \mathbb{R}^n \times \mathbb{R}^m$.

If the function $(t,x,u) \to f(t,x,u) : [t_0,\theta] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous with respect to (x,u), then the conditions 2.B and 2.C are satisfied.

Let us define the trajectory of the system (2.1) generated by a control function $u_*(\cdot) \in L_2\left([t_0,\theta];\mathbb{R}^m\right)$. An absolutely continuous function $x_*(\cdot) : [t_0,\theta] \to \mathbb{R}^n$ satisfying the equation $\dot{x}_*(t) = f\left(t, x_*\left(t\right), u_*(t)\right)$ for almost all $t \in [t_0,\theta]$ and initial condition $x_*(t_0) = x_0$ is said to be a trajectory of the system (2.1) generated by the control function $u_*(\cdot) \in L_2\left([t_0,\theta];\mathbb{R}^m\right)$. The sets of trajectories of the system (2.1) generated by all admissible control functions $u(\cdot) \in U_\varepsilon$, $u(\cdot) \in U_\varepsilon^\alpha$ and $u(\cdot) \in U_0^*$ are denoted by $X_\varepsilon(t_0,x_0)$, $X_\varepsilon^\alpha(t_0,x_0)$ and $X_0^*(t_0,x_0)$ respectively. It is obvious that the inclusions

$$X_\varepsilon^\alpha(t_0,x_0) \subset X_\varepsilon(t_0,x_0), \quad X_0^*(t_0,x_0) \subset X_\varepsilon(t_0,x_0)$$

are verified for every $\varepsilon \in [0,1]$ and $\alpha > 0$.

For fixed $t \in [t_0,\theta]$ we set

$$X_\varepsilon(t;t_0,x_0) = \left\{x(t) \in \mathbb{R}^n : x(\cdot) \in X_\varepsilon(t_0,x_0)\right\}, \tag{2.3}$$

$$X_\varepsilon^\alpha(t;t_0,x_0) = \left\{x(t) \in \mathbb{R}^n : x(\cdot) \in X_\varepsilon^\alpha(t_0,x_0)\right\}, \tag{2.4}$$

$$X_0^*(t;t_0,x_0) = \left\{x(t) \in \mathbb{R}^n : x(\cdot) \in X_0^*(t_0,x_0)\right\}. \tag{2.5}$$

The sets $X_\varepsilon(t;t_0,x_0)$, $X_\varepsilon^\alpha(t;t_0,x_0)$ and $X_0^*(t;t_0,x_0)$ are called the attainable sets of the system (2.1) at the instant of time t, generated by all admissible control functions from the sets U_ε, U_ε^α and U_0^* respectively.

It is obvious that the attainable sets consist of points to which arrive the trajectories of the system (2.1) at the instant of time t.

By symbol $C\left([t_0,\theta];\mathbb{R}^n\right)$ we denote the space of continuous functions $x(\cdot) : [t_0,\theta] \to \mathbb{R}^n$ with norm

$$\|x(\cdot)\|_C = \max\left\{\|x(t)\| : t \in [t_0,\theta]\right\},$$

$h_n(\cdot,\cdot)$ and $h_C(\cdot,\cdot)$ stand for the Hausdorff distance between the subsets of the spaces $\mathbb{R}^n$ and $C\left([t_0,\theta];\mathbb{R}^n\right)$ respectively.

Let us formulate the propositions which will be used in following arguments.

Proposition 2. *Each control function $u(\cdot) \in L_2\left([t_0,\theta];\mathbb{R}^m\right)$ generates unique trajectory of the system* (2.1).

Denote

$$\alpha_* = \kappa\left[(\theta-t_0) + (\theta-t_0)^{1/2}c_*\right] \cdot \exp\kappa\left[(\theta-t_0) + c_*(\theta-t_0)^{1/2}\right] \tag{2.6}$$

where c_* is defined by (2.2).

The following proposition characterizes boundedness of the set of trajectories.

Proposition 3. *For every $\varepsilon \in [0,1]$ and $x(\cdot) \in X_\varepsilon(t_0,x_0)$ the inequality $\|x(\cdot)\|_C \leq \alpha_*$ holds.*

P r o o f. Let us choose an arbitrary $\varepsilon > 0$ and $x(\cdot) \in X_\varepsilon(t_0,x_0)$, generated by the control function $u(\cdot) \in U_\varepsilon$. According to the Condition 2.C, inequality (2.2) and Cauchy–Schwarz inequality we have

$$\|x(t)\| \leq \kappa\int_{t_0}^{t}(\|x(\tau)\|+1)(\|u(\tau)\|+1)\,d\tau$$

$$\leq \kappa\int_{t_0}^{t}(\|u(\tau)\|+1)\|x(\tau)\|\,d\tau + \kappa\left[(\theta-t_0) + (\theta-t_0)^{1/2}c_*\right]$$

for every $t \in [t_0, \theta]$. Applying Bellman–Gronwall inequality and Cauchy–Schwarz inequality and taking into consideration (2.2) and (2.6) we conclude from the last inequality

$$\begin{aligned} \|x(t)\| &\leq \kappa \left[(\theta - t_0) + (\theta - t_0)^{1/2} c_* \right] \cdot \exp \left[\kappa \int_{t_0}^{\theta} (\|u(\tau)\| + 1) \, d\tau \right] \\ &\leq \kappa \left[(\theta - t_0) + (\theta - t_0)^{1/2} c_* \right] \cdot \exp \kappa \left[(\theta - t_0) + c_* (\theta - t_0)^{1/2} \right] = \alpha_* \end{aligned} \tag{2.7}$$

for every $t \in [t_0, \theta]$. The inequality (2.7) completes the proof. □

Let

$$\psi(\delta) = \kappa(\alpha_* + 1) \left(\delta + c_* \delta^{1/2} \right), \quad \delta \geq 0. \tag{2.8}$$

It is obvious that $\psi(\delta) \to 0$ as $\delta \to 0^+$.

Proposition 4. *For every $\varepsilon \in [0,1]$, $x(\cdot) \in X_\varepsilon(t_0, x_0)$, $t_1 \in [t_0, \theta]$ and $t_2 \in [t_0, \theta]$ the inequality*

$$\|x(t_1) - x(t_2)\| \leq \psi(|t_1 - t_2|)$$

is verified, and hence

$$h_n(X_\varepsilon(t_1; t_0, x_0), X_\varepsilon(t_2; t_0, x_0)) \leq \psi(|t_1 - t_2|)$$

where $\psi(\cdot)$ is defined by (2.8).

P r o o f. Without loss of generality let us assume that $t_2 > t_1$. Choose an arbitrary $\varepsilon > 0$ and $x(\cdot) \in X_\varepsilon(t_0, x_0)$, generated by the control function $u(\cdot) \in U_\varepsilon$. According to the Condition 2.C, Proposition 3, (2.2) and (2.8) we have

$$\begin{aligned} \|x(t_2) - x(t_1)\| &\leq \kappa \int_{t_1}^{t_2} (\|x(\tau)\| + 1)(\|u(\tau)\| + 1) \, d\tau \leq \kappa(\alpha_* + 1) \int_{t_1}^{t_2} (\|u(\tau)\| + 1) \, d\tau \\ &\leq \kappa(\alpha_* + 1) \left[(t_2 - t_1) + (t_2 - t_1)^{1/2} c_* \right] = \psi(|t_2 - t_1|) \, . \end{aligned}$$

The proposition is proved. □

Proposition 3, Proposition 4 and Arzela-Ascoli theorem (see, e.g., [10, p. 102]) imply the validity of the following theorem.

Theorem 1. *For each $\varepsilon \in [0,1]$ the set of trajectories $X_\varepsilon(t_0, x_0)$ of the system (2.1) is a precompact subset of the space $C([t_0, \theta]; \mathbb{R}^n)$.*

Note that in general, the set of trajectories $X_\varepsilon(t_0, x_0)$ and $X_0^*(t_0, x_0)$ are not closed subsets of the space $C([t_0, \theta]; \mathbb{R}^n)$ (see, [3, 6]). Denote

$$\begin{aligned} B_n(\alpha_*) &= \{ x \in \mathbb{R}^n : \|x\| \leq \alpha_* \} \, , \\ D_n(\alpha_*) &= \{ (t, x) \in [t_0, \theta] \times \mathbb{R}^n : x \in B_n(\alpha_*) \} \, , \end{aligned}$$

where α_* is defined by equality (2.6).

Here and henceforth we will have in mind the cylinder $D_n(\alpha_*)$ as the set D in Condition 2.B.

3. Properties of the set of trajectories

Denote

$$\beta_* = \gamma_1(\theta - t_0) + 2\gamma_2 c_*(\theta - t_0)^{1/2}, \tag{3.1}$$

$$g_* = \gamma_3 c_*(\theta - t_0)^{1/2} \cdot \exp(\beta_*), \tag{3.2}$$

$$B_C(1) = \left\{x(\cdot) \in C\left([t_0,\theta];\mathbb{R}^n\right) : \|x(\cdot)\|_C \le 1\right\}, \tag{3.3}$$

where c_* is defined in (2.2).

Proposition 5. *For every $\varepsilon \in [0,1]$ the inequality*

$$h_C(X_\varepsilon(t_0,x_0), X_0(t_0,x_0)) \le g_*\left(1 - \frac{1}{\sqrt{1+\varepsilon}}\right)$$

holds.

P r o o f. Let us choose an arbitrary $x(\cdot) \in X_\varepsilon(t_0,x_0)$ generated by the control function $u(\cdot) \in U_\varepsilon$. Define new control function $u_0(\cdot) : [t_0,\theta] \to \mathbb{R}^m$, setting

$$u_0(t) = \frac{1}{\sqrt{1+\varepsilon}} u(t), \quad t \in [t_0,\theta]. \tag{3.4}$$

The equality (3.4) yields that $u_0(\cdot) \in U_0$. Now, from (2.2), (3.4) and Cauchy-Schwarz inequality it follows that

$$\|u(\cdot) - u_0(\cdot)\|_1 = \int_{t_0}^{\theta}\left(1 - \frac{1}{\sqrt{1+\varepsilon}}\right)\|u(\tau)\|\, d\tau \le c_*(\theta - t_0)^{1/2}\left(1 - \frac{1}{\sqrt{1+\varepsilon}}\right). \tag{3.5}$$

Let $x_0(\cdot) : [t_0,\theta] \to \mathbb{R}^n$ be the trajectory of the system (2.1) generated by the control function $u_0(\cdot) \in U_0$. Then $x_0(\cdot) \in X_0(t_0,x_0)$. From Condition 2.B, (2.1) and (3.5) it follows that

$$\begin{aligned} \|x(t) - x_0(t)\| \le \int_{t_0}^{t} &\left[\gamma_1 + \gamma_2(\|u(\tau)\| + \|u_0(\tau)\|)\right]\|x(\tau) - x_0(\tau)\|\, d\tau \\ &+ \gamma_3 c_*(\theta - t_0)^{1/2}\left(1 - \frac{1}{\sqrt{1+\varepsilon}}\right) \end{aligned} \tag{3.6}$$

for every $t \in [t_0,\theta]$.

Taking into consideration the inequality (2.2), Gronwall–Bellman inequality and Cauchy–Schwarz inequality, from (3.1), (3.2) and (3.6) we obtain

$$\begin{aligned} \|x(t) - x_0(t)\| &\le \gamma_3 c_*(\theta - t_0)^{1/2}\left(1 - \frac{1}{\sqrt{1+\varepsilon}}\right) \cdot \exp\left[\int_{t_0}^{\theta}\left[\gamma_1 + \gamma_2(\|u(\tau)\| + \|u_0(\tau)\|)\right] d\tau\right] \\ &\le \gamma_3 c_*(\theta - t_0)^{1/2}\left(1 - \frac{1}{\sqrt{1+\varepsilon}}\right) \cdot \exp(\beta_*) = g_*\left(1 - \frac{1}{\sqrt{1+\varepsilon}}\right) \end{aligned}$$

for every $t \in [t_0,\theta]$, and hence

$$\|x(\cdot) - x_0(\cdot)\|_C \le g_*\left(1 - \frac{1}{\sqrt{1+\varepsilon}}\right).$$

Since $x(\cdot) \in X_\varepsilon(t_0,x_0)$ is an arbitrarily chosen trajectory, $x_0(\cdot) \in X_0(t_0,x_0)$, then the last inequality implies that

$$X_\varepsilon(t_0,x_0) \subset X_0(t_0,x_0) + g_*\left(1 - \frac{1}{\sqrt{1+\varepsilon}}\right) B_C(1) \tag{3.7}$$

where $B_C(1)$ is defined by (3.3). From inclusion $X_0(t_0,x_0)\subset X_\varepsilon(t_0,x_0)$ and (3.7) we obtain the proof of the proposition. □

From Proposition 5 it follows the validity of the following corollaries.

Corollary 1. *$h_C(X_\varepsilon(t_0,x_0),X_0(t_0,x_0))\to 0$ as $\varepsilon\to 0^+$.*

Corollary 2. *For every $\varepsilon\in[0,1]$ and $t\in[t_0,\theta]$ the inequality*

$$h_n(X_\varepsilon(t;t_0,x_0),X_0(t;t_0,x_0))\le g_*\left(1-\frac{1}{\sqrt{1+\varepsilon}}\right)$$

is verified where the sets $X_\varepsilon(t;t_0,x_0)$, $\varepsilon\in[0,1]$, are defined by (2.3).

Denote

$$r_*=2\gamma_3 c_*^2\cdot\exp(\beta_*) \tag{3.8}$$

where β_* is defined by (3.1).

Proposition 6. *For every $\varepsilon\in[0,1]$ and $\alpha>0$ the inequality*

$$h_C(X_\varepsilon(t_0,x_0),X_\varepsilon^\alpha(t_0,x_0))\le\frac{r_*}{\alpha}$$

is satisfied where r_* is defined by (3.8).

P r o o f. Let us choose an arbitrary $\varepsilon\in[0,1]$ and $y(\cdot)\in X_\varepsilon(t_0,x_0)$ generated by the control function $v(\cdot)\in U_\varepsilon$. Define new control function $v_*(\cdot):[t_0,\theta]\to\mathbb{R}^m$, setting

$$v_*(t)=\begin{cases} v(t) & \text{if } \|v(t)\|\le\alpha\,,\\ \dfrac{v(t)}{\|v(t)\|}\cdot\alpha & \text{if } \|v(t)\|>\alpha.\end{cases} \tag{3.9}$$

Let

$$A_*=\{t\in[t_0,\theta]:\|v(t)\|>\alpha\}.$$

Then from (2.2) we have

$$\alpha^2\mu(A_*)\le\int_{A_*}\|v(\tau)\|^2\,d\tau\le\int_{t_0}^{\theta}\|v(\tau)\|^2\,d\tau\le c_*^2,$$

and hence

$$\mu(A_*)\le\frac{c_*^2}{\alpha^2} \tag{3.10}$$

where $\mu(A_*)$ stands for the Lebesgue measure of the set A_*.

Since $v(\cdot)\in U_\varepsilon$ and $\|v(\tau)\|>\alpha$ for every $\tau\in A_*$, then (3.9) implies that

$$\begin{aligned}&\int_{t_0}^{\theta}\langle B(\tau)v_*(\tau),v_*(\tau)\rangle\,d\tau\\ &=\int_{[t_0,\theta]\setminus A_*}\langle B(\tau)v(\tau),v(\tau)\rangle\,d\tau+\int_{A_*}\langle B(\tau)v(\tau),v(\tau)\rangle\cdot\frac{\alpha^2}{\|v(\tau)\|^2}\,d\tau\\ &\le\int_{[t_0,\theta]\setminus A_*}\langle B(\tau)v(\tau),v(\tau)\rangle\,d\tau+\int_{A_*}\langle B(\tau)v(\tau),v(\tau)\rangle\,d\tau\\ &=\int_{t_0}^{\theta}\langle B(\tau)v(\tau),v(\tau)\rangle\,d\tau\le 1+\varepsilon.\end{aligned} \tag{3.11}$$

Now, (3.9) and (3.11) yield that $v_*(\cdot) \in U_\varepsilon^\alpha$. Let $y_*(\cdot) : [t_0, \theta] \to \mathbb{R}^n$ be the trajectory of the system (2.1) generated by the control function $v_*(\cdot) \in U_\varepsilon^\alpha$. Then $y_*(\cdot) \in X_\varepsilon^\alpha(t_0, x_0)$. Now the condition 2.B, inclusions $v(\cdot) \in U_\varepsilon$, $v_*(\cdot) \in U_\varepsilon^\alpha$, (2.2), (3.9) and (3.10) imply that

$$
\begin{aligned}
\|y(t) - y_*(t)\| &\leq \int_{t_0}^{t} \left[\gamma_1 + \gamma_2(\|v(\tau)\| + \|v_*(\tau)\|)\right] \|y(\tau) - y_*(\tau)\| \, d\tau + \gamma_3 \int_{A_*} \|v(\tau) - v_*(\tau)\| \, d\tau \\
&\leq \int_{t_0}^{t} \left[\gamma_1 + \gamma_2(\|v(\tau)\| + \|v_*(\tau)\|)\right] \|y(\tau) - y_*(\tau)\| \, d\tau + \gamma_3 \cdot [\mu(A_*)]^{1/2} \left[\|v(\cdot)\|_2 + \|v_*(\cdot)\|_2\right] \\
&\leq \int_{t_0}^{t} \left[\gamma_1 + \gamma_2(\|v(\tau)\| + \|v_*(\tau)\|)\right] \|y(\tau) - y_*(\tau)\| \, d\tau + \frac{2\gamma_3 c_*^2}{\alpha}
\end{aligned}
$$

for every $t \in [t_0, \theta]$. The last inequality, the inclusions $v(\cdot) \in U_\varepsilon$, $v_*(\cdot) \in U_\varepsilon^\alpha$, Bellman–Gronwall inequality and (3.8) yield

$$
\|y(t) - y_*(t)\| \leq \frac{2\gamma_3 c_*^2}{\alpha} \cdot \exp\left(\int_{t_0}^{\theta} \left[\gamma_1 + \gamma_2(\|v(\tau)\| + \|v_*(\tau)\|)\right] d\tau\right) \leq \frac{2\gamma_3 c_*^2}{\alpha} \cdot \exp(\beta_*) = \frac{r_*}{\alpha}
$$

for every $t \in [t_0, \theta]$, and hence

$$
\|y(\cdot) - y_*(\cdot)\|_C \leq \frac{r_*}{\alpha}. \tag{3.12}
$$

Since $y(\cdot) \in X_\varepsilon(t_0, x_0)$ is an arbitrarily chosen trajectory, $y_*(\cdot) \in X_\varepsilon^\alpha(t_0, x_0)$, then from (3.12) it follows that

$$
X_\varepsilon(t_0, x_0) \subset X_\varepsilon^\alpha(t_0, x_0) + \frac{r_*}{\alpha} \cdot B_C(1), \tag{3.13}
$$

where the set $B_C(1)$ is defined by (3.3). Taking into consideration that $X_\varepsilon^\alpha(t_0, x_0) \subset X_\varepsilon(t_0, x_0)$, we obtain from (3.13) the proof of the proposition. □

Corollary 3. *$h_C(X_\varepsilon(t_0, x_0), X_\varepsilon^\alpha(t_0, x_0)) \to 0$ as $\alpha \to +\infty$ uniformly with respect to the $\varepsilon \in [0, 1]$.*

From Propositions 5 and 6 it follows the validity of the following theorem.

Theorem 2. *For every $\varepsilon \in [0, 1]$ and $\alpha > 0$ the inequality*

$$
h_C(X_0(t_0, x_0), X_\varepsilon^\alpha(t_0, x_0)) \leq g_* \left(1 - \frac{1}{\sqrt{1 + \varepsilon}}\right) + \frac{r_*}{\alpha}
$$

is satisfied where g_* and r_* are defined by (3.2) and (3.8) respectively.

Corollary 4. *For every $\varepsilon \in [0, 1]$, $\alpha > 0$ and $t \in [t_0, \theta]$ the inequality*

$$
h_n(X_0(t; t_0, x_0), X_\varepsilon^\alpha(t; t_0, x_0)) \leq g_* \left(1 - \frac{1}{\sqrt{1 + \varepsilon}}\right) + \frac{r_*}{\alpha}
$$

is satisfied where $X_\varepsilon^\alpha(t; t_0, x_0)$ is defined by (2.4).

4. Robustness of the trajectories

Let us discuss the robustness of the trajectories with respect to the fast consumption of the remaining control resource.

Proposition 7. *Let $\nu > 0$ be a given number, $Q_* \subset [a,b]$ be Lebesgue measurable set, $z(\cdot) \in X_0(t_0,x_0)$ be a trajectory of the system (2.1) generated by the control function $w(\cdot) \in U_0$,*

$$\int_{t_0}^{\theta} \langle B(\tau)w(\tau), w(\tau)\rangle d\tau = \sigma_* < 1,$$

the control function $w_*(\cdot) \in L_2([t_0,\theta];\mathbb{R}^m)$ be such that

$$\int_{t_0}^{\theta} \langle B(\tau)w_*(\tau), w_*(\tau)\rangle d\tau = 1, \quad w_*(t) = w(t), \quad t \in [t_0,\theta] \setminus Q_*,$$

and $z_*(\cdot)$ be the trajectory of the system (2.1) generated by the control function $w_*(\cdot)$. If

$$\mu(Q_*) \leq \left[\frac{\nu}{2c_*\gamma_3 \exp(\beta_*)}\right]^2, \tag{4.1}$$

then

$$\|z(\cdot) - z_*(\cdot)\|_C \leq \nu$$

where c_* is defined by (2.2), β_* is defined by (3.1).

P r o o f. Let us underline that the equality

$$\int_{t_0}^{\theta} \langle B(\tau)w_*(\tau), w_*(\tau)\rangle d\tau = 1$$

implies that $w_*(\cdot) \in U_0^*$ and hence $z_*(\cdot) \in X_0^*(t_0,x_0)$. From Condition 2.B, inclusions $w(\cdot) \in U_0$, $w_*(\cdot) \in U_0^*$, (2.2) and definition of the control function $w_*(\cdot)$ it follows that

$$\begin{aligned}
\|z(t) - z_*(t)\| &\leq \int_{t_0}^{t} \left[\gamma_1 + \gamma_2(\|w(\tau)\| + \|w_*(\tau)\|)\right] \|z(\tau) - z_*(\tau)\| \, d\tau + \gamma_3 \int_{Q_*} \|w(\tau) - w_*(\tau)\| \, d\tau \\
&\leq \int_{t_0}^{t} \left[\gamma_1 + \gamma_2(\|w(\tau)\| + \|w_*(\tau)\|)\right] \|z(\tau) - z_*(\tau)\| \, d\tau + \gamma_3 \cdot [\mu(Q_*)]^{1/2} \left[\|w(\cdot)\|_2 + \|w_*(\cdot)\|_2\right] \\
&\leq \int_{t_0}^{t} \left[\gamma_1 + \gamma_2(\|w(\tau)\| + \|w_*(\tau)\|)\right] \|z(\tau) - z_*(\tau)\| \, d\tau + 2\gamma_3 c_* \cdot [\mu(Q_*)]^{1/2}
\end{aligned}$$

for every $t \in [t_0,\theta]$. The last inequality, the inclusions $w(\cdot) \in U_0$, $w_*(\cdot) \in U_0^*$, Bellman–Gronwall inequality, (3.1) and (4.1) imply

$$\begin{aligned}
\|z(t) - z_*(t)\| &\leq 2\gamma_3 c_* [\mu(Q_*)]^{1/2} \cdot \exp\left(\int_{t_0}^{\theta} \left[\gamma_1 + \gamma_2(\|w(\tau)\| + \|w_*(\tau)\|)\right] d\tau\right) \\
&\leq 2\gamma_3 c_* [\mu(Q_*)]^{1/2} \cdot \exp(\beta_*) \leq \nu
\end{aligned}$$

for every $t \in [t_0,\theta]$, and consequently $\|z(\cdot) - z_*(\cdot)\|_C \leq \nu$.
The proof is completed. □

Theorem 3. *The equality*

$$h_C(X_0(t_0,x_0), X_0^*(t_0,x_0)) = 0$$

holds.

P r o o f. Let $\nu > 0$ be an arbitrary fixed number and let us choose an arbitrary trajectory $x(\cdot) \in X_0(t_0, x_0)$ of the system (2.1) generated by the control function $u(\cdot) \in U_0$. Assume that

$$\int_{t_0}^{\theta} \langle B(\tau)u(\tau), u(\tau)\rangle d\tau = \sigma_* < 1.$$

Let $Q_* \subset [t_0, \theta]$ be such that

$$\mu(Q_*) \leq \left[\frac{\nu}{2c_*\gamma_3 \exp(\beta_*)}\right]^2, \tag{4.2}$$

where c_* is defined by (2.2), β_* is defined by (3.1) and let

$$\int_{[t_0,\theta]\setminus Q_*} \langle B(\tau)u(\tau), u(\tau)\rangle \, d\tau = \sigma_1, \tag{4.3}$$

$$B_* = \int_{Q_*} B(\tau) \, d\tau. \tag{4.4}$$

It is obvious that B_* is positive definite $m \times m$ matrix and $\sigma_1 \leq \sigma_* < 1$.

Define new control function $v_0(\cdot) : [t_0, \theta] \to \mathbb{R}^m$, setting

$$v_0(t) = \begin{cases} u(t) & \text{if} \quad t \in [t_0, \theta] \setminus Q_*, \\ u_0 & \text{if} \quad t \in Q_*, \end{cases} \tag{4.5}$$

where $u_0 \in \mathbb{R}^m$ is such that

$$\langle B_* u_0, u_0\rangle \ = \ 1 - \sigma_1. \tag{4.6}$$

From (4.3), (4.4), (4.5) and (4.6) it follows that

$$\int_{t_0}^{\theta} \langle B(\tau)v_0(\tau), v_0(\tau) >\rangle \, d\tau = \int_{[t_0,\theta]\setminus Q_*} \langle B(\tau)u(\tau), u(\tau)\rangle \, d\tau \ + \int_{Q_*} \langle B(\tau)u_0, u_0\rangle \, d\tau$$
$$= \sigma_1 + \Big\langle \Big(\int_{Q_*} B(\tau) \, d\tau \Big) u_0, u_0 \Big\rangle = \sigma_1 + \langle B_* u_0, u_0\rangle = \sigma_1 + 1 - \sigma_1 = 1,$$

and hence $v_0(\cdot) \in U_0^*$. Let $x_0(\cdot)$ be the trajectory of the system (2.1) generated by the control function $v_0(\cdot) \in U_0^*$. Then $x_0(\cdot) \in X_0^*(t_0, x_0)$ and from (4.2) and Proposition 7 we obtain

$$\|x(\cdot) - x_0(\cdot)\|_C \leq \nu \, .$$

Since $x(\cdot) \in X_0(t_0, x_0)$ is an arbitrarily chosen trajectory, $x_0(\cdot) \in X_0^*(t_0, x_0)$, then the last inequality implies that

$$X_0(t_0, x_0) \subset X_0^*(t_0, x_0) + \nu \cdot B_C(1) \tag{4.7}$$

where $B_C(1)$ is defined by (3.3). Taking into consideration that $X_0^*(t_0, x_0) \subset X_0(t_0, x_0)$, from (4.7) we obtain that

$$h_C(X_0^*(t_0, x_0), X_0(t_0, x_0)) \leq \nu \, . \tag{4.8}$$

Since $\nu > 0$ is an arbitrarily fixed number, then (4.8) yields the proof of the theorem. □

From Theorem 3 we obtain the validity of the following corollaries.

Corollary 5. *The equality*

$$\mathrm{cl}\,(X_0(t_0, x_0)) = \mathrm{cl}\,(X_0^*(t_0, x_0))$$

is verified where cl *denotes the closure of a set.*

The Corollary 5 means that every trajectory $x(\cdot) \in X_0(t_0, x_0)$ of the system (2.1) can be approximated by the trajectory which is generated by full consumption of the control resource.

Corollary 6. *For every $t \in [t_0, \theta]$ the equality*

$$\mathrm{cl}\,(X_0(t; t_0, x_0)) = \mathrm{cl}\,(X_0^*(t; t_0, x_0))$$

is satisfied where $X_0(t; t_0, x_0)$ and $X_0^*(t; t_0, x_0)$ are defined by (2.3) *and* (2.5) *respectively.*

From Theorems 2 and 3 we obtain the validity of the following theorem.

Theorem 4. *For every $\varepsilon \in [0, 1]$ and $\alpha > 0$ the inequality*

$$h_C(X_0^*(t_0, x_0), X_\varepsilon^\alpha(t_0, x_0)) \leq g_* \left(1 - \frac{1}{\sqrt{1+\varepsilon}}\right) + \frac{r_*}{\alpha}$$

is satisfied where g_* and r_* are defined by (3.2) *and* (3.8) *respectively.*

5. Conclusion

The results asserting that a small perturbations in the quadratic integral constraints inspire a small deviation on the set of trajectories can be applied in mathematical modelling of the control systems where the total control resource is measured with small errors. According to the obtained results, it is possible to introduce a norm type geometric constraint along with quadratic type integral constraint where upper bound of the norm type geometric constraint is sufficiently large. Since the integrally constrained control functions are not geometrically constrained, this fact simplifies the structure of the set of control functions and allows to avoid geometrical unboundedness of the admissible control functions.

Robustness of the trajectories with respect to the fast and full consumption implies that it is reasonable to spend the control resource in economical mode, i.e. it is advisable to consume the control resource on the domains with sufficiently small Lebesgue measures in small portions. This yields that if you have a superfluous control resource and you want to get rid of this resource, then by spending all of the resource on the domain with sufficiently small Lebesgue measure, you will get a small deviation from the original system's trajectory.

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