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GRACEFUL CHROMATIC NUMBER OF SOME CARTESIAN PRODUCT GRAPHS[1]

I Nengah Suparta[a,†], **Mathiyazhagan Venkathacalam**[b,††], **I Gede Aris Gunadi**[a,†††], **Putu Andi Cipta Pratama**[a,††††]

[a]Department of Mathematics, Universitas Pendidikan Ganesha,
Jl. Udayana 11, Singaraja-Bali 81117, Indonesia

[b]Department of Mathematics, Kongunadu Arts and Science College,
Coimbatore–641029, Tamil Nadu, India

[†]nengah.suparta@undiksha.ac.id [††]venkatmaths@kongunaducollege.ac.in
[†††]igedearisgunadi@undiksha.ac.id [††††]andicipta25@gmail.com

Abstract: A graph $G(V,E)$ is a system consisting of a finite non empty set of vertices $V(G)$ and a set of edges $E(G)$. A (*proper*) *vertex colouring* of G is a function $f: V(G) \to \{1,2,\ldots,k\}$, for some positive integer k such that $f(u) \neq f(v)$ for every edge $uv \in E(G)$. Moreover, if $|f(u)-f(v)| \neq |f(v)-f(w)|$ for every adjacent edges $uv, vw \in E(G)$, then the function f is called *graceful colouring* for G. The minimum number k such that f is a graceful colouring for G is called the *graceful chromatic number* of G. The purpose of this research is to determine graceful chromatic number of Cartesian product graphs $C_m \times P_n$ for integers $m \geq 3$ and $n \geq 2$, and $C_m \times C_n$ for integers $m, n \geq 3$. Here, C_m and P_m are cycle and path with m vertices, respectively. We found some exact values and bounds for graceful chromatic number of these mentioned Cartesian product graphs.

Keywords: Graceful colouring, Graceful chromatic number, Cartesian product.

1. Introduction

A graph $G(V,E)$ is a system consisting of a finite non empty set of vertices $V(G)$ and a set of edges $E(G)$. Let G and H be two disjoint graphs. The *Cartesian product* of G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$, and edges $xy, uv \in V(G) \times V(H)$ are adjacent in $G \times H$, if $x = u$ and $yv \in E(H)$ or $y = v$ and $xu \in E(G)$. A (*proper*) *vertex colouring* of G is a way of colouring vertices in G such that each adjacent vertices are assigned to different colours.

If for a vertex colouring of G we have that every adjacent edges in G have different induced colours, then the vertex colouring is called *graceful*. We may think a graceful colouring of G as a function $f: V(G) \to \{1,2,\ldots,k\}$, for some positive integer k, such that for every edge $uv \in E(G)$ we have $f(u) \neq f(v)$, and for any vertex $u \in V(G)$ we have $|f(u)-f(v)| \neq |f(u)-f(w)|$ for every vertices $v, w \in V(G)$ which are adjacent to u. The absolute value $|f(u)-f(v)|$ for every $uv \in E(G)$, is the *induced label* of the edge $uv \in$E(G). In this sense, the terms colour and label are interchangeable. The smallest value of k for which the function f is a graceful vertex colouring of G is called the *graceful chromatic number* of G. The graceful colouring is a variation of graceful labeling which was introduced by Alexander Rosa in 1967 (see Gallian in [5]). Whereas, the notion of graceful colouring was introduced by Gary Chartrand in 2015, as a variant of the proper vertex k-colouring problem (see [3]). Since then, researches on graceful colouring numbers started to be celebrated.

Byers in [3] derived exact values for the graceful chromatic number of some graphs: path, cycle, wheel, and caterpillar; and introduced some bounds for certain connected regular graphs.

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Moreover, English, et al. in [4] invented graceful chromatic number of some classes of trees, and gave a lower bound for the graceful chromatic number of connected graphs with certain minimum degree. Mincu et al. in [6] derived graceful chromatic number of some well-known graph classes, such as diamond graph, Petersen graph, Moser spindle graph, Goldner-Harary graph, friendship graphs, and fan graphs. Graceful chromatic number of some particular unicyclic class graphs were presented by Alfarisi et al. (2019) in [1].

Furthermore, in 2022, Asy'ari et al. in [2] presented graceful chromatic numbers of several types of graphs, including star graphs, diamond graphs, book graphs. In addition, Asy'ari, et al. also stated some open problems. One of the problems is to determine the graceful chromatic number of some Cartesian product of certain graphs. Here we derive graceful chromatic number of Cartesian product graph $C_m \times P_n$, $m \geq 3, n \geq 2$, where C_m is the cycle with m vertices and P_n is the path with n vertices. The Cartesian product graph $C_m \times P_n$ is known as *prism* for $n = 2$ and as *generalized prism* for $n \geq 3$. We also introduce bounds for Cartesian product graph $C_m \times C_n$, $m, n \geq 3$.

To proceed with the main results, we need to introduce some introductory facts which will be beneficial for our further discussion.

Let G be a graph and x be a vertex of G. All vertex which are adjacent to x are called *the neighbors of* x, and denoted by $N(x)$. The degree of the vertex x, denoted by $\deg(x)$, is equal to the cardinality of $N(x)$, $\deg(x) = |N(x)|$. We will start with the following lemma.

Lemma 1. *Let G be a graph and u be a vertex in G with degree $d \geq 1$. Let f be a graceful colouring for G. If $f(u) = a$, $1 \leq a \leq d$, then there is a vertex $v \in N(u)$ with colour $f(v) \geq d + a$.*

P r o o f. Let $f(u) = a$ with $1 \leq a \leq d$. If $a = 1$, the smaller possible colours we can assign for the all d neighbors $v \in N(u)$ of u, are $2, 3, \ldots, d$ and the colour $d + 1$. This means that, there is a vertex $v \in N(u)$ with $f(v) \geq d + 1 = d + a$. We are done for the case $a = 1$.

Now, assume $f(u) = a$, $1 < a \leq d$. Note that the colours k and $2a-k$, for every $k, 1 \leq k \leq a-1$, can not be assigned simultaneously for the vertices in $N(u)$, since they give the same difference from the colour a. Therefore, the maximum number of colours we may assign from the first $2(a-1)$ smallest colours $\{k, 2a-k : 1 \leq k \leq a-1\}$ is equal to $a-1$. It implies that the remaining vertices in $N(u)$ which are not coloured yet, is at least $d-(a-1)$ vertices. The colours we need for these vertices are started from a colour $\geq 2a$. This means that the next $d-(a-1)$ smallest colours we should assign are $2a, 2a+1, \ldots, 2a+(d-(a-1)-1)$. So, there is a vertex $v \in N(u)$ such that its colour $f(v) \geq 2a + (d-(a-1)-1) = d + a$. □

In a specific case, the colour of a vertex u is equal to the degree of u, $f(u) = deg(u)$, we have the following corollary.

Corollary 1. *In a graph G with graceful colouring f, if the vertex u has degree $d \geq 1$ and colour d, then there is a vertex $v \in N(u)$ with colour $f(v) \geq 2d$.*

P r o o f. Let G be a graph and u be a vertex of G with $deg(u) = d$. Let f be a graceful colouring for G where $f(u) = d$. By Lemma 1, we found a neighbor v of u such that $f(v) \geq d + d = 2d$. □

The following result was introduced by Byers (2018) in [3].

Lemma 2 (Byers in [3]). *The graceful chromatic number of cycle C_n on $n \geq 3$ vertices is*

$$\chi_g(C_n) = \begin{cases} 4, & if \quad n \neq 5, \\ 5, & if \quad n = 5. \end{cases} \tag{1.1}$$

Then, we will introduce some terminologies related with certain ladder graphs.

A *ladder* of $2m$ vertices, $m \geq 2$, denoted by L_m, is the Cartesian product graph of the path on m vertices and the path on two vertices. The ladder L_2 is the cycle graph of four vertices. Assume that the vertices of L_m are $v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_m$ such that its edges are v_iv_{i+1}, w_iw_{i+1} : $1 \leq i \leq m-1, v_iw_i : 1 \leq i \leq m$. For $m \geq 4$, if the vertices v_1 and v_m, and the vertices w_1 and w_m are identified, then we obtain a prism $C_{m-1} \times P_2$. In this resulting $C_{m-1} \times P_2$, $v_1 = v_m$, $w_1 = w_m$, and edge $v_1w_1 = v_mw_m$. Due to this, we may call the ladder L_m as the *open* graph of $C_{m-1} \times P_2$ about the edge v_1w_1.

On the other side, let $C_m \times P_2, m \geq 3$, be a prism. This prism has vertex set $\{v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_m\}$ and edge set

$$\{v_iv_{i+1}, w_iw_{i+1} : 1 \leq i \leq m-1\} \cup \{v_1v_m, w_1w_m,\} \cup \{v_iw_i : 1 \leq i \leq m\}.$$

After opening $C_m \times P_2$ about the edge v_1w_1 into the ladder L_{m+1}, the vertices v_1 and w_1 copy themselves into two copies each; the first copy of v_1(resp. w_1) is adjacent with v_2(resp. w_2), and the second copy of v_1(resp. w_1) is adjacent with v_m(resp. w_m). These last vertex copies in the ladder L_{m+1} are named as v_{m+1} and w_{m+1}, respectively. Therefore, if f a colouring for the prism $C_m \times P_2$, then in the ladder L_{m+1} we have $f(v_1) = f(v_{m+1}$ as well as $f(w_1) = f(w_{m+1})$. In this case, we may also call $C_m \times P_2$ as the *closed* graph of L_{m+1} about the edges v_1w_1 and v_mw_m.

In the following lemma we will show that a ladder of $2m$ vertices, with $m \not\equiv 0 \pmod 4$, can not be gracefully coloured using 4 colours.

Lemma 3. *Using four different colours, the graph $C_m \times P_2$, with $m \geq 3$, $m \not\equiv 0 \pmod 4$, can not be gracefully coloured.*

P r o o f. Let a, b, c and d be four different colours, and let $m = 4k + r$, $1 \leq r \leq 3$. Consider the ladder L_{m+1} as the opened graph of $C_m \times P_2$. Let the vertex and edge sets of the ladder L_{m+1} be $\{v_i, w_i : 1 \leq i \leq m+1\}$ and $\{v_iv_{i+1}, w_iw_{i+1} : 1 \leq i \leq m, v_iw_i : 1 \leq i \leq m+1\}$, respectively. Observe that the colour of v_j (resp. w_j) must be the same with the colour of w_{j+2} (resp. v_{j+2}) or of w_{j-2} (resp. v_{j-2}) for realizable integer j (realizable means in the range of discussion). Without loss of generality, let the colour of v_1 is a. Therefore, the colour of w_{4s+3} and of v_{4t+1} is a, for some realizable non-negative integers s, t. Now let us see cases: $r = 1$, $r = 2$, and $r = 3$. Suppose that f is a graceful colouring for $C_m \times P_2$.

Case $r = 1$. If we take $t = k$, then we have $f(v_1) = a = f(v_{4k+1}) = f(v_m)$. Note that $v_{m+1} = v_{4k+2}$ is adjacent with v_m. Thus, $f(v_{m+1})$ can not be a to maintain proper colouring property. But, in $C_m \times P_2$, vertices v_1 and v_{m+1} are identical which insist $f(v_{m+1}) = f(v_1) = a$. This implies a contradiction. So, for $r = 1$ the graph $C_m \times P_2$ can not be gracefully coloured.

Case $r = 2$. Applying a similar argument, by assuming the colour of v_1 is a, we have that $f(w_{m+1}) = f(w_{4k+3}) = f(v_1) = a$. In graph $C_m \times P_2$, vertices w_1 and w_{m+1} are identical. On the other side, w_1 is adjacent with v_1, so that they can not get the same colour. Thus, a contradiction occurs.

Case $r = 3$. Again by using a similar reason, we have that $f(w_m) = f(w_{4k+3}) = f(v_1) = a$. We know that w_{m+1} in $C_m \times P_2$ is identified with w_1, and therefore is adjacent with both w_m and v_1. This implies that the induced edge colours of $v_1w_1 (= v_1w_{m+1})$ and w_1w_m are the same which then contradicts the gracefulness property.

In any case we have proven that $C_m \times P_2$, $m \not\equiv 0 (\text{mod } 4)$, can not be gracefully coloured using only 4 colours. □

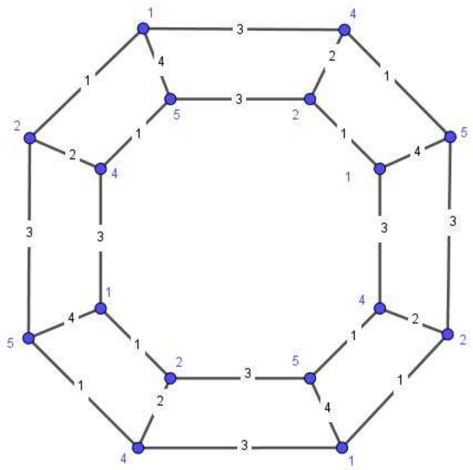

Figure 1. A graceful colouring of $C_8 \times P_2$.

2. Results on prism and generalized prism graphs

In this section, we will be dealing with the graceful chromatic number of prism $C_m \times P_2$ first, $m \geq 3$, and then with the graceful chromatic number of generalized prism graphs $C_m \times P_n$, $m, n \geq 3$. As for some consequences, we also derive some bounds for graceful chromatic number of graph $C_m \times C_n$, $m, n \geq 3$, for some specific values of m and n.

Our main discussion will be separated into two subsections: For $C_m \times P_2, m \geq 3$ and for $C_m \times P_n$, with $m, n \geq 3$.

2.1. Prism graph $C_m \times P_2$ for $m \geq 3$.

Theorem 1. *If* $m \equiv 0 \pmod 4$*, then the graceful chromatic number of graph* $C_m \times P_2$ *is equal to* 5.

P r o o f. Note that the graph $C_m \times P_2$ contains subgraph C_4. Based on Lemma 2, we may conclude that $\chi_g(C_m \times P_2) \geq 4$. Since all vertices of $C_m \times P_2$ has degree 3, if the colour 3 is used, then by Corollay 1, the colour greater than 6 should occur. Therefore, the four colours we will use are $1, 2, 4$, and 5. Now we will prove that using these four colours, we are able to colour $C_m \times P_2$ gracefully. To confirm this, we will do by introducing the following graceful colouring technique for $C_m \times P_2$ using only labels $1, 2, 4$, and 5.

Let the vertices of $C_m \times P_2$ is the set

$$\{v_{1+i}, v_{2+i}, v_{3+i}, v_{4+i}, w_{1+i}, w_{2+i}, w_{3+i}, w_{4+i} : \ i = 4k, \ k = 0, 1, 2, \ldots, m/4 - 1\}$$

and its edge set is

$$\{v_1 v_m, w_1 w_m, v_m w_m, v_i v_{i+1}, w_i w_{i+1}, v_i w_i : \ i = 1, 2, \ldots, m - 1\}.$$

Define a colouring f for $C_m \times P_2$ as follows.

$$f(v_i) = \begin{cases} 1, & \text{if} \quad i \equiv 1 \pmod 4, \\ 4, & \text{if} \quad i \equiv 2 \pmod 4, \\ 5, & \text{if} \quad i \equiv 3 \pmod 4, \\ 2, & \text{if} \quad i \equiv 0 \pmod 4, \end{cases} \qquad f(w_i) = \begin{cases} 5, & \text{if} \quad i \equiv 1 \pmod 4, \\ 2, & \text{if} \quad i \equiv 2 \pmod 4, \\ 1, & \text{if} \quad i \equiv 3 \pmod 4, \\ 4, & \text{if} \quad i \equiv 0 \pmod 4. \end{cases} \tag{2.1}$$

Based on the above function f, it is clear that for every adjacent vertices u and v we have $f(u) \neq f(v)$. We can immediately observe that for any adjacent edges uw and wv in C_m we have

$$\big\{|f(u) - f(w)|,\ |f(w) - f(v)|\big\} = \{1, 3\}.$$

Furthermore, we also have

$$\big\{|f(v_i) - f(w_i)| :\ 1 \leq i \leq m\big\} = \{2, 4\}.$$

Remember that each vertex u in $C_m \times P_2$ has degree 3; say x_1, x_2, and x_3 are the vertices adjacent to u. From the function f we can immediately conclude that the set

$$\big\{|f(u) - f(x_1)|,\ |f(u) - f(x_2)|,\ |f(u) - f(x_3)|\big\}$$

is equal to $\{1, 2, 3\}$ or to $\{1, 3, 4\}$. Thus, the function f satisfies the property to become graceful colouring for $C_m \times P_2$. Therefore, $\chi_g(C_m \times P_2) = 5$. □

Theorem 2. *If* $m \not\equiv 0 \pmod 4$*, then the graceful chromatic number of graph* $C_m \times P_2$ *is equal to* 6*.*

P r o o f. The proof of Theorem 2 will make use of the result described in the proof of Theorem 1.

For some positive integer $k \geq 1$, consider $C_{4k} \times P_2$ which is coloured as in (2.1). Let the ladder L_{4k+1} be the open graph of $C_{4k} \times P_2$ about v_1w_1. Since $C_m \times P_2$ contains subgraph C_4, to colour it gracefully, one needs at least 4 colours. But, when $m \equiv 1, 2$ or $3 \pmod 4$, based on Lemma 3, we can not colour the graph $C_{4k} \times P_2$ gracefully using only 4 colours. Therefore, we have to use at least 5 colours. The smallest five colours are $1, 2, 3, 4$, and 5. But, based on Corollary 1, whenever we apply 3 for a vertex colour, the colour 6 or greater colour must occur. Thus, the graceful chromatic number of $C_m \times P_2$ is at least 6. To conclude that $\chi_g(C_m \times P_2) = 6$, we will proceed by showing that a graceful colouring exist with maximum colour 6, as follows.

Case 1: $m \equiv 1 \pmod 4$. First, consider $C_5 \times P_2$ with vertex set $\{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5\}$ and with edge set $\{a_1a_5, b_1b_5, a_ia_{i+1}, b_ib_{i+1} : i = 1 \leq i \leq 4\} \cup \{a_ib_i : 1 \leq i \leq 5\}$. Now, we colour vertices using the following function f:

$$f(a_i) = \begin{cases} 1, & \text{if} \quad i = 1, \\ 4, & \text{if} \quad i = 2, \\ 3, & \text{if} \quad i = 3, \\ 5, & \text{if} \quad i = 4, \\ 2, & \text{if} \quad i = 5, \end{cases} \qquad f(b_i) = \begin{cases} 5, & \text{if} \quad i = 1, \\ 2, & \text{if} \quad i = 2, \\ 6, & \text{if} \quad i = 3, \\ 1, & \text{if} \quad i = 4, \\ 4, & \text{if} \quad i = 5. \end{cases}$$

The coloured $C_5 \times P_2$ will be used as the seed of our general construction for Case 1, and its diagram is depicted in Fig. 2.

Consider the opened ladder L_6 from the coloured $C_5 \times P_2$ above about a_1b_1. In L_6, the colours of a_1, a_2, a_3, a_4, a_5, and a_6 are $1, 2, 5, 3, 4$, and 1, while the colours of b_1, b_2, b_3, b_4, b_5, and b_6 are $5, 4, 1, 6, 2$, and 5.

Then, consider the open ladder L_{4k+1}, for some positive integer $k \geq 1$, from the coloured $C_{4k} \times P_2$ in Theorem 1 about v_1w_1. Here, the colours of v_1 and w_1 are also 1 and 5, respectively. The same colours are also for v_{4k+1} which is 1, and for w_{4k+1} which is 5. Based on (2.1), we have $f(v_{4k}) = 2$, and $f(w_{4k}) = 4$. By identifying v_{4k+1} with a_6 and w_{4k+1} with b_6, and maintaining the

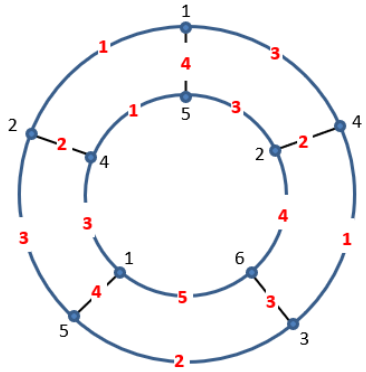

Figure 2. A graceful colouring of $C_5 \times P_2$.

other vertex colours, then we get a new ladder on $4(k+1)+2$ vertices, $L_{4(k+1)+2}$, with graceful colouring.

Furthermore, we know that $f(v_2) = 4$, $f(w_2) = 2$, $f(a_2) = 2$, and $f(b_2) = 4$. Thus by identifying v_1 with a_1 and w_1 with b_1 in the ladder $L_{4(k+1)+2}$, we obtain $C_{4(k+1)+1} \times P_2$ with a graceful colouring.

From here, we may infer that the graceful chromatic number of the graph $C_m \times P_2$, for $m \equiv 1 \pmod 4$ is equal to 6.

Case 2: $m \equiv 2 \pmod 4$. First, consider $C_6 \times P_2$ with vertex set

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\},$$

and with edge set

$$\{a_1a_6, b_1b_6, a_ia_{i+1}, b_ib_{i+1} : i = 1 \le i \le 5, \quad a_ib_i : 1 \le i \le 6\}.$$

As a seed graph, we define the following colouring for $C_6 \times P_2$ as follows.

$$f(a_i) = \begin{cases} 1, & \text{if} \quad i = 1, \\ 3, & \text{if} \quad i = 2, \\ 4, & \text{if} \quad i = 3, \\ 1, & \text{if} \quad i = 4, \\ 3, & \text{if} \quad i = 5, \\ 4, & \text{if} \quad i = 6, \end{cases} \qquad f(b_i) = \begin{cases} 5, & \text{if} \quad i = 1, \\ 6, & \text{if} \quad i = 2, \\ 2, & \text{if} \quad i = 3, \\ 5, & \text{if} \quad i = 4, \\ 6, & \text{if} \quad i = 5, \\ 2, & \text{if} \quad i = 6. \end{cases}$$

By inspection we can verify that the above colouring for $C_6 \times P_2$ is graceful. The diagram of the coloured graph is shown in Fig. 3.

Let the ladder of 7 vertices, L_7, is the open graph from the $C_6 \times P_2$ above about v_1w_1. We emphasize here that in this ladder L_7, vertices a_7 and b_7 have colours 1 and 5, respectively; the same as the colours of a_1 and b_1, respectively.

We use again the same ladder L_{4k+1}, $k \ge 1$, as in Case 1. Now we identify v_{4k+1} with a_7 and w_{4k+1} with b_7, and maintaining the other vertex colours. Then we get a new ladder on $4(k+1)+3$ vertices, $L_{4(k+1)+3}$, with graceful colouring.

Furthermore, we identify v_1 with a_1 and w_1 with b_1 in the ladder $L_{4(k+1)+3}$. Based on the previous colours, we know that the colours of $v_2, w_2, a_2, b_2, v_1 = a_1, w_1 = b_1$, are $4, 2, 3, 6, 1, 5$, respectively. This means that after the last identification, the gracefulness colouring of $C_{4(k+1)+2}$ are maintained. Thus, we may conclude that $C_{4(k+1)+2} \times P_2$ is with graceful colouring.

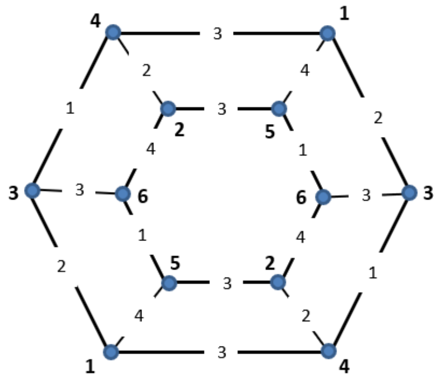

Figure 3. A graceful colouring of $C_6 \times P_2$.

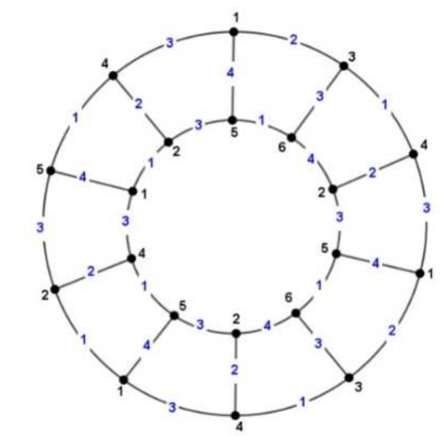

Figure 4. A graceful colouring of $C_{10} \times P_2$.

A graceful labeled $C_{10} \times P_2$ which is constructed using this method is depicted in Fig. 4.

From here, we may infer that the graceful chromatic number of the graph $C_m \times P_2$, for $m \equiv 2 \pmod 4$ is equal to 6.

Case 3: $m \equiv 3 \pmod 4$. Here we will introduce a construction for graceful colouring of $C_m \times P_2$ with $m \equiv 3 \pmod 4$. We start with $C_3 \times P_2$ with vertex set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ and edge set $\{a_3a_1, a_1a_2, a_2a_3, b_3b_1, b_1b_2, b_2b_3, a_1b_1, a_2b_2, a_3b_3\}$. Then we colour $C_3 \times P_2$ using the following colouring f.

$$f(a_i) = \begin{cases} 1, & \text{if} \quad i = 1, \\ 3, & \text{if} \quad i = 2, \\ 4, & \text{if} \quad i = 3, \end{cases} \qquad f(b_i) = \begin{cases} 5, & \text{if} \quad i = 1, \\ 6, & \text{if} \quad i = 2, \\ 2, & \text{if} \quad i = 3. \end{cases}$$

We can immediately check that this colouring f is graceful. The diagram of the gracefully coloured graph $C_3 \times P_2$ is shown in Fig. 5. We can verify that the graceful chromatic number of this graph is 6.

We should mention again that this above colouring of $C_3 \times P_2$ is graceful. As we did for Case 1 and Case 2, first we will observe the open ladder L_4 from $C_3 \times P_2$ about a_1b_1. In this L_4, the

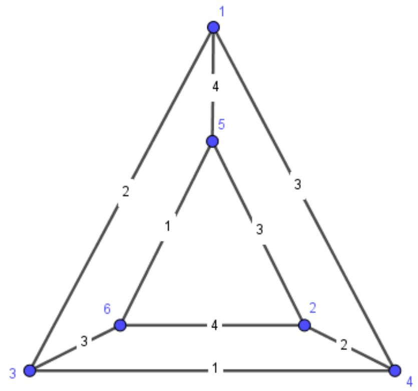

Figure 5. A graceful colouring of $C_3 \times P_2$.

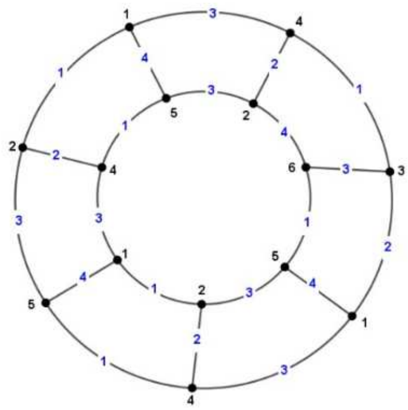

Figure 6. A graceful colouring of $C_7 \times P_2$.

colour of vertices $a_4 = a_1 = 1$ and $b_4 = b_1 = 5$. Observe back the open ladder L_{4k+1} in Case 1 (and Case 2).

Now we identify v_{4k+1} with a_4 and w_{4k+1} with b_4 to obtain a graceful colouring ladder L_{4k+4}. Let us denote the colouring as α. We can easily see that in this ladder we have $\alpha(a_1) = \alpha(v_1) = 1$ and $\alpha(b_1) = \alpha(w_1) = 5$. Moreover, we have also $\alpha(a_2) = f(a_2) = 3$, $\alpha(b_2) = f(b_2) = 6$, $\alpha(v_2) = 4$, and $\alpha(w_2) = 2$. Thus, by identifying v_1 with a_1 and w_1 with b_1, we get a graceful colouring $C_{4k+3} \times P_2$, with graceful chromatic number is 6. See the labeled graph $C_7 \times P_2$ in Fig. 6 as an example of the graph resulted from the construction.

Therefore, we may conclude that the graceful chromatic number of the graph $C_m \times P_2$, with $m \equiv 3 \pmod 4$ is also 6.

Since in all cases of m we proved that $C_m \times P_2$ has graceful chromatic number 6, we may conclude that $\chi_g(C_m \times P_2) = 6$. □

2.2. Results on generalized prism graphs $C_m \times P_n$, $m, n \geq 3$.

For a graph G, let f be a graceful colouring for G. It is obvious that for a vertex $u \in V(G)$, if $v, w \in N(u)$, then $f(v) \neq f(w)$. Therefore, we can immediately observe that the graph $P_3 \times P_3$ can not be coloured by only four different colours. This observation gives

$$\chi_g(P_3 \times P_3) \geq 5.$$

But, if we use only five colours $1, 2, 3, 4$ and 5, the center vertex of $P_3 \times P_3$ must be 1 or 5. Then, by inspection we can show that using only five colours, we can not colour $P_3 \times P_3$ gracefully. This gives the following lemma.

Lemma 4. *The graceful chromatic number of the graph $P_3 \times P_3$, $\chi_g(P_3 \times P_3) \geq 6$.*

The following Lemma 5 will be an important tool for the proofs of our main results encountered in this section.

Lemma 5. *The graceful chromatic number of the graph $P_5 \times P_5$, $\chi_g(P_5 \times P_5) \geq 7$.*

Proof. Let the vertices of $P_5 \times P_5$ be $V(P_5 \times P_5) = \{v_{ij} : i, j = 0, 1, 2, 3, 4\}$ and $E(P_5 \times P_5) = \{v_{ij}v_{i(j+1)}, v_{ij}v_{(i+1)j} : i, j = 0, 1, 2, 3\}$. Now, observe the subgraph $P_3 \times P_3$ with $V(P_3 \times P_3) = \{v_{ij} : i, j = 1, 2, 3\}$ and

$$E(P_3 \times P_3) = \{v_{ij}v_{(i+1)j}, v_{ij}v_{(i)(j+1)} : i, j = 1, 2\}.$$

In $P_5 \times P_5$, every vertex of the subgraph $P_3 \times P_3$ has degree 4. Based on Lemma 4, for gracefully colouring $P_3 \times P_3$, we need at least five colours. If the colour 3 or 4 is assigned for a vertex of $P_3 \times P_3$, then based on Lemma 1 the colour greater than or equal to $4 + 3 = 7$ must appear in $P_5 \times P_5$. If the colors 3 and 4 both are not assigned for any vertex of $P_3 \times P_3$, then, since we need at least five colours, we need some color greater than or equal to 7 for gracefully colouring $P_5 \times P_5$. □

Now, observe the graph $P_4 \times P_3$. We will make use of this observation for facilitating the result which will be formulated in Lemma 6. Let $V(P_4 \times P_3) = \{v_{ij} : i = 0, 1, 2, 3; j = 0, 1, 2\}$, and $E(P_4 \times P_3) = \{v_{ij}v_{i(j+1)} : i = 0, 1, 2, 3; j = 0, 1\} \cup \{v_{ij}v_{(i+1)j} : i = 0, 1, 2; j = 0, 1, 2\}$. The picture in Fig. 7 is the diagram of graph $P_4 \times P_3$ with vertex names.

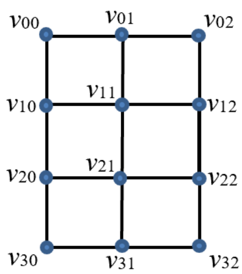

Figure 7. The graph $P_4 \times P_3$ with vertex names.

In here, we will restrict a vertex colouring α for $P_4 \times P_3$ as $\alpha(v_{0j}) = \alpha(v_{3j}), \forall j = 0, 1, 2$. We will show that under this restriction, using only six colours, the vertex colouring α can not be graceful.

Let the six colours be $1, 2, 3, 4, 5$ and 6. Based on Lemma 1, since the degree of vertices v_{11} and v_{21} each is four, the colours 3 and 4 both can not be used for these two vertices. So, there are four colours: $1, 2, 5$, and 6 that can be assigned for the vertices v_{11} and v_{21}. In total, there are six different combinations for colouring these two vertices: $\{\alpha(v_{11}), \alpha(v_{21})\} = \{a, b\}$, $a, b \in \{1, 2, 5, 6\}$, with $a \neq b$. We can check by inspection that any one of these combinations results in the colouring α is not graceful. But, for the space consideration, we will only describe the detail process for combination $\{\alpha(v_{11}), \alpha(v_{21})\} = \{1, 2\}$ as in Fig. 8. Note that the case $\alpha(v_{11}) = a$ and $\alpha(v_{21}) = b$ is similar to the case $\alpha(v_{11}) = b$ and $\alpha(v_{21}) = a$.

The explanation of the colouring process in Fig. 8 is the following:

1) The colours $\alpha(v_{11}) = 1$ and $\alpha(v_{21}) = 2$ are fixed as the initial combination.
2) The next vertex colouring follows the following vertices order: $v_{20}, v_{10}, v_{00}, v_{01}, v_{02}, v_{12}, v_{22}$. Note that $\alpha(v_{3j}) := \alpha(v_{0j}), \forall j = 0, 1, 2$, based on the restriction imposed for α.
3) For some colours x, y and z, a notation $x/\mathbf{y}/z$ means that we assign the colour y(indicated with bold face) for the related vertex among the possible colours x, y and z.
4) The colour which stands alone (written in red bold face), indicates that the colour is the only possible colour for the related vertex.
5) The red cross sign **X** informs that the colouring process is discontinue at the related vertex, since there is no possible choice of colours to colour the vertex. The appearance of **X** indicates that the colouring fails to be graceful.

From Fig. 8 we can see that each colouring process ends to be not graceful which is indicated by the appearance of the sign **X**. Thus, we may conclude that under the restriction $\alpha(v_{1j}) = \alpha(v_{4j}), j = 0, 1, 2$, using exactly six different colours, we can not colour the graph $P_4 \times P_3$ gracefully.

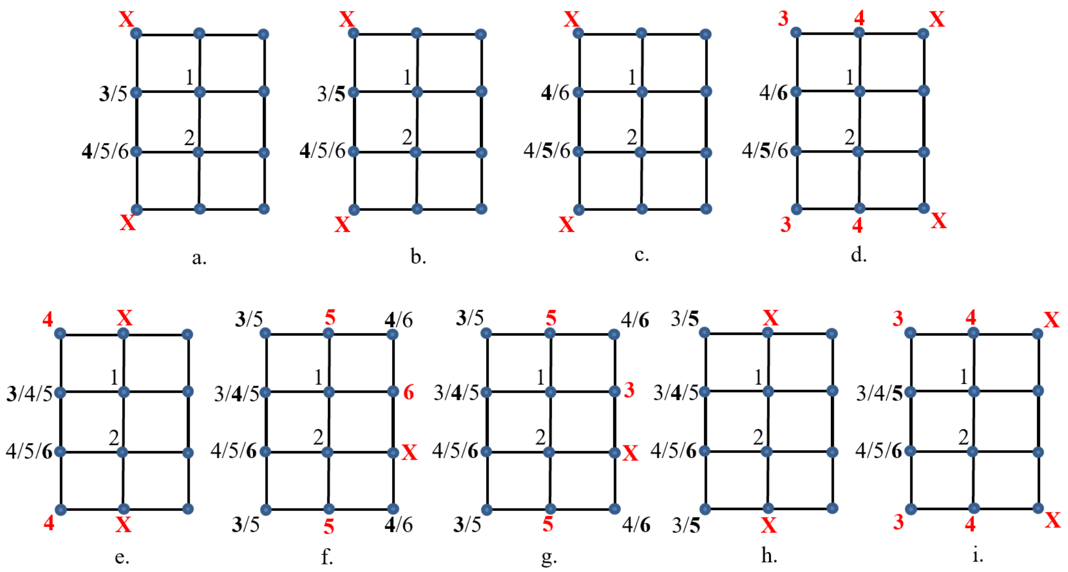

Figure 8. The colouring process for $P_4 \times P_3$ with $\alpha(v_{11}) = 1$ and $\alpha(v_{21}) = 2$.

If we extend this last observation to graph $P_4 \times P_n$, $n \geq 3$, with

$$V(P_4 \times P_n) = \{v_{ij} : i=0, 1, 2, 3; j=0, 1, \ldots, n-1\},$$

and

$$E(P_m \times P_n) = \{v_{ij}v_{i(j+1)} : i=0, 1, 2, 3; j=0, 1, \ldots, n-2\} \cup \{v_{ij}v_{(i+1)j} : i = 0, 1, 2; j = 0, 1, \ldots, n-1\},$$

under restriction that $\alpha(v_{0j}) = \alpha(v_{3j}), j = 0, 1, \ldots, n-1$, we may also conclude that we need at least seven colours to maintain the colouring α becomes graceful for $P_4 \times P_n$.
From this last observation we can formulate the following result.

Lemma 6. *For $n \geq 3$, the graceful chromatic number of the graph $C_3 \times P_n$, $\chi_g(C_3 \times P_n) \geq 7$.*

P r o o f. The generalized prism graph $C_3 \times P_n, n \geq 3$, can be obtained by identifying vertices v_{0j} and v_{3j} for every $j = 0, 1, 2, \ldots, n-1$ as it is in the last observation. By considering a graceful colouring α for the graph $P_4 \times P_n$ under the above mentioned restriction, we are done. □

For facilitating the discussion of our main results in this section, we need the following definition, as we defined a ladder as an open graph of $C_m \times P_2$ in the previous section. Here we will define a similarone as an open graph from the graph $C_m \times P_n$, $m, n \geq 3$. Let the vertex set of graph $C_m \times P_n$, $m, n \geq 3$, be

$$\{v_{ij}, 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\},$$

and its edge set be

$$\{v_{ij}v_{kl}, \text{ if } i = k \quad \text{and} \quad |j-l| = 1 \quad \text{or} \quad j = l \quad \text{and} \quad |i-k| \equiv 1 \pmod{m}\}.$$

Consider the open graph of $C_m \times P_n$, $m, n \geq 3$, about the path P which has end vertices v_{00} and v_{0n}, and has vertex set and edge set $\{v_{0j}, j = 0, 1, \ldots, n-1\}$ and $\{v_{0j}v_{0(j+1)}, j = 0, 1, \ldots, n-2\}$, respectively. Denote this open graph by $\mathcal{L}_{m+1,n}$. This graph is a grid graph having $(m+1) \times n$ vertices which involves two copies of path P. These two copies of path P, each has vertices $v_{0j}, j = 0, 1, \ldots, n-1$ and edges $v_{0j}v_{0(j+1)}, j = 0, 1, \ldots, n-2$. In the open graph $\mathcal{L}_{m+1,n}$, the vertices and edges of the second copy of P will be denoted by $v_{mj}, j = 0, 1, \ldots, n-1$, and $v_{mj}v_{(m)(j+1)}$, $j = 0, 1, \ldots, n-1$, respectively. It is clear that the vertex v_{mj} is adjacent with $v_{(m-1)j}$ for every $j = 0, 1, \ldots, n-2$. In this case, $C_m \times P_n$ can be reconstructed from $\mathcal{L}_{m+1,n}$ by identifying vertex v_{0j} and v_{mj} for every $j = 0, 1, \ldots, n-1$.

Theorem 3. *For any positive integers $m, n \geq 3$, with $m \equiv 0 \pmod 3$, $\chi_g(C_m \times P_n) = 7$.*

P r o o f. From Lemma 4 we know that the graceful chromatic number of $C_m \times P_n$ is at least seven. Now we will show that a graceful colouring exists for $C_m \times P_n$ such that it uses only seven different colours, and therefore $\chi_g(C_m \times P_n) = 7$.

Let the vertex set of $C_m \times P_n$ is $\{v_{ij} | 0 \leq i \leq m-1;\ 0 \leq j \leq n-1\}$, and edge set

$$\big\{v_{ij}v_{rs} | i = r \quad \text{and} \quad |s-j| \equiv 1 \pmod{n} \quad \text{or} \quad j = s \quad \text{and} \quad |i-r| \equiv 1 \pmod{m}\big\}.$$

To this end, here we define a colouring function f for $C_m \times P_n$ as follows.

$$f(v_{ij}) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 3, \quad j \equiv 0 \pmod 6, \\ 5, & \text{if } i \equiv 0 \pmod 3, \quad j \equiv 1 \pmod 6, \\ 6, & \text{if } i \equiv 0 \pmod 3, \quad j \equiv 2 \pmod 6, \\ 2, & \text{if } i \equiv 0 \pmod 3, \quad j \equiv 3 \pmod 6, \\ 3, & \text{if } i \equiv 0 \pmod 3, \quad j \equiv 4 \pmod 6, \\ 7, & \text{if } i \equiv 0 \pmod 3, \quad j \equiv 5 \pmod 6, \\ 3, & \text{if } i \equiv 1 \pmod 3, \quad j \equiv 0 \pmod 6, \\ 7, & \text{if } i \equiv 1 \pmod 3, \quad j \equiv 1 \pmod 6, \\ 1, & \text{if } i \equiv 1 \pmod 3, \quad j \equiv 2 \pmod 6, \\ 5, & \text{if } i \equiv 1 \pmod 3, \quad j \equiv 3 \pmod 6, \\ 6, & \text{if } i \equiv 1 \pmod 3, \quad j \equiv 4 \pmod 6, \\ 2, & \text{if } i \equiv 1 \pmod 3, \quad j \equiv 5 \pmod 6, \\ 6, & \text{if } i \equiv 2 \pmod 3, \quad j \equiv 0 \pmod 6, \\ 2, & \text{if } i \equiv 2 \pmod 3, \quad j \equiv 1 \pmod 6, \\ 3, & \text{if } i \equiv 2 \pmod 3, \quad j \equiv 2 \pmod 6, \\ 7, & \text{if } i \equiv 2 \pmod 3, \quad j \equiv 3 \pmod 6, \\ 1, & \text{if } i \equiv 2 \pmod 3, \quad j \equiv 4 \pmod 6, \\ 5, & \text{if } i \equiv 2 \pmod 3, \quad j \equiv 5 \pmod 6. \end{cases} \tag{2.2}$$

An example of a graceful coloured graph $C_6 \times P_n$ using (2.2) is shown in Fig. 9. In this figure we may also see the related open graph $\mathcal{L}_{7,n}$ of $C_6 \times P_n$.

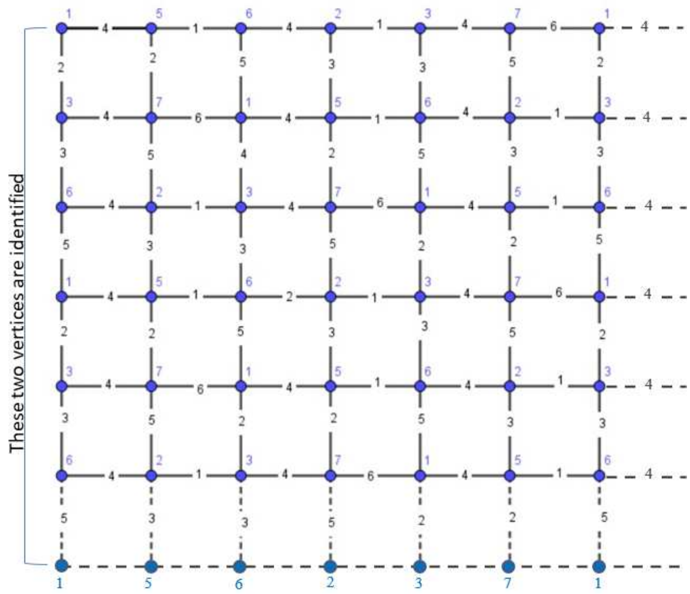

Figure 9. A graceful colouring of $C_6 \times P_n$, $n \geq 3$.

Fig. 9 also helps us to be able to check by inspection that f is a graceful colouring for the graph $C_m \times P_n$, with $m \equiv 0 \pmod 3$. Therefore, we may conclude that this graph has chromatic number 7. □

Furthermore, based on (2.2) we see that for every i, $0 \leq i \leq m-1$, we have $f(v_{ij}) = f(v_{ik})$ provided $|j-k| \equiv 0 \pmod 6$.

Corollary 2. *For any positive integers $m, n \geq 3$, with $m \equiv 0 \pmod 3$ and with $n \equiv 0 \pmod 6$, $\chi_g(C_m \times C_n) = 7$.*

P r o o f. The proof of this corollary may be derived from (2.2). From Theorem 3 we conclude that $\chi_g(C_m \times P_n) = 7$, if $m \equiv 0 \pmod 3$, and $n \geq 3$. From (2.2) we know that $f(v_{ij}) = f(v_{ik})$ whenever $|j-k| \equiv 0 \pmod 6$. Thus, if $n \equiv 0 \pmod 6$, then if we identify vertex v_{i0} and v_{in} for every $i, 0 \leq i \leq m-1$ in $C_m \times P_n$, then we get a graceful coloured graph $C_m \times C_n$, $m \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 6$. Therefore, we may conclude that $\chi_g(C_m \times C_n) = 7$ where $m \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 6$. □

In the remaining part of this section we will see the graceful colouring number for $C_m \times P_n$, with $m \not\equiv 0 \pmod 3$, $n \geq 3$. We start to observe the case $m \equiv 1 \pmod 3$ as we formulate in the following theorem.

Theorem 4. *If $m \equiv 1 \pmod 3$, then $7 \leq \chi_g(C_m \times P_n) \leq 8$.*

P r o o f. We will make use of prism graph $C_4 \times P_n$ as the seed of our graceful colouring construction. We first introduce a colouring for the graph $C_4 \times P_n$, $n \geq 3$.

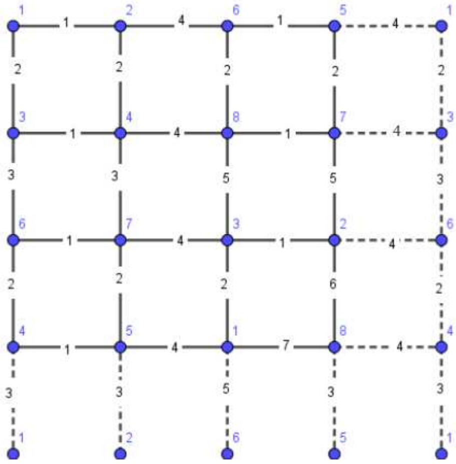

Figure 10. A graceful colouring of $C_4 \times P_n$.

Let the vertex set of $C_4 \times P_n$ is

$$\{v_{ij} | 0 \le i \le 3;\ 0 \le j \le n-1\},$$

and edge set

$$\{v_{ij}v_{rs} |\ i = r \quad \text{and} \quad |s-j| \equiv 1 \pmod{n} \quad \text{or} \quad j = s \quad \text{and} \quad |i-r| \equiv 1 \pmod{4}\}.$$

To this end, we define a colouring function f as follows.

$$f(v_{ij}) = \begin{cases} 1, & \text{if} \quad i \equiv 0 \pmod{4}, \quad j \equiv 0 \pmod{4}, \\ 2, & \text{if} \quad i \equiv 0 \pmod{4}, \quad j \equiv 1 \pmod{4}, \\ 6, & \text{if} \quad i \equiv 0 \pmod{4}, \quad j \equiv 2 \pmod{4}, \\ 5, & \text{if} \quad i \equiv 0 \pmod{4}, \quad j \equiv 3 \pmod{4}, \\ 3, & \text{if} \quad i \equiv 1 \pmod{4}, \quad j \equiv 0 \pmod{4}, \\ 4, & \text{if} \quad i \equiv 1 \pmod{4}, \quad j \equiv 1 \pmod{4}, \\ 8, & \text{if} \quad i \equiv 1 \pmod{4}, \quad j \equiv 2 \pmod{4}, \\ 7, & \text{if} \quad i \equiv 1 \pmod{4}, \quad j \equiv 3 \pmod{4}, \\ 6, & \text{if} \quad i \equiv 2 \pmod{4}, \quad j \equiv 0 \pmod{4}, \\ 7, & \text{if} \quad i \equiv 2 \pmod{4}, \quad j \equiv 1 \pmod{4}, \\ 3, & \text{if} \quad i \equiv 2 \pmod{4}, \quad j \equiv 2 \pmod{4}, \\ 2, & \text{if} \quad i \equiv 2 \pmod{4}, \quad j \equiv 3 \pmod{4}, \\ 4, & \text{if} \quad i \equiv 3 \pmod{4}, \quad j \equiv 0 \pmod{4}, \\ 5, & \text{if} \quad i \equiv 3 \pmod{4}, \quad j \equiv 1 \pmod{4}, \\ 1, & \text{if} \quad i \equiv 3 \pmod{4}, \quad j \equiv 2 \pmod{4}, \\ 8, & \text{if} \quad i \equiv 3 \pmod{4}, \quad j \equiv 3 \pmod{4}. \end{cases} \tag{2.3}$$

For an illustration one can see in Fig. 10

Fig. 10 helps us to see that (2.3) gives a graceful colouring for $C_4 \times P_n$ for every $n \ge 3$ with $\chi_g(C_4 \times P_n) \le 8$. Therefore, based on Lemma 4, we may conclude that $7 \le \chi_g(C_4 \times P_n) \le 8$.

Furthermore, the graceful colouring of $C_m \times P_n$, with $m \equiv 1 \pmod{3}$ and $n \ge 3$ in general, is obtained by extending graceful coloured graph $C_4 \times P_n$ using the prism graph $C_3 \times P_n$ which has colouring as we will show below.

Let the vertex set of $C_3 \times P_n$ is

$$\{v_{ij} |\ 0 \le i \le 2;\ 0 \le j \le n-1\},$$

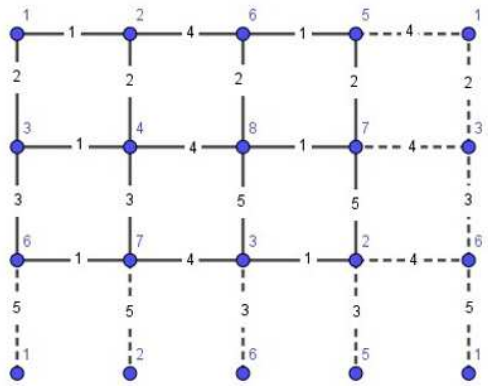

Figure 11. A graceful colouring of $C_3 \times P_n$.

and edge set

$$\{v_{ij}v_{rs}|\ i = r \quad \text{and} \quad |s-j| \equiv 1 \pmod n \quad \text{or} j = s \text{ and } |i-r| \equiv 1 \pmod 3\}.$$

To this end, we define a colouring function f as follows.

$$f(v_{ij}) = \begin{cases} 1, & \text{if} \quad i \equiv 0 \pmod 3, \quad j \equiv 0 \pmod 4, \\ 2, & \text{if} \quad i \equiv 0 \pmod 3, \quad j \equiv 1 \pmod 4, \\ 6, & \text{if} \quad i \equiv 0 \pmod 3, \quad j \equiv 2 \pmod 4, \\ 5, & \text{if} \quad i \equiv 0 \pmod 3, \quad j \equiv 3 \pmod 4, \\ 3, & \text{if} \quad i \equiv 1 \pmod 3, \quad j \equiv 0 \pmod 4, \\ 4, & \text{if} \quad i \equiv 1 \pmod 3, \quad j \equiv 1 \pmod 4, \\ 8, & \text{if} \quad i \equiv 1 \pmod 3, \quad j \equiv 2 \pmod 4, \\ 7, & \text{if} \quad i \equiv 1 \pmod 3, \quad j \equiv 3 \pmod 4, \\ 6, & \text{if} \quad i \equiv 2 \pmod 3, \quad j \equiv 0 \pmod 4, \\ 7, & \text{if} \quad i \equiv 2 \pmod 3, \quad j \equiv 1 \pmod 4, \\ 3, & \text{if} \quad i \equiv 2 \pmod 3, \quad j \equiv 2 \pmod 4, \\ 2, & \text{if} \quad i \equiv 2 \pmod 3, \quad j \equiv 3 \pmod 4. \end{cases} \tag{2.4}$$

The diagram of coloured graph $\mathcal{L}_{4,n}$ from $C_3 \times P_n$ is depicted in Fig. 11. The coloured graph $C_3 \times P_n$ is obtained by identifying v_{0j} and v_{3j} for all $j, 0 \le j \le n-1$. We can immediately observe that (2.4) gives a graceful colouring for the prism graph $C_3 \times P_n$ with $\chi_g(C_3 \times P_n) \le 8$. Again based on Lemma 4, we conclude that $7 \le \chi_g(C_3 \times P_n) \le 8$.

For producing a graceful colouring for $C_m \times P_n$, $m \equiv 1 \pmod 3$ we use $\mathcal{L}_{5,n}$ from $C_4 \times P_n$ and $\mathcal{L}_{4,n}$ from $C_3 \times P_n$, by identifying v_{5j} of $\mathcal{L}_{5,n}$ and v_{0j} of $\mathcal{L}_{4,n}$ for all j, $0 \le j \le n-1$. This identification results in a graceful coloured grid graph $\mathcal{L}_{8,n}$. Then, if we identify v_{8j} of $\mathcal{L}_{8,n}$ and v_{0j} of $\mathcal{L}_{4,n}$ for all $j, 0 \le j \le n-1$, we get a graceful coloured grid graph $\mathcal{L}_{11,n}$. Continuing the same procedure, then we get a graceful coloured grid graph $\mathcal{L}_{(m+1),n}$. Then by identifying vertex v_{0j} and v_{mj} from $\mathcal{L}_{(m+1),n}$ we obtain $C_m \times P_n$ with $m \equiv 1 \pmod 3$ and $n \ge 3$. □

As one consequence, as we formulated Corollary 2 based on Theorem 3, we also formulate a corollary based on Theorem 4 as the following.

Corollary 3. *If* $m \equiv 1 \pmod 3$ *and* $n \equiv 0 \pmod 4$, *then* $7 \le \chi_g(C_m \times C_n) \le 8$.

Now we go to the next case $m \equiv 2 \pmod 3$. The result is formulated in the following theorem.

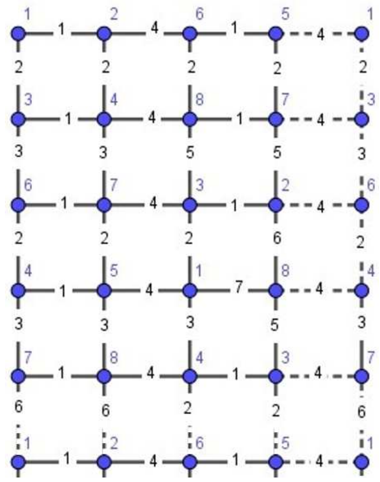

Figure 12. A graceful colouring of grid graph $\mathcal{L}_{6,n}$ of $C_5 \times P_n$.

Theorem 5. *If* $m \equiv 2 \pmod 3$, *then* $7 \le \chi_g(C_m \times P_n) \le 8$.

P r o o f. To proof this theorem, we will start with a graceful colouring for $C_5 \times P_n$, $m \equiv 2 \pmod 3$, $n \ge 3$. We introduce the following colouring for the graph $C_5 \times P_n$, $n \ge 3$.

$$f(v_{ij}) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 5, \; j \equiv 0 \pmod 4, \\ 2, & \text{if } i \equiv 0 \pmod 5, \; j \equiv 1 \pmod 4, \\ 6, & \text{if } i \equiv 0 \pmod 5, \; j \equiv 2 \pmod 4, \\ 5, & \text{if } i \equiv 0 \pmod 5, \; j \equiv 3 \pmod 4, \\ 3, & \text{if } i \equiv 1 \pmod 5, \; j \equiv 0 \pmod 4, \\ 4, & \text{if } i \equiv 1 \pmod 5, \; j \equiv 1 \pmod 4, \\ 8, & \text{if } i \equiv 1 \pmod 5, \; j \equiv 2 \pmod 4, \\ 7, & \text{if } i \equiv 1 \pmod 5, \; j \equiv 3 \pmod 4, \\ 6, & \text{if } i \equiv 2 \pmod 5, \; j \equiv 0 \pmod 4, \\ 7, & \text{if } i \equiv 2 \pmod 5, \; j \equiv 1 \pmod 4, \\ 3, & \text{if } i \equiv 2 \pmod 5, \; j \equiv 2 \pmod 4, \\ 2, & \text{if } i \equiv 2 \pmod 5, \; j \equiv 3 \pmod 4, \\ 4, & \text{if } i \equiv 3 \pmod 5, \; j \equiv 0 \pmod 4, \\ 5, & \text{if } i \equiv 3 \pmod 5, \; j \equiv 1 \pmod 4, \\ 1, & \text{if } i \equiv 3 \pmod 5, \; j \equiv 2 \pmod 4, \\ 8, & \text{if } i \equiv 3 \pmod 5, \; j \equiv 3 \pmod 4, \\ 7, & \text{if } i \equiv 4 \pmod 5, \; j \equiv 0 \pmod 4, \\ 8, & \text{if } i \equiv 4 \pmod 5, \; j \equiv 1 \pmod 4, \\ 4, & \text{if } i \equiv 4 \pmod 5, \; j \equiv 2 \pmod 4, \\ 3, & \text{if } i \equiv 4 \pmod 5, \; j \equiv 3 \pmod 4. \end{cases} \tag{2.5}$$

The diagram for coloured grid graph $\mathcal{L}_{6,n}$ of $C_5 \times P_n$, which is derived from (2.5), can be seen in Fig. 12. Using this diagram we may conclude that the colouring is graceful. It is clear that $\chi_g(C_5 \times P_n) \le 8$.

The process of expanding to get coloured graph $C_m \times P_n$, $m \equiv 2 \pmod 3$, $n \ge 3$, is similar to the previous process as was described in the proof of Theorem 4. Here we use graceful coloured grid graph $\mathcal{L}_{6,n}$ from graceful coloured graph $C_5 \times P_n$, and graceful coloured grid graph $\mathcal{L}_{4,n}$ from graceful coloured graph $C_3 \times P_n$. Again by considering Lemma 4,we then conclude that

$7 \leq \chi_g(C_5 \times P_n) \leq 8.$ □

Similar to the previous corollaries, here we formulate the following corollary as a consequence of Theorem 5.

Corollary 4. *If* $m \equiv 2 \pmod 3$ *and* $n \equiv 0 \pmod 4$, *then* $7 \leq \chi_g(C_m \times C_n) \leq 8$.

3. Conclusion

In the discussion above, it has been proven that prism graph $C_m \times P_2$ has a chromatic number equal to 5 when $m \equiv 0 \pmod 4$, and equal to 6 when $m \not\equiv 0 \pmod 4$. While for generalized prism $C_m \times P_n$ we found that its chromatic number is equal to 7 while $m \equiv 0 \pmod 3$. Whereas for $m \not\equiv 0 \pmod 3$, we got that $7 \leq \chi_g(C_m \times P_n) \leq 8$. Based on these results, we could also derive some exact and bound values of graceful chromatics number of $C_m \times C_n$ for certain $m, n \geq 3$. Regarding this last observation, we propose the following open problem and conjecture.

Conjecture. If $m \not\equiv 0 \pmod 3$ and $n \geq 3$, then $\chi_g(C_m \times P_n) = 8$.

Open problem. What is $\chi_g(C_m \times C_n)$, if $m \not\equiv 0 \pmod 3$, $m, n \geq 3$?

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