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A PRESENTATION FOR A SUBMONOID OF THE SYMMETRIC INVERSE MONOID

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Abstract: In the present paper, we study a submonoid of the symmetric inverse semigroup I_n. Specifically, we consider the monoid of all order-, fence-, and parity-preserving transformations of I_n. While the rank and a set of generators of minimal size for this monoid are already known, we will provide a presentation for this monoid.

Keywords: Symmetric inverse monoid, Order-preserving, Fence-preserving, Presentation.

1. Introduction

Let $\overline{n}$ be a finite chain with n elements, where n is a positive integer, denoted by $\overline{n} = \{1 < 2 < \cdots < n\}$. We denote by PT_n the monoid (under composition) of all partial transformations on $\overline{n}$. A partial transformation α on the set $\overline{n}$ is a mapping from a subset A of $\overline{n}$ into $\overline{n}$. The domain (respectively, image or range) of α is denoted by $dom(\alpha)$ (respectively, $im(\alpha)$). The empty transformation is denoted by ε. A transformation $\alpha \in PT_n$ is called order-preserving if $x < y$ implies $x\alpha \leq y\alpha$ for all $x, y \in dom(\alpha)$. It is worth noting that we write mappings on the right of their arguments and perform composition from left to right. Furthermore, an $\alpha \in PT_n$ is called a partial injection when α is injective. The set of all partial injections forms a monoid, the symmetric inverse semigroup I_n, as introduced by Wagner [17]. We denote by POI_n the submonoid of I_n, consisting of all order-preserving partial injections on $\overline{n}$. This monoid has already been well-studied (see e.g., [6]).

A non-linear order that is closed to a linear order in some sense is the so-called zig-zag order. The pair $(\overline{n}, \preceq)$ is called a zig-zag poset or fence if

$$1 \prec 2 \succ \cdots \prec n-1 \succ n \text{ if n is odd and } 1 \prec 2 \succ \cdots \succ n-1 \prec n \text{ if n is even, respectively.}$$

The definition of the partial order $\preceq$ is self-explanatory. A transformation $\alpha \in PT_n$ is referred to as fence-preserving if it preserves the partial order $\preceq$, meaning that for all $x, y \in dom(\alpha)$ with $x \prec y$, we have $x\alpha \preceq y\alpha$. The set of fence-preserving transformations on $\overline{n}$ was initially explored by Currie, Visentin, and Rutkowski. In [2, 14], the authors investigated the number of order-preserving maps of a finite fence. In particular, a formula for the number of order-preserving self-mappings of a fence was introduced. It is noteworthy that every element of a fence is either minimal or maximal. For all $x, y \in \overline{n}$ with $x \prec y$, we have $y \in \{x-1, x+1\}$. We denote by PFI_n the submonoid of I_n, consisting of all fence-preserving partial injections of $\overline{n}$. We denote by IF_n the inverse submonoid of PFI_n of all regular elements in PFI_n. It is easy to see that IF_n is the set of all $\alpha \in PFI_n$ with $\alpha^{-1} \in PFI_n$. It is worth mentioning that several properties of a variety of monoids of fence-preserving transformations were studied [3, 7, 9, 11, 12, 16].

In the present paper, we focus on a submonoid of $IOF_n = IF_n \bigcap POI_n$. Let $a \in dom(\alpha)$ for some $\alpha \in IOF_n$. If $a+1 \in dom(\alpha)$ or $a-1 \in dom(\alpha)$ then it is easy to verify that a and $a\alpha$ have the same parity. In other words, a is odd if and only if $a\alpha$ is odd. However, if $a-1$ and $a+1$ are not in $dom(\alpha)$, then a and $a\alpha$ can have different parity. In order to exclude this case, we require that the image of any $a \in dom(\alpha)$ has the same parity as $a\alpha$. In this context, we refer to α as parity-preserving. In our paper, we consider the monoid IOF_n^{par} of all parity-preserving transformations of IOF_n. Notably, for any $\alpha \in IOF_n^{par}$, the inverse partial injection α^{-1} exists and possesses order-preserving, fence-preserving, and parity-preserving. This observation implies that IOF_n^{par} is an inverse submonoid of I_n, as explained in [15]. Furthermore, in the same paper [15], the authors provided a characterization of the monoid IOF_n^{par}:

Proposition 1 [15]. *Let $p \leq n$ and let*

$$\alpha = \begin{pmatrix} d_1 & < & d_2 & < & \cdots & < & d_p \\ m_1 & & m_2 & & \cdots & & m_p \end{pmatrix} \in I_n.$$

Then $\alpha \in IOF_n^{par}$ if and only if the following four conditions hold:

(i) $m_1 < m_2 < ... < m_p$;
(ii) *d_1 and m_1 have the same parity;*
(iii) *$d_{i+1} - d_i = 1$ if and only if $m_{i+1} - m_i = 1$ for all $i \in \{1, ..., p-1\}$;*
(iv) *$d_{i+1} - d_i$ is even if and only if $m_{i+1} - m_i$ is even for all $i \in \{1, ..., p-1\}$.*

Also in [15], a set of generators of IOF_n^{par} of minimal size is given. This leads to the question of a presentation of IOF_n^{par}. In this paper, we will exhibit a monoid presentation for IOF_n^{par}. A monoid presentation is represented as an ordered pair $\langle X \mid R \rangle$, where X is a set, referred to as the alphabet (a set whose elements are called letters), and R is a binary relation on the free monoid generated by X, denoted by X^*. A pair $(u, v) \in X^* \times X^*$ is represented by $u \approx v$ and is called relation. We state that $u \approx v$, for $u, v \in X^*$, is a consequence of R if $(u, v) \in \rho_R$, where ρ_R denotes the congruence on X^* generated by R. We say that the momoid IOF_n^{par} has (monoid) presentation $\langle X \mid R \rangle$ if IOF_n^{par} is isomorphic to the factor semigroup X^*/ρ_R. For a more comprehensive understanding of semigroups, presentations, and standard notation see [1, 8, 10, 13].

Given that IOF_n^{par} is a finite monoid, we can always exhibit a presentation for it. A usual method to establish a good presentations is the Guess and Prove Method, which is described by the following theorem, adapted to monoids from Ruškuc (1995, Proposition 3.2.2).

Theorem 1 [13]. *Let X be a generating set for IOF_n^{par}, let $R \subseteq X^* \times X^*$ be a set of relations and let $W \subseteq X^*$ that the following conditions are satisfied:*

1. *The generating set X of IOF_n^{par} satisfies all the relations from R;*
2. *For each word $w \in X^*$, there exists a word $w' \in W$ such that the relation $w \approx w'$ is a consequence of R;*
3. $|W| \leq |IOF_n^{par}|$.

Then IOF_n^{par} is defined by the presentation $\langle X \mid R \rangle$.

In the next section, we introduce the alphabet (generating set) denoted as X_n and the binary relation R on X_n^*. Furthermore, we will demonstrate that X_n fulfills all the relations in R as outlined in Theorem 1, item 1. Following the guidance of item 2 in Theorem 1, we will establish a set of forms, denoted as P, in Section 3. Finally, in the last section, we will provide a proof for item 3 of Theorem 1.

2. The generator and relations

In this section, we will define the alphabet X_n and introduce a binary relation R on X_n^*. We will also demonstrate that the corresponding generating set satisfies all the relations in R. Let $\overline{v}_i$ be the partial identity with the domain $\overline{n}\backslash\{i\}$ for all $i \in \{1, ..., n\}$. Additionally, let us define

$$\overline{u}_i = \begin{pmatrix} 1 & \cdots & i & i+1 & i+2 & i+3 & i+4 & \cdots & n \\ 3 & \cdots & i+2 & - & - & - & i+4 & \cdots & n \end{pmatrix}$$

and $\overline{x}_i = (\overline{u}_i)^{-1}$ for all $i \in \{1, ..., n-2\}$. By Proposition 1, it is easy to verify that $\overline{u}_i$ as well as $\overline{x}_i$, $i \in \{1, ..., n-2\}$, belong to IOF_n^{par}. In [15], the authors have shown that $\{\overline{v}_1, \overline{v}_2, ..., \overline{v}_n, \overline{u}_1, \overline{u}_2, ..., \overline{u}_{n-2}, \overline{x}_1, \overline{x}_2, ..., \overline{x}_{n-2}\}$ is a generating set of IOF_n^{par}. In order to use Theorem 1, we define an alphabet

$$X_n = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_{n-2}, x_1, x_2, ..., x_{n-2}\},$$

which corresponds to the set of generators of IOF_n^{par}. For $w = w_1...w_m$ with $w_1, ..., w_m \in X_n$, where m being a positive integer, we write w^{-1} for the word $w^{-1} = w_m...w_1$.

We fix a particular sequence of letters as follows: $x_{i,j} = x_i x_{i+2} ... x_{i+2j-2}$ and $u_{i,j} = u_i u_{i+2} ... u_{i+2j-2}$ for $i \in \{1, ..., n-2\}$, $j \in \{1, ..., \lfloor (n-i)/2 \rfloor\}$ and obtain the following sets of words:

$$W_x = \left\{ x_{i,j} : i \in \{1, ..., n-2\},\ j \in \left\{1, ..., \left\lfloor \frac{n-i}{2} \right\rfloor \right\} \right\},$$

$$W_x^{-1} = \{x_{i,j}^{-1} : x_{i,j} \in W_x\},$$

$$W_u = \left\{ u_{i,j} : i \in \{1, ..., n-2\},\ j \in \left\{1, ..., \left\lfloor \frac{n-i}{2} \right\rfloor \right\} \right\}.$$

Let w be any word of the form $w = w_1...w_m$ with $w_1, ..., w_m \in W_x \cup W_u$ and m is a positive integer. For $k \in \{1, ..., m\}$, the word w_k is of the form

$$w_k = \begin{cases} u_{i_k, j_k} & \text{if} \quad w_k \in W_u; \\ x_{i_k, j_k} & \text{if} \quad w_k \in W_x \end{cases}$$

for some $i_k \in \{1, ..., n-2\}$, $j_k \in \{1, ..., \lfloor (n-i)/2 \rfloor\}$. We observe $j_k = |w_k|$, i.e. j_k is the length of the word w_k. We define two sequences $1_x, 2_x, ..., m_x$ and $1_u, 2_u, ..., m_u$ of indicators: for $k \in \{1, ..., m\}$ let

$$k_x = \begin{cases} i_k + 2|w_k| + 2|W_u^k| - 2|W_x^k| & \text{if} \quad w_k \in W_u; \\ i_k & \text{if} \quad w_k \in W_x \end{cases}$$

and

$$k_u = \begin{cases} i_k + 2|w_k| - 2|W_u^k| + 2|W_x^k| & \text{if} \quad w_k \in W_x; \\ i_k & \text{if} \quad w_k \in W_u, \end{cases}$$

where W_u^s (respectively, W_x^s) means the word $w_{s+1}...w_m$ without the letters in $\{x_1,...,x_{n-2}\}$ respectively, in $\{u_1,...,u_{n-2}\}$) for $s \in \{0,1,...,m-1\}$ and $W_u^m = W_x^m = \epsilon$, where ϵ is the empty word. Let Q_0 be the set of all words $w = w_1...w_m$ with $w_1,...,w_m \in W_x \cup W_u$ and m being a positive integer such that:

(1_q) If $w_k, w_l \in W_x$ then $i_k + 2j_k + 1 < i_l$ for $k < l \leq m$;
(2_q) If $w_k, w_l \in W_u$ then $i_k + 2j_k + 1 < i_l$ for $k < l \leq m$;
(3_q) If $w_k \in W_u$ then $i_k + 2j_k + 2 \leq (k+1)_u$ for $k \in \{1,...,m-1\}$ and $(k+1)_x - k_x \geq 2$;
(4_q) If $w_k \in W_x$ then $i_k + 2j_k + 2 \leq (k+1)_x$ for $k \in \{1,...,m-1\}$ and $(k+1)_u - k_u \geq 2$.

Let now $w = w_1...w_m \in Q_0$ and let $w^* = W_u^0(W_x^0)^{-1}$. Further, we define recursively a set A_w:

(5_q) If $m_u > m_x$ and $m_u + 2 \leq n$ then $A_m = \{m_u+2,...,n\}$,
if $m_u < m_x$ and $m_x + 2 \leq n$ then $A_m = \{m_x+2,...,n\}$,
otherwise $A_m = \emptyset$;
(6_q) If $w_k \in W_u$ then $A_k = A_{k+1} \cup \{i_k + 2j_k + 2,...,(k+1)_u - 1\}$ for $k \in \{1,...,m-1\}$,
if $w_k \in W_x$ then $A_k = A_{k+1} \cup \{k_u + 2,...,(k+1)_u - 1)\}$ for $k \in \{1,...,m-1\}$;
(7_q) If $1 \in \{1_x, 1_u\}$ then $A_w = A_1$,
if $1 < 1_u \leq 1_x$ then $A_w = A_1 \cup \{1,...,1_u - 1\}$,
if $1 < 1_x < 1_u$ then $A_w = A_1 \cup \{1_u - 1_x + 1,...,1_u - 1\}$.

For a set $A = \{i_1 < i_2 < \cdots < i_k\} \subseteq \overline{n}$, let $v_A = v_{i_1}v_{i_2}...v_{i_k}$ for some $k \in \{1,...,n\}$. Note that $v_\emptyset$ means the empty word ϵ. For convenience, we put $v_i = \epsilon$ for $i \geq n+1$. Let

$$W_n = \{v_A w^* : w \in Q_0, A \subseteq A_w\} \cup \{v_A : A \subseteq \overline{n}\}.$$

On the other hand, we will define now a set of relations. For this, let W_t be the set of all words of the form $u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}x_{j_{m+1}}$ with the following four properties:

(i) $l \in \{0,...,n-2\}$, and $m \in \{0,...,n-3\}$;
(ii) $i_0 < i_1 < \cdots < i_l \in \{1,...,n-2\}$;
(iii) $j_1 > j_2 > \cdots > j_m > j_{m+1} \in \{1,...,n-2\}$;
(iv) if $k \in \{i_0,...,i_{l-1}\}$ (respectively, $k \in \{j_2,...,j_{m+1}\}$) then $k+1, k+3 \notin \{i_1,...,i_l\}$ (respectively, $k+1, k+3 \notin \{j_1,...,j_m\}$) for all $k \in \{1,...,n-3\}$.

Then we define a sequence R of relations on X_n^* as follows: for $i,j \in \{1,...,n\}$ and $k = i + 2j - 2$, let

$$(E)\ x_iu_j \approx \begin{cases} v_1v_2v_{i+3}...v_{j+3}, & \text{if } i < j, j-i = 2,3; \\ v_1v_2v_{j+3}...v_{i+3}, & \text{if } i > j, i-j = 2,3; \\ v_1v_2v_{j+3}v_{j+4}, & \text{if } i > j, i-j = 1; \\ v_1v_2v_{j+2}v_{j+3}, & \text{if } i < j, j-i = 1; \\ v_1v_2v_{i+3}, & \text{if } i = j; \\ v_1v_2u_jx_{i+2}, & \text{if } i < j, j-i \geq 4; \\ v_1v_2u_{j+2}x_i, & \text{if } i > j, i-j \geq 4; \end{cases}$$

($L1$) $u_2u_1 \approx u_1u_2 \approx x_1x_2 \approx x_2x_1 \approx u_2^2 \approx x_2^2 \approx v_1v_2v_3v_4v_5$;
($L2$) $u_3u_2 \approx x_2x_3 \approx v_1v_2v_3v_4v_5v_6$;
($L3$) $u_iu_1 \approx v_1v_2u_i$ and $x_1x_i \approx v_3v_4x_i, i \geq 3$;
($L4$) $u_iu_2 \approx v_1v_2v_3u_i$ and $x_2x_i \approx v_3v_4v_5x_i, i \geq 4$;
($L5$) $u_iu_{i-1} \approx v_{i+3}u_{i-3}u_{i-1}$ and $x_{i-1}x_i \approx v_{i+3}x_{i-1}x_{i-3}, i \geq 4$;
($L6$) $u_iu_j \approx u_{j-2}u_i$ and $x_jx_i \approx x_ix_{j-2}, i > j \geq 3, i-j \geq 2$;

$(R1)$ $v_i^2 \approx v_i$, $i \in \{1, ..., n\}$;
$(R2)$ $v_i v_j \approx v_j v_i$, $i, j \in \{1, ..., n\}, i \neq j$;
$(R3)$ $v_i u_j \approx u_j v_i$ and $v_i x_j \approx x_j v_i$, $i \in \{j + 4, ..., n\}$;
$(R4)$ $v_i u_j \approx u_j v_{i+2}$ and $v_{i+2} x_j \approx x_j v_i$, $1 \leq i \leq j$;
$(R5)$ $v_i u_j \approx u_j$ and $x_j v_i \approx x_j$, $i \in \{j + 1, j + 2, j + 3\}$;
$(R6)$ $u_j v_i \approx u_j$ and $v_i x_j \approx x_j$, $i \in \{1, 2, j + 3\}$;
$(R7)$ $u_1^2 \approx x_1^2 \approx v_1 ... v_4$;
$(R8)$ $u_i^2 \approx u_{i-2} u_i$ and $x_i^2 \approx x_i x_{i-2}$, $i \geq 3$;
$(R9)$ $u_i u_{i+1} \approx u_{i-1} u_{i+1}$ and $x_{i+1} x_i \approx x_{i+1} x_{i-1}$, $i \in \{2, ..., n - 5\}$;
$(R10)$ $u_i u_{i+3} \approx v_{i+6} u_i u_{i+2}$ and $x_{i+3} x_i \approx v_{i+6} x_{i+2} x_i$, $i \leq n - 5$;
$(R11)$ $w \approx v_{i_0+1} v_{i_0+2} v_{i_0+3} u_{i_1} ... u_{i_l} x_{j_1} ... x_{j_m}$, $w = u_{i_0} u_{i_1} ... u_{i_l} x_{j_1} ... x_{j_m} x_{j_{m+1}} \in W_t$ with $j_{m+1} = i_0 + 2l - 2m$;
$(R12)$ $w \approx v_{i_0} v_{i_0+1} v_{i_0+2} v_{i_0+3} u_{i_1} ... u_{i_l} x_{j_1} ... x_{j_m}$, $w = u_{i_0} u_{i_1} ... u_{i_l} x_{j_1} ... x_{j_m} x_{j_{m+1}} \in W_t$ with $j_{m+1} = i_0 + 2l - 2m - 1$;
$(R13)$ $w \approx v_{i_0+1} v_{i_0+2} v_{i_0+3} v_{i_0+4} u_{i_1} ... u_{i_l} x_{j_1} ... x_{j_m}$, $w = u_{i_0} u_{i_1} ... u_{i_l} x_{j_1} ... x_{j_m} x_{j_{m+1}} \in W_t$ with $j_{m+1} = i_0 + 2l - 2m + 1$;
$(R14)$ $w \approx u_{i_0} u_{i_1} ... u_{i_l} x_{j_1} ... x_{j_m}$, $w = u_{i_0} u_{i_1} ... u_{i_l} x_{j_1} ... x_{j_m} x_{j_{m+1}} \in W_t$ with $j_{m+1} < 2l - 2m$;
$(R15)$ $w \approx u_{i_1} ... u_{i_l} x_{j_1} ... x_{j_m} x_{j_{m+1}}$, $w = u_{i_0} u_{i_1} ... u_{i_l} x_{j_1} ... x_{j_m} x_{j_{m+1}} \in W_t$ with $i_0 < 2m - 2l$;
$(R16)$ $v_1 ... v_i u_{i,j} \approx v_1 ... v_{k+3}$, $i \in \{1, ..., n - 2\}$;
$(R17)$ $v_{k-i+3} ... v_{k+2} x_{i,j}^{-1} \approx v_1 ... v_{k+3}$, $i \in \{1, ..., n - 2\}$;
$(R18)$ $v_i u_{i,j} \approx v_{k+3} u_{i-1,j}$, $i \in \{2, ..., n - 2\}$;
$(R19)$ $v_{k+2} x_{i,j}^{-1} \approx v_{k+3} x_{i-1,j}^{-1}$, $i \in \{2, ..., n - 2\}$.

Lemma 1. *The relations from R hold as equations in IOF_n^{par}, when the letters are replaced by the corresponding transformations.*

P r o o f. We show the statement diagrammatically. This method was also used in [4, 5]. We give an example calculation for the relation $(R10)$ $u_i u_{i+3} \approx v_{i+6} u_i u_{i+2}$, $i \leq n - 5$, in Figures 1 and 2 below. Note we can show $x_{i+3} x_i \approx v_{i+6} x_{i+2} x_i$ in a similar way. □

By Figures 1 and 2, we have that $\overline{u}_i \overline{u}_{i+3} = \overline{v}_{i+6} \overline{u}_i \overline{u}_{i+2}$.

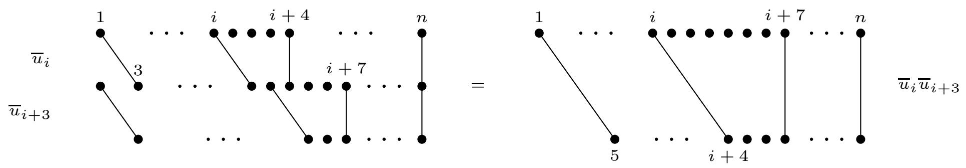

Figure 1. $\overline{u}_i \overline{u}_{i+3}$.

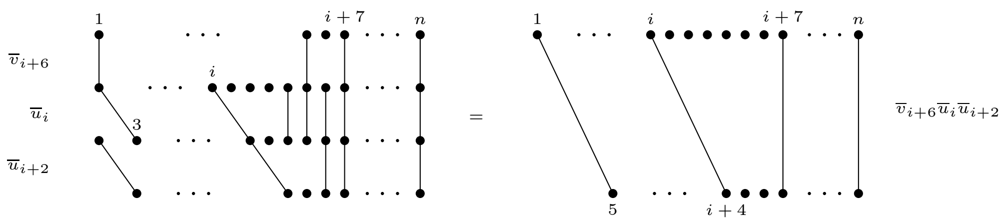

Figure 2. $\overline{v}_{i+6} \overline{u}_i \overline{u}_{i+2}$.

Next, we will verify consequences of R, which are important by technical reasons.

Lemma 2. *(i) For* $w = u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}x_{j_{m+1}} \in W_t$ *with* $j_{m+1} = 2l - 2m$, *we have* $w \approx v_1 u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}$.

(ii) For $w = u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}x_{j_{m+1}} \in W_t$ *with* $i_0 = 2m - 2l$, *we have* $w \approx v_{i_0+3}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}x_{j_{m+1}}$.

P r o o f. (i) We have

$$u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}x_{j_{m+1}} \overset{(R14)}{\approx} u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}x_{j_{m+1}-1}x_{j_{m+1}}.$$

Suppose $j_{m+1} = 2l - 2m \geq 4$. Then

$$\begin{aligned} u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}x_{j_{m+1}-1}x_{j_{m+1}} &\overset{(L5)}{\approx} u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}v_{j_{m+1}+3}x_{j_{m+1}-1}x_{j_{m+1}-3} \\ &\overset{(R4)}{\approx} v_1u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}x_{j_{m+1}-1}x_{j_{m+1}-3} \overset{(R14)}{\approx} v_1u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}. \end{aligned}$$

Suppose $j_{m+1} = 2l - 2m < 4$, i.e. $j_{m+1} = 2$. We prove that

$$u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}x_{j_{m+1}} \approx v_1u_{i_0}u_{i_1}...u_{i_l}x_{j_1}...x_{j_m}$$

by using $(L1)$ and $(R4)$–$(R6)$ in a similar way.

(ii) The proof is similar to (i), by using $(R15)$ and $(L5)$ if $i_0 \geq 4$ and $(R15)$, $(L1)$, and $(R4)$–$(R6)$ if $i_0 = 2$. □

3. Set of forms

In this section, we introduce an algorithm, which transforms any word $w \in X_n^*$ to a word in W_n using R, with other words, we show that for all $w \in X_n^*$, there is $w' \in W_n$ such that $w \approx w'$ is a consequence of R. First, the algorithm transforms each $w \in X_n^*$ to a "new" word w'. All these "new" words will be collected in a set. Later, we show that this set belongs to W_n. Let $w \in X_n^* \backslash \{\epsilon\}$.

- Using $(R1)$–$(R6)$, we can move any v_i for $i \in \{1, 2, ..., n\}$, at the beginning of the word or we can cancel it. So we obtain $w \approx \tilde{v}\tilde{w}$, where $\tilde{v} \in \{v_1, ..., v_n\}^*$ and $\tilde{w} \in \{u_1, u_2, ..., u_{n-2}, x_1, x_2, ..., x_{n-2}\}^*$.

- Moreover, we separate the u_i's and x_i's for $i \in \{1, ..., n-2\}$ by (E) and $(R1)$–$(R6)$. Then $\tilde{v}\tilde{w} \approx \overline{v}\overline{B}\overline{C}$, where $\overline{v} \in \{v_1, ..., v_n\}^*$, $\overline{B} \in \{u_1, u_2, ..., u_{n-2}\}^*$, and $\overline{C} \in \{x_1, x_2, ..., x_{n-2}\}^*$.

- By $(L1)$–$(L6)$ and $(R1)$–$(R6)$, we get $\overline{v}\overline{B}\overline{C} \approx v'B'C'$, where $v' \in \{v_1, ..., v_n\}^*$, $B' \in \{u_1, u_2, ..., u_{n-2}\}^*$, and $C' \in \{x_1, x_2, ..., x_{n-2}\}^*$ such that the indices of the letters in the word B' are ascending and in the word C' are descending (reading from the left to the right).

- By $(L1)$, $(R7)$–$(R10)$, and $(R1)$–$(R6)$, we replace subwords of $B'C'$ of the form $x_{i+3}x_i, x_{i+1}x_i, x_i^2, u_i^2, u_iu_{i+3}$, and u_iu_{i+1} until $v'B'C' \approx v''w_1...w_p$ with $v'' \in \{v_1, ..., v_n\}^*$ and $w_1, ..., w_p \in W_x^{-1} \cup W_u$ such that

  if $u_i \in var(w_1...w_p)$ (respectively, $x_i \in var(w_1...w_p)$) then $u_{i+1}, u_{i+3} \notin var(w_1...w_p)$ (respectively, $x_{i+1}, x_{i+3} \notin var(w_1...w_p)$) for all $i \in \{1, ..., n-2\}$ and each letter in $w_1...w_p$ is unique. $(*)$

  Note that this is possible since each of the relations $(L1)$, $(R7)$–$(R10)$, and $(R1)$–$(R6)$ does not increase the index of any letter in $\{u_1, u_2, ..., u_{n-2}, x_1, x_2, ..., x_{n-2}\}$ in the "new" word.

- Using $(R11)$–$(R15)$, Lemmas 2, and $(R1)$–$(R6)$, we remove letters x_i and u_i, respectively, until one can not more remove a letter x_i or u_i for $i \in \{1, 2, ..., n-2\}$. We obtain $v''w_1...w_p \approx v'''w'_1...w'_{p'}$, where $v''' \in \{v_1, ..., v_n\}^*$ and $w'_1, ..., w'_{p'} \in W_x^{-1} \cup W_u$. Note that is possible since each of the relations $(R11)$–$(R15)$ as well as Lemmas 2 only removes letters (and add letters in $\{v_1, ..., v_n\}$, respectively).

- We decrease the indices of the letters in $\{u_1, u_2, ..., u_{n-2}, x_1, x_2, ..., x_{n-2}\}$ (if possible) by $(R16)$–$(R19)$ as well as $(R1)$–$(R6)$ and obtain $v'''w'_1...w'_{p'} \approx v^*B^*C^*$ with $v^* \in \{v_1, ..., v_n\}^*$, $B^* \in \{u_1, u_2, ..., u_{n-2}\}^*$, and $C^* \in \{x_1, x_2, ..., x_{n-2}\}^*$. Note that the indices of the letters in B^* (respectively, in C^*) are ascending (respectively, are descending).

We repeat all steps. The procedure terminates if the word will not change more in all steps. We obtain $v^*B^*C^* \approx v_A\hat{w}_1...\hat{w}_{\hat{p}}$, where $\hat{w}_1, ..., \hat{w}_{\hat{p}} \in W_x^{-1} \cup W_u$ and $A \subseteq \overline{n}$ such that no v_j $(j \in A)$ can be canceled by using $(R1)$–$(R6)$. This case has to happen since the number of the letters from $\{u_1, u_2, ..., u_{n-2}, x_1, x_2, ..., x_{n-2}, v_1, ..., v_n\}$ decreases or is kept and the indices of the u_i's and x_i's decrease or are kept in each step.

We denote by P the set of all words obtained from $w \in X_n^*$ by that algorithm.

By $(*)$, we obtain immediately from the algorithm.

Remark 1. Let $\hat{w} = v_A\hat{w}_1...\hat{w}_m \in P$ and let $1 \leq k < k' \leq m$.
If $\hat{w}_k, \hat{w}_{k'} \in W_u$ then $i_k + 2|\hat{w}_k| + 2 \leq i_{k'}$.
If $\hat{w}_k, \hat{w}_{k'} \in W_x$ then $i_{k'} + 2|\hat{w}_{k'}| + 2 \leq i_k$.

Let fix a word $\hat{w} = v_A\hat{w}_1...\hat{w}_m \in P$. There are $a, b \in \{0, ..., n\}$ with $a + b = m$, $t_1, ..., t_{a+b} \in \{1, ..., m\}$, $w_{t_1}, ..., w_{t_a} \in W_u$ and $w_{t_{a+1}}, ..., w_{t_{a+b}} \in W_x$ such that

$$\hat{w} = v_A\hat{w}_1...\hat{w}_m = v_Aw_{t_1}...w_{t_a}w_{t_{a+1}}^{-1}...w_{t_{a+b}}^{-1},$$

where $\{w_{t_1}, ..., w_{t_a}\} = \emptyset$ or $\{w_{t_{a+1}}, ..., w_{t_{a+b}}\} = \emptyset$ (i.e. $a = 0$ or $b = 0$) is possible. We observe that $\{\hat{w}_1, ..., \hat{w}_m\} = \{w_{t_1}, ..., w_{t_a}, w_{t_{a+1}}^{-1}, ..., w_{t_{a+b}}^{-1}\}$ and $\{t_1, ..., t_a, t_{a+1}, ..., t_{a+b}\} = \{1, ..., m\}$. We define an order on $\{t_1, ..., t_a, t_{a+1}, ..., t_{a+b}\}$ by $t_1 < \cdots < t_a$ and $t_{a+b} < \cdots < t_{a+1}$. If $a, b \geq 1$, the order between $t_1, ..., t_a$ and $t_{a+1}, ..., t_{a+b}$ is given by the following rule:

Let $k \in \{1, ..., a\}$ and $l \in \{1, ..., b\}$
if $i_{t_k} + 2|w_{t_k}| - 2 + 2|w_{t_{k+1}}...w_{t_a}| - 2|w_{t_{a+1}}^{-1}...w_{t_{a+l-1}}^{-1}| < i_{t_{a+l}} + 2|w_{t_{a+l}}^{-1}| - 2$ then $t_k < t_{a+l}$ and

if $i_{t_k} + 2|w_{t_k}| - 2 + 2|w_{t_{k+1}}...w_{t_a}| - 2|w_{t_{a+1}}^{-1}...w_{t_{a+l-1}}^{-1}| > i_{t_{a+l}} + 2|w_{t_{a+l}}^{-1}| - 2$ then $t_k > t_{a+l}$.
The case

$$i_{t_k} + 2|w_{t_k}| - 2 + 2|w_{t_{k+1}}...w_{t_a}| - 2|w_{t_{a+1}}^{-1}...w_{t_{a+l-1}}^{-1}| = i_{t_{a+l}} + 2|w_{t_{a+l}}^{-1}| - 2$$

is not possible, since otherwise we can cancel $u_{i_{t_k}+2|w_{t_k}|-2}$ and $x_{i_{t_{a+l}}+2|w_{t_{a+l}}^{-1}|-2}$ in $\hat{w}$ by $(R11)$. Our next aim is to describe the relationships between $k_u, (k+1)_u$ and $k_x, (k+1)_x$ for all $k \in \{1, ..., m-1\}$ for the word $w = w_1...w_m$.

Lemma 3. *For all $k \in \{1, ..., m-1\}$, we have $k_u < (k+1)_u$ and $k_x < (k+1)_x$.*

P r o o f. Let $k \in \{1, ..., m-1\}$. Suppose $w_k, w_{k+1} \in W_u$. We obtain $k_u < (k+1)_u$ and

$$k_x = i_k + 2|w_k| + 2|W_u^k| - 2|W_x^k|,$$
$$(k+1)_x = i_{k+1} + 2|w_{k+1}| + 2|W_u^{k+1}| - 2|W_x^{k+1}|.$$

By Remark 1, we have $i_k + 2|w_k| + 2 \le i_{k+1}$. This gives

$$i_k + 2|w_k| + 2|W_u^k| - 2|W_x^k| < i_{k+1} + 2|W_u^k| - 2|W_x^k| = i_{k+1} + 2|w_{k+1}| + 2|W_u^{k+1}| - 2|W_x^{k+1}|$$

(since $w_{k+1} \in W_u$ implies $2|W_x^k| = 2|W_x^{k+1}|$). Then $k_x < (k+1)_x$. For the case $w_k, w_{k+1} \in W_x$, we can show that $k_u < (k+1)_u$ and $k_x < (k+1)_x$ in a similar way.

Suppose $w_k \in W_u$ and $w_{k+1} \in W_x$. First, we will show $k_u < (k+1)_u$. We have $k_u = i_k$ and

$$(k+1)_u = i_{k+1} + 2|w_{k+1}| + 2|W_x^{k+1}| - 2|W_u^{k+1}|.$$

Since $k \in \{t_1, ..., t_a\}$ and $k+1 \in \{t_{a+1}, ..., t_{a+b}\}$, we obtain

$$i_k + 2|w_k| - 2 + 2|W_u^k| - 2|W_x^{k+1}| < i_{k+1} + 2|w_{k+1}| - 2.$$

Then

$$i_k < i_k + 2|w_k| < i_{k+1} + 2|w_{k+1}| + 2|W_x^{k+1}| - 2|W_u^{k+1}|$$

(since $w_{k+1} \in W_x$ implies $|W_u^k| = |W_u^{k+1}|$). Then $k_u < (k+1)_u$. Moreover, we prove $k_x < (k+1)_x$ similarly. The case $w_k \in W_x$ and $w_{k+1} \in W_u$ can be shown in a similar way as above. □

Of course, the next goal should be the proof of $w = w_1 ... w_m \in Q_0$, i.e. we will show that w satisfies (1_q)–(4_q).

Lemma 4. *We have $w = w_1 ... w_m \in Q_0$.*

P r o o f. Exactly, w satisfies (1_q) and (2_q). This is trivially checked by Remark 1.

Let $k \in \{1, ..., m-1\}$ and let $w_k \in W_u, w_{k+1} \in W_x$. This provides $k \in \{t_1, ..., t_a\}$, $k+1 \in \{t_{a+1}, ..., t_{a+b}\}$. We have

$$i_k + 2|w_k| - 2 + 2|W_u^k| - 2|W_x^{k+1}| < i_{k+1} + 2|w_{k+1}| - 2.$$

Since $w_{k+1} \in W_x$, we have

$$2|W_u^k| = 2|W_u^{k+1}|.$$

So

$$i_k + 2|w_k| - 2 + 2|W_u^{k+1}| - 2|W_x^{k+1}| < i_{k+1} + 2|w_{k+1}| - 2.$$

We observe that

$$i_k + 2|w_k| - 2 + 2|W_u^{k+1}| - 2|W_x^{k+1}| + 1 \le i_{k+1} + 2|w_{k+1}| - 2.$$

If

$$i_k + 2|w_k| - 2 + 2|W_u^{k+1}| - 2|W_x^{k+1}| + 1 = i_{k+1} + 2|w_{k+1}| - 2,$$

we can cancel $u_{i_k + 2|w_k| - 2}, x_{i_{k+1} + 2|w_{k+1}| - 2}$ by $(R13)$ in $\hat{w}$. This contradicts $\hat{w} \in P$. Then

$$i_k + 2|w_k| - 2 + 2|W_u^{k+1}| - 2|W_x^{k+1}| + 2 \le i_{k+1} + 2|w_{k+1}| - 2,$$

i.e.

$$i_k + 2|w_k| + 2 \le i_{k+1} + 2|w_{k+1}| - 2|W_u^{k+1}| + 2|W_x^{k+1}| = (k+1)_u.$$

Next, to show that $(k+1)_x - k_x \ge 2$. Lemma 3 gives $(k+1)_x - k_x \ge 1$.

If $(k+1)_x - k_x = 1$ then

$$i_{k+1} - i_k - 2|w_k| - 2|W_u^k| + 2|W_x^k| = 1.$$

This implies

$$i_{k+1}+2|w_{k+1}|-2=i_k+2|w_k|-2+2|W_u^k|-2|W_x^{k+1}|+1$$

since

$$2|W_x^k|=2|W_{k+1}|+2|W_x^{k+1}|.$$

We can cancel $u_{i_k+2|w_k|-2}, x_{i_{k+1}+2|w_{k+1}|-2}$ in $\hat{w}$ by $(R13)$. This contradicts $\hat{w}\in P$. Thus, $(k+1)_x-k_x\geq 2$. In case $w_k, w_{k+1}\in W_u$, by using Remark 1, we easily get

$$i_k+2|w_k|+2\leq (k+1)_u.$$

To show $(k+1)_x-k_x\geq 2$, it is routine to calculate directly. Together with Remark 1, we will get that $(k+1)_x-k_x\geq 2$. Altogether, w satisfies (3_q). We prove that w satisfies (4_q) in a similar way. Therefore, $w\in Q_0$. □

We have shown $w\in Q_0$. This leads us to the next step, showing that $A\subseteq A_w$. First, we point out subsets of $\overline{n}$, which do not contain any element of A.

Lemma 5. *Let $q\in\{1,...,a\}$ and let*

$$\rho\in\{i_{t_q}+1,...,i_{t_q}+2|w_{t_q}|+1\}\cap\overline{n}.$$

Then $\rho\notin A$.

P r o o f. Assume $\rho\in A$. Then

$$v_\rho w_{t_1}...w_{t_q}...w_{t_a}w_{t_{a+1}}^{-1}...w_{t_{a+b}}^{-1}\overset{(R3)}{\approx} w_{t_1}...v_\rho w_{t_q}...w_{t_a}w_{t_{a+1}}^{-1}...w_{t_{a+b}}^{-1}.$$

If $\rho\in\{i_{t_q}+1, i_{t_q}+2, i_{t_q}+3\}\cap\overline{n}$ then

$$v_\rho u_{i_{t_q}}\overset{(R5)}{\approx} u_{i_{t_q}}.$$

If $\rho=i_{t_q}+h+t$ for some $h\in\{2,4,...,2|w_{t_q}|-2\}$ and $t\in\{2,3\}$ then

$$\begin{aligned}
w_{t_1}...v_\rho w_{t_q}...w_{t_a}w_{t_{a+1}}^{-1}...w_{t_{a+b}}^{-1} &= w_{t_1}...v_\rho u_{i_{t_q}}u_{i_{t_q}+2}...u_{i_{t_q}+2|w_{t_q}|-2}w_{t_{q+1}}...w_{t_a}w_{t_{a+1}}^{-1}...w_{t_{a+b}}^{-1}\\
&\overset{(R3)}{\approx} w_{t_1}...u_{i_{t_q}}...v_{(i_{t_q}+h+t)}u_{i_{t_q}+h}...u_{i_{t_q}+2|w_{t_q}|-2}w_{t_{q+1}}...w_{t_a}w_{t_{a+1}}^{-1}...w_{t_{a+b}}^{-1}\\
&\overset{(R5)}{\approx} w_{t_1}...u_{i_{t_q}}...u_{i_{t_q}+h}...u_{i_{t_q}+2|w_{t_q}|-2}w_{t_{q+1}}...w_{t_a}w_{t_{a+1}}^{-1}...w_{t_{a+b}}^{-1},
\end{aligned}$$

i.e. we can cancel v_ρ in $\hat{w}$ using $(R3)$ and $(R5)$, a contradiction. □

Lemma 6. *Let $\rho\in A$ and let $q\in\{1,...,a\}$ such that $t_q\neq m$. If $\rho\in\{(t_q)_u+1,...,(t_q+1)_u-1\}$ then*

$$\rho\in\left\{(t_q)_u+2|w_{t_q}|+2,...,(t_q+1)_u-1\right\}\subseteq A_w.$$

P r o o f. We have $(t_q)_u=i_{t_q}$. It is a consequence of Lemma 5 that

$$\rho\in\left\{i_{t_q}+2|w_{t_q}|+2,...,(t_q+1)_u-1\right\}$$

and by (6_q), we have

$$\left\{i_{t_q}+2|w_{t_q}|+2,...,(t_q+1)_u-1\right\}\subseteq A_w.$$

□

Lemma 7. *Let $\rho \in A$, if $t_a = m$ and $\rho \in \{i_m + 1, ..., n\}$ then $\rho \in \{m_x + 2, ..., n\} \subseteq A_w$.*

P r o o f. Assume $\rho \in \{i_m + 1, ..., m_x + 1\}$. We have $m_x + 1 = i_{t_a} + 2|w_{t_a}| + 1$. Then $\rho \in \{i_{t_a} + 1, ..., i_{t_a} + 2|w_{t_a}| + 1\}$. By Lemma 5, we have $\rho \notin A$. Therefore, $\rho \in \{m_x + 2, ..., n\} \subseteq A_w$ by (5_q). □

Lemma 8. *Let $\rho \in A$, then $\rho \neq (t_{a+l})_u + 1$ for all $l \in \{1, ..., b\}$.*

P r o o f. Let $l \in \{1, ..., b\}$. Assume $\rho = (t_{a+l})_u + 1$. Suppose that there exists $q \in \{1, ..., a\}$ with $t_q > t_{a+l}$. Then

$$v_\rho w_{t_1} ... w_{t_q} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}} \overset{(R3)}{\approx} w_{t_1} ... v_\rho w_{t_q} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}}$$
$$\overset{(R4)}{\approx} w_{t_1} ... w_{t_q} ... w_{t_a} v_{\rho+2|w_{t_q}...w_{t_a}|} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}}.$$

Since

$$(t_{a+l})_u + 1 = i_{t_{a+l}} + 2|w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+l}}| - 2|w_{t_q} ... w_{t_a}| + 1,$$

we have

$$\rho + 2|w_{t_q} ... w_{t_a}| = i_{t_{a+l}} + 2|w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+l}}| + 1.$$

Suppose $t_q < t_{a+l}$ for all $q \in \{1, ..., a\}$. Then we have

$$(t_{a+l})_u + 1 = i_{t_{a+l}} + 2|w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+l}}| + 1,$$

i.e.

$$v_\rho w_{t_1} ... w_{t_q} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}} \overset{(R3)}{\approx} w_{t_1} ... w_{t_q} ... w_{t_a} v_\rho w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}}.$$

Both cases imply

$$w_{t_1} ... w_{t_q} ... w_{t_a} v_{i_{t_{a+l}} + 2|w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+l}}| + 1} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}}$$
$$\overset{(R4)}{\approx} w_{t_1} ... w_{t_q} ... w_{t_a} w^{-1}_{t_{a+1}} ... v_{i_{t_{a+l}} + 2|w^{-1}_{t_{a+l}}| + 1} w^{-1}_{t_{a+l}} ... w^{-1}_{t_{a+b}} \overset{(R6)}{\approx} w_{t_1} ... w_{t_q} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+l}} ... w^{-1}_{t_{a+b}},$$

i.e. we can cancel v_ρ in $\hat{w}$ using $(R3)$, $(R4)$, and $(R6)$, a contradiction. □

Lemma 9. *Let $\rho \in A$ and let $l \in \{1, ..., b\}$ such that $t_{a+l} \neq m$. If $\rho \in \{(t_{a+l})_u + 1, ..., (t_{a+l} + 1)_u - 1\}$ then*

$$\rho \in \left\{(t_{a+l})_u + 2, ..., (t_{a+l} + 1)_u - 1\right\} \subseteq A_w.$$

P r o o f. It is a consequence of Lemma 8 that $\rho \in \{(t_{a+l})_u + 2, ..., (t_{a+l} + 1)_u - 1\}$ and by (6_q), we have $\{(t_{a+l})_u + 2, ..., (t_{a+l} + 1)_u - 1\} \subseteq A_w$. □

Lemma 10. *Let $\rho \in A$. If $t_{a+1} = m$ and $\rho \in \{m_u + 1, ..., n\}$ then $\rho \in \{m_u + 2, ..., n\} \subseteq A_w$.*

P r o o f. Suppose $\rho = m_u + 1 = (t_{a+1})_u + 1$. By Lemma 8, we have $\rho \notin A$. Therefore, $\rho \in \{m_u + 2, ..., n\} \subseteq A_w$ by (5_q). □

Lemma 11. *If* $1 < 1_x < 1_u$ *then* $\rho \notin A$ *for all* $\rho \in \{1, ..., 1_u - 1_x\}$.

P r o o f. Let $\rho \in \{1, ..., 1_u - 1_x\}$. Assume $\rho \in A$. We observe that

$$1_u - 1_x = 2|w_{t_{a+b}}^{-1} ... w_{t_{a+1}}^{-1}| - 2|w_{t_1} ... w_{t_a}| = 2k$$

for some positive integer k. We put $\mathcal{U} = w_{t_1} ... w_{t_a}$ and $\mathcal{X} = w_{t_{a+b}}^{-1} ... w_{t_{a+1}}^{-1}$, i.e. $2k = 2|\mathcal{X}| - 2|\mathcal{U}|$ and $|\mathcal{X}| = |\mathcal{U}| + k$. Let

$$w_{t_{a+1}}^{-1} ... w_{t_{a+b}}^{-1} = y_1 ... y_{|\mathcal{U}|} y_{|\mathcal{U}|+1} ... y_{|\mathcal{U}|+k},$$

where $y_1, ..., y_{|\mathcal{U}|+k} \in \{x_1, ..., x_{n-2}\}$. Then

$$v_\rho w_{t_1} ... w_{t_a} y_1 ... y_{|\mathcal{U}|} y_{|\mathcal{U}|+1} ... y_{|\mathcal{U}|+k} \overset{(R4)}{\approx} w_{t_1} ... w_{t_a} v_{\rho + 2|w_{t_1} ... w_{t_a}|} y_1 ... y_{|\mathcal{U}|} y_{|\mathcal{U}|+1} ... y_{|\mathcal{U}|+k}.$$

Using Remark 1, it is routine to calculate that

$$2|w_{t_{a+b}}^{-1} ... w_{t_{a+1}}^{-1}| < i_{t_{a+1}} + 2|w_{t_{a+1}}^{-1}|,$$

i.e.

$$(1_u - 1_x) + 2|w_{t_1} ... w_{t_a}| = 2|w_{t_{a+b}}^{-1} ... w_{t_{a+1}}^{-1}| < i_{t_{a+1}} + 2|w_{t_{a+1}}^{-1}|.$$

This implies

$$\rho + 2|w_{t_1} ... w_{t_a}| \leq i_{t_{a+1}} + 2|w_{t_{a+1}}^{-1}|.$$

Then

$$w_{t_1} ... w_{t_a} v_{\rho + 2|w_{t_1} ... w_{t_a}|} y_1 ... y_{|\mathcal{U}|} y_{|\mathcal{U}|+1} ... y_{|\mathcal{U}|+k} \overset{(R4)}{\approx} w_{t_1} ... w_{t_a} y_1 ... y_{|\mathcal{U}|} v_\rho y_{|\mathcal{U}|+1} ... y_{|\mathcal{U}|+k}.$$

Note that $1_u - 1_x$ is even and there is $i \in \{2, 4, ..., 1_u - 1_x\}$ such that $\rho \in \{i - 1, i\}$. If $\rho = i - 1$ then

$$\rho - 2|y_{|\mathcal{U}|+1} ... y_{|\mathcal{U}|+i/2-1}| = 1.$$

If $\rho = i$ then

$$\rho - 2|y_{|\mathcal{U}|+1} ... y_{|\mathcal{U}|+i/2-1}| = 2.$$

Thus,

$$\begin{aligned} & w_{t_1} ... w_{t_a} y_1 ... y_{|\mathcal{U}|} v_\rho y_{|\mathcal{U}|+1} ... y_{|\mathcal{U}|+k} \\ \overset{(R4)}{\approx} \; & w_{t_1} ... w_{t_a} y_1 ... y_{|\mathcal{U}|} y_{|\mathcal{U}|+1} ... v_{\rho - 2|y_{|\mathcal{U}|+1} ... y_{|\mathcal{U}|+i/2-1}|} y_{|\mathcal{U}|+i/2} ... y_{|\mathcal{U}|+(1_u - 1_x)/2} \\ = \; & w_{t_1} ... w_{t_a} y_1 ... y_{|\mathcal{U}|} y_{|\mathcal{U}|+1} ... v_{\hat{\rho}} y_{|\mathcal{U}|+i/2} ... y_{|\mathcal{U}|+(1_u - 1_x)/2} \end{aligned}$$

(where $\hat{\rho} \in \{1, 2\}$)

$$\overset{(R6)}{\approx} w_{t_1} ... w_{t_a} y_1 ... y_{|\mathcal{U}|} y_{|\mathcal{U}|+1} ... y_{|\mathcal{U}|+i/2} ... y_{|\mathcal{U}|+(1_u - 1_x)/2},$$

i.e. we can cancel v_ρ in $\hat{w}$ using $(R4)$ and $(R6)$, a contradiction. □

Lemma 12. *Let* $\rho \in A$ *with* $\rho \in \{1, ..., 1_u - 1\}$. *If* $1 < 1_u \leq 1_x$ *then* $\rho \in \{1, ..., 1_u - 1\} \subseteq A_w$ *and if* $1 < 1_x < 1_u$ *then* $\rho \in \{1_u - 1_x + 1, ..., 1_u - 1\} \subseteq A_w$.

P r o o f. If $1 < 1_u \leq 1_x$ then $\{1, ..., 1_u - 1\} \subseteq A_w$ by (7_q). If $1 < 1_x < 1_u$, it is a consequence of Lemma 11 that $\rho \in \{1_u - 1_x + 1, ..., 1_u - 1\}$ and by (7_q), we have $\{1_u - 1_x + 1, ..., 1_u - 1\} \subseteq A_w$. □

Lemma 13. *We have* $(t_q)_u \notin A$ *for all* $q \in \{1, ..., a\}$.

Proof. Let $q \in \{1, ..., a\}$. We have

$$w_{t_q} = u_{i_{t_q}} u_{i_{t_q}+2} ... u_{i_{t_q}+2|w_{t_q}|-2}$$

and $(t_q)_u = i_{t_q}$. Assume $(t_q)_u \in A$. If $i_{t_q} \geq 2$ then

$$\begin{aligned} v_{i_{t_q}} w_{t_1} ... w_{t_q} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}} &\overset{(R3)}{\approx} w_{t_1} ... v_{i_{t_q}} u_{i_{t_q}} u_{i_{t_q}+2} ... u_{i_{t_q}+2|w_{t_q}|-2} w_{t_{q+1}} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}} \\ &\overset{(R18)}{\approx} w_{t_1} ... v_{i_{t_q}+2|w_{t_q}|+1} u_{i_{t_q}-1} u_{i_{t_q}+1} ... u_{i_{t_q}+2|w_{t_q}|-3} w_{t_{q+1}} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}}. \end{aligned}$$

If $i_{t_q} = 1$ then $q = 1$ and

$$\begin{aligned} v_{i_{t_1}} w_{t_1} w_{t_2} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}} &= v_1 u_1 u_3 ... u_{1+2|w_{t_1}|-2} w_{t_2} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}} \\ &\overset{(R16)}{\approx} v_1 v_2 ... v_{1+2|w_{t_1}|+1} w_{t_2} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}}. \end{aligned}$$

We observe that we can replace several letters in $\hat{w}$ by letters with decreasing index by $(R18)$ and the letters $u_1, u_3, ..., u_{1+2|w_{t_1}|-2}$ were canceled in $\hat{w}$ by $(R16)$, respectively, a contradiction. □

Lemma 14. *We have* $(t_{a+l})_u \notin A$ *for all* $l \in \{1, ..., b\}$.

Proof. Let $l \in \{1, ..., b\}$. Now assume that $(t_{a+l})_u \in A$. We will have the following two cases. In the first case, we suppose that there exists $q \in \{1, ..., a\}$ with $t_q > t_{a+l}$ and, of course, for the trivial second case is supposed $t_q < t_{a+l}$ for all $q \in \{1, ..., a\}$. Using $(R3)$ and $(R4)$ in the first case and $(R4)$ in the second case, together with a few tedious calculations, both cases imply

$$v_{(t_{a+l})_u} w_{t_1} ... w_{t_q} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}} \approx w_{t_1} ... w_{t_a} v_{i_{t_{a+l}}+2|w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+l}}|} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}}.$$

It is routine to calculate that

$$w_{t_1} ... w_{t_a} v_{i_{t_{a+l}}+2|w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+l}}|} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}} \overset{(R4)}{\approx} w_{t_1} ... w_{t_a} w^{-1}_{t_{a+1}} ... v_{i_{t_{a+l}}+2|w^{-1}_{t_{a+l}}|} w^{-1}_{t_{a+l}} ... w^{-1}_{t_{a+b}}.$$

If $i_{t_{a+l}} + 2|w^{-1}_{t_{a+l}}| > 3$ then

$$\begin{aligned} & w_{t_1} ... w_{t_a} w^{-1}_{t_{a+1}} ... v_{i_{t_{a+l}}+2|w^{-1}_{t_{a+l}}|} w^{-1}_{t_{a+l}} ... w^{-1}_{t_{a+b}} \\ &= w_{t_1} ... w_{t_a} w^{-1}_{t_{a+1}} ... v_{i_{t_{a+l}}+2|w^{-1}_{t_{a+l}}|} x_{i_{t_{a+l}}+2|w_{t_{a+l}}|-2} x_{i_{t_{a+l}}+2|w_{t_{a+l}}|-4} ... x_{i_{t_{a+l}}} w^{-1}_{t_{a+l+1}} ... w^{-1}_{t_{a+b}} \\ &\overset{(R19)}{\approx} w_{t_1} ... w_{t_a} w^{-1}_{t_{a+1}} ... v_{i_{t_{a+l}}+2|w^{-1}_{t_{a+l}}|+1} x_{i_{t_{a+l}}+2|w_{t_{a+l}}|-3} x_{i_{t_{a+l}}+2|w_{t_{a+l}}|-5} ... x_{i_{t_{a+l}}-1} w^{-1}_{t_{a+l+1}} ... w^{-1}_{t_{a+b}}. \end{aligned}$$

If $i_{t_{a+l}} + 2|w^{-1}_{t_{a+l}}| = 3$ then $w^{-1}_{t_{a+b}} = x_1$. Thus,

$$\begin{aligned} & w_{t_1} ... w_{t_a} v_{i_{t_{a+l}}+2|w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+l}}|} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b}} \\ &\overset{(R4)}{\approx} w_{t_1} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b-1}} v_3 x_1 \overset{(R17)}{\approx} w_{t_1} ... w_{t_a} w^{-1}_{t_{a+1}} ... w^{-1}_{t_{a+b-1}} v_1 v_2 v_3 v_4. \end{aligned}$$

We observe that we can replace several letters in $\hat{w}$ by letters with decreasing index by $(R19)$ and the letter x_1 can be canceled in $\hat{w}$ by $(R17)$, respectively, a contradiction. □

If we summarize the previous lemmas, then we obtain:

Lemma 15. *We have $A \subseteq A_w$.*

P r o o f. Let $\rho \in A$. Then it is easy to verify that $\rho \in \{1, ..., 1_u\}$ or $\rho \in \{k_u + 1, ..., (k+1)_u\}$ for some $k \in \{1, ..., m-1\}$ or $\rho \in \{m_u + 1, ..., n\}$. Suppose that $\rho \in \{k_u + 1, ..., (k+1)_u - 1\}$ for some $k \in \{1, ..., m-1\}$. Lemmas 13 and 14 show that $k_u \notin A$. Then we can conclude that $\rho \in A_w$ by Lemmas 6 and 9. Suppose $\rho \in \{m_u + 1, ..., n\}$. Then we can conclude that $\rho \in A_w$ by Lemmas 7 and 10. Finally, we suppose that $\rho \in \{1, ..., 1_u - 1\}$. Then we can conclude that $\rho \in A_w$ by Lemma 12. Eventually, we have $\rho \in A_w$ for all $\rho \in A$. Therefore, $A \subseteq A_w$. □

Lemmas 4 and 15 prove that $\hat{w} = v_A \hat{w}_1 ... \hat{w}_m \in W_n$. Consequently, we have:

Proposition 2. *$P \subseteq W_n$.*

By the definition of the set P and Proposition 2, it is proved:

Corollary 1. *Let $w \in X_n^*$. Then there is $w' \in P \subseteq W_n$ with $w \approx w'$.*

4. A presentation for IOF_n^{par}

In this section, we exhibit a presentation for IOF_n^{par}. Concerning the results from the previous sections, it remains to show that $|W_n| \leq |IOF_n^{par}|$. For this, we construct a word w_α, for all $\alpha \in IOF_n^{par}$, in the following way.

Let

$$\alpha = \begin{pmatrix} d_1 & < & d_2 & < & \cdots & < & d_p \\ m_1 & & m_2 & & \cdots & & m_p \end{pmatrix} \in IOF_n^{par} \backslash \{\varepsilon\}$$

for a positive integer $p \leq n$. There are a unique $l \in \{0, 1, ..., p-1\}$ and a unique set $\{r_1, ..., r_l\} \subseteq \{1, ..., p-1\}$ such that (i)–(iii) are satisfied:

(i) $r_1 < ... < r_l$;
(ii) $d_{r_i+1} - d_{r_i} \neq m_{r_i+1} - m_{r_i}$ for $i \in \{1, ..., l\}$;
(iii) $d_{i+1} - d_i = m_{i+1} - m_i$ for $i \in \{1, ..., p-1\} \backslash \{r_1, ..., r_l\}$.

Note that $l = 0$ means $\{r_1, ..., r_l\} = \emptyset$. Further, we put $r_{l+1} = p$. For $i \in \{1, ..., l\}$, we define

$$w_i = \begin{cases} x_{m_{r_i}, ((m_{r_i+1} - m_{r_i}) - (d_{r_i+1} - d_{r_i}))/2} & \text{if} \quad m_{r_i+1} - m_{r_i} > d_{r_i+1} - d_{r_i}; \\ u_{d_{r_i}, ((d_{r_i+1} - d_{r_i}) - (m_{r_i+1} - m_{r_i}))/2} & \text{if} \quad m_{r_i+1} - m_{r_i} < d_{r_i+1} - d_{r_i}. \end{cases}$$

Obviously, we have $w_i \in W_x \cup W_u$ for all $i \in \{1, ..., l\}$. If $m_p = d_p$ then we put $w_{l+1} = \epsilon$. If $m_p \neq d_p$, we define additionally

$$w_{l+1} = \begin{cases} x_{m_p, (d_p - m_p)/2} & \text{if} \quad d_p > m_p; \\ u_{d_p, (m_p - d_p)/2} & \text{if} \quad d_p < m_p. \end{cases}$$

Clearly, $w_{l+1} \in W_x \cup W_u$. We consider the word

$$w = w_1 ... w_{l+1}.$$

From this word, we construct a new word w_α^* by arranging the subwords $s \in W_x$ in reverse order at the end, replacing s by s^{-1}. In other words, we consider the word

$$w_\alpha^* = w_{s_1} ... w_{s_a} w_{s_{a+1}}^{-1} ... w_{s_{a+b}}^{-1}$$

such that $w_{s_1}, ..., w_{s_a} \in W_u$, $w_{s_{a+1}}, ..., w_{s_{a+b}} \in W_x$ and

$$\{w_{s_1}, ..., w_{s_a}, w_{s_{a+1}}, ..., w_{s_{a+b}}\} = \{w_1, ..., w_{a+b}\},$$

where $s_1 < ... < s_a, s_{a+b} < ... < s_{a+1}$, and $a, b \in \overline{n} \cup \{0\}$ with

$$a + b = \begin{cases} l & \text{if} \quad d_p = m_p; \\ l+1 & \text{if} \quad d_p \neq m_p. \end{cases}$$

For convenience, $a = 0$ means $w^*_\alpha = w^{-1}_{s_{a+1}} ... w^{-1}_{s_{a+b}}$ and $b = 0$ means $w^*_\alpha = w_{s_1} ... w_{s_a}$. Now, we add recursively letters from the set $\{v_1, ..., v_n\} \subseteq X_n$ to the word w^*_α, obtaining new words $\lambda_0, \lambda_1, ..., \lambda_p$.

(1) For $d_p \leq n - 2$:

(1.1) if $m_p < d_p$ then $\lambda_0 = v_{d_p+2} ... v_n w^*_\alpha$;
(1.2) if $n - 1 > m_p > d_p$ then $\lambda_0 = v_{m_p+2} ... v_n w^*_\alpha$;
(1.3) if $m_p = d_p$ then $\lambda_0 = v_{m_p+1} ... v_n w^*_\alpha$;
otherwise $\lambda_0 = w^*_\alpha$.

(2) If $d_p = m_p = n - 1$ then $\lambda_0 = v_n w^*_\alpha$. Otherwise $\lambda_0 = w^*_\alpha$.

(3) For $k \in \{2, ..., p\}$:

(3.1) if $2 \leq m_k - m_{k-1} = d_k - d_{k-1}$ then $\lambda_{p-k+1} = v_{d_{k-1}+1} ... v_{d_k-1} \lambda_{p-k}$;
(3.2) if $2 < m_k - m_{k-1} < d_k - d_{k-1}$ then $\lambda_{p-k+1} = v_{d_k-(m_k-m_{k-1}-2)} ... v_{d_k-1} \lambda_{p-k}$;
(3.3) if $m_k - m_{k-1} > d_k - d_{k-1} > 2$ then $\lambda_{p-k+1} = v_{d_{k-1}+2} ... v_{d_k-1} \lambda_p$;
otherwise $\lambda_{p-k+1} = \lambda_{p-k}$.

(4) If $d_1 = 1$ or $m_1 = 1$ then $\lambda_p = \lambda_{p-1}$.

(5) If $1 < d_1 \leq m_1$ then $\lambda_p = v_1 ... v_{d_1-1} \lambda_{p-1}$.

(6) If $1 < m_1 < d_1$ then $\lambda_p = v_{d_1-m_1+1} ... v_{d_1-1} \lambda_{p-1}$.

The word λ_p induces a set $A = \{a \in \overline{n} : v_a \text{ is a letter in } \lambda_p\}$ and it is easy to verify that $\rho \notin A$ for all $\rho \in dom(\alpha)$. We put $w_\alpha = \lambda_p$. The word w_α has the form $w_\alpha = v_A w^*_\alpha$.

Our next aim is to present the relationship between cardinality of W_n and IOF^{par}_n. This leads us to assume the existence of a map $f : IOF^{par}_n \backslash \{\varepsilon\} \to W_n \backslash \{v_{\overline{n}}\}$, where $f(\alpha) = w_\alpha$ for all $\alpha \in IOF^{par}_n \backslash \{\varepsilon\}$. We start by constructing the transformation $\alpha_{v_A w^*}$ for any $v_A w^* \in W_n$, different from $v_{\overline{n}}$. Let $v_A w^* \in W_n \backslash \{v_{\overline{n}}\}$. We have $w \in Q_0, A \subseteq A_w$, and there are $w_1, ..., w_m \in W_u \cup W_x$ such that $w = w_1 ... w_m$ for some positive integer m. For $k \in \{1, ..., m\}$, we define $a_k = k_u + 2$ and $b_k = i_k + 2j_k + 2$, whenever $w_k \in W_x$. On the other hand, we define $a_k = i_k + 2j_k + 2$ and $b_k = k_x + 2$, whenever $w_k \in W_u$. It is easy to verify that $a_m = b_m$. We put

$$\alpha_{v_A w^*} = \overline{v}_A \begin{pmatrix} 1 + 1_u - \min\{1_u, 1_x\} ... 1_u & a_1 ... 2_u & \cdots & a_{m-1} ... m_u & a_m ... n \\ 1 + 1_x - \min\{1_u, 1_x\} ... 1_x & b_1 ... 2_x & \cdots & b_{m-1} ... m_x & b_m ... n \end{pmatrix}.$$

For convenience, we also give

$$\alpha_{v_A w^*} = \begin{pmatrix} d_1 & d_2 & \cdots & d_p \\ m_1 & m_2 & \cdots & m_p \end{pmatrix}$$

for some positive integer $p \leq n$. In the following, we show that $\alpha_{v_A w^*}$ is well-defined in the sense that the construction of $\alpha_{v_A w^*}$ gives a transformation.

Lemma 16. *$\alpha_{v_A w^*}$ is well-defined.*

P r o o f. Let $k \in \{1, ..., m-1\}$. Suppose $w_k, w_{k+1} \in W_u$. We have

$$k_u = i_k, \quad k_x = i_k + 2|w_k| + 2|W_u^k| - 2|W_x^k|,$$
$$(k+1)_u = i_{k+1}, (k+1)_x = i_{k+1} + 2|w_{k+1}| + 2|W_u^{k+1}| - 2|W_x^{k+1}|,$$

and $a_k = i_k + 2j_k + 2, b_k = k_x + 2$. Then

$$(k+1)_u - a_k = i_{k+1} - (i_k + 2j_k + 2),$$
$$(k+1)_x - b_k = i_{k+1} + 2|w_{k+1}| + 2|W_u^{k+1}| - 2|W_x^{k+1}| - k_x - 2$$
$$= i_{k+1} + 2|w_{k+1}| + 2|W_u^{k+1}| - 2|W_x^{k+1}| - i_k - 2|w_k| - 2|W_u^k| + 2|W_x^k| - 2$$
$$= i_{k+1} - i_k - 2j_k - 2 = i_{k+1} - (i_k + 2j_k + 2).$$

Therefore, $(k+1)_u - a_k = (k+1)_x - b_k$.

For the rest cases ($w_k \in W_u$ and $w_{k+1} \in W_x$, $w_k \in W_x$ and $w_{k+1} \in W_u$ as well as $w_k, w_{k+1} \in W_x$), a proof similar as above will eventually show that $(k+1)_u - a_k = (k+1)_x - b_k$. Furthermore, suppose $d_p = m_p$. Let $k \in \{1, ..., m\}$ and $w_k \in W_u$. We have

$$a_k - k_u = i_k + 2j_k + 2 - k_u = i_k + 2j_k + 2 - i_k = 2j_k + 2,$$
$$b_k - k_x = k_x + 2 - k_x = 2.$$

Thus, $a_k - k_u \neq b_k - k_x$.

For the case $w_k \in W_x$, we can show $a_k - k_u \neq b_k - k_x$ in the same way.

Continuously, suppose $d_p \neq m_p$. By the previous part of the proof, we have $a_k - k_u \neq b_k - k_x$ for all $k \in \{1, ..., m-1\}$. Moreover, we observe that $d_p \notin \{a_m, ..., n\}$ and $m_p \notin \{b_m, ..., n\}$ because $n - a_m = n - b_m$. This implies $d_p = m_u$ and $m_p = m_x$. By any of the above, we can conclude that $\alpha_{v_A w^*}$ is well-defined. □

The proof of Lemma 16 shows $(k+1)_u - a_k = (k+1)_x - b_k$ for all $k \in \{1, ..., m-1\}$. Then $a_k - k_u \neq b_k - k_x$ for all $k \in \{1, ..., m\}$, whenever $d_p = m_p$, and $a_k - k_u \neq b_k - k_x$ for all $k \in \{1, ..., m-1\}$ and $d_p = m_u, m_p = m_x$, whenever $d_p \neq m_p$. Furthermore, observing by trivial calculation, $a_k - k_u \geq 2$ and $b_k - k_x \geq 2$. Therefore, if there exists $i \in \{1, ..., p-1\}$, where $d_{i+1} - d_i \neq m_{i+1} - m_i$, then $d_i \in \{1_u, ..., (m-1)_u\}(\cup\{m_u\})$, $m_i \in \{1_x, ..., (m-1)_x\}(\cup\{m_x\})$ and we put $k_u = d_{r_k}, k_x = m_{r_k}$ for all $k \in \{1, ..., m-1\}(\cup\{m\})$ (we put $r_m = p$, whenever $d_p \neq m_p$). This gives the unique set $\{r_1, ..., r_m\}$ as required by the definition of $w_{\alpha_{v_A w^*}}$. Moreover, we need to show that $\alpha_{v_A w^*} \in IOF_n^{par} \backslash \{\varepsilon\}$ by checking (i)-(iv) of Proposition 1. We will now show that $\alpha_{v_A w^*} \in IOF_n^{par}$ as well as $w_{\alpha_{v_A w^*}} = v_A w^*$. This gives the tools to calculate that $|W_n| \leq |IOF_n^{par}|$.

Lemma 17. $\alpha_{v_A w^*} \in IOF_n^{par} \backslash \{\varepsilon\}$.

P r o o f. Clearly, $\alpha_{v_A w^*} \neq \varepsilon$. We will prove that $\alpha_{v_A w^*}$ satisfies the conditions (i)–(iv) in Proposition 1. We observe that $d_1 < d_2 < \cdots < d_p$ and $m_1 < m_2 < \cdots < m_p$ by definition of $\alpha_{v_A w^*}$. We have $1_u - d_1 = 1_x - m_1$, i.e. $1_u - 1_x = d_1 - m_1$. By the definition of k_u and k_x, for $k \in \{1, ..., m\}$, we observe that $1_u - 1_x$ is even, i.e. $d_1 - m_1$ is even. Thus, d_1 and m_1 have the same parity.

Let $d_{i+1} - d_i = 1$ for some $i \in \{1, ..., p-1\}$. Then $d_i \in dom(\alpha) \backslash \{1_u, ..., m_u\}$ implies $m_{i+1} - m_i = d_{i+1} - d_i = 1$.

Let $m_{i+1} - m_i = 1$ for some $i \in \{1, ..., p-1\}$. Then $m_i \in im(\alpha) \backslash \{1_x, ..., m_x\}$ implies $d_{i+1} - d_i = m_{i+1} - m_i = 1$.

Let $d_{i+1} - d_i$ is even. Suppose $d_{i+1} - d_i \neq m_{i+1} - m_i$. This gives $d_i = k_u$ and $m_i = k_x$ for some $k \in \{1, ..., m-1\}$. By the definition of k_u and k_x, we observe that $k_u - k_x$ is even.

Moreover, $(k+1)_u - d_{i+1} = (k+1)_x - m_{i+1}$ since $(k+1)_u - (k+1)_x$ is even, we have $d_{i+1} - m_{i+1}$ is even. Then d_{i+1}, d_i and d_i, m_i as well as d_{i+1}, m_{i+1} have the same parity. This implies that m_{i+1}, m_i have the same parity, i.e. $m_{i+1} - m_i$ is even. Conversely, we can prove similarly that, if $m_{i+1} - m_i$ is even then $d_{i+1} - d_i$ is even. By Proposition 1, we get $\alpha_{v_A w^*} \in IOF_n^{par}$. □

We can construct $f(\alpha_{v_A w^*}) = w_{\alpha_{v_A w^*}}$, where $w_{\alpha_{v_A w^*}} = v_{\tilde{A}} \hat{w}^*_{\alpha_{v_A w^*}}$ with $\hat{w} = \hat{w}_1 ... \hat{w}_m$ for $\hat{w}_1, ..., \hat{w}_m \in W_u \cup W_x$ and $\tilde{A} \subseteq \overline{n}$. We will prove that f is surjective in the next lemma.

Lemma 18. *Let $v_A w^* \in W_n \backslash \{v_{\overline{n}}\}$. Then there is $\alpha \in IOF_n^{par} \backslash \{\varepsilon\}$ with $v_A w^* = w_\alpha$.*

P r o o f. We have $w_{\alpha_{v_A w^*}} = v_{\tilde{A}} \hat{w}^*_{\alpha_{v_A w^*}}$, where $\hat{w} = \hat{w}_1 ... \hat{w}_m$ with $\hat{w}_1, ..., \hat{w}_m \in W_u \cup W_x$ and $\tilde{A} \subseteq \overline{n}$. First, our goal is to show that $\hat{w} = w$. Suppose $d_p = m_p$ and let $k \in \{1, ..., m\}$ such that $b_k - k_x > a_k - k_u$. By the definition of $\hat{w}_k$, we have $\hat{w}_k = x_{k_x, ((b_k - k_x) - (a_k - k_u))/2}$ and $k_x = i_k$. Then

$$\frac{(b_k - k_x) - (a_k - k_u)}{2} = \frac{i_k + 2j_k + 2 - i_k - k_u - 2 + k_u}{2} = j_k,$$

i.e. $\hat{w}_k = x_{i_k, j_k} = w_k$. For the case $b_k - k_x < a_k - k_u$, we can prove that $\hat{w}_k = w_k$ in a similar way. This gives $\hat{w}_1 ... \hat{w}_m = w_1 ... w_m$.

Suppose $d_p \neq m_p$. We have $a_k - k_u \neq b_k - k_x$ for all $k \in \{1, ..., m-1\}$ and by a similar proof as above, we have $\hat{w}_1 ... \hat{w}_{m-1} = w_1 ... w_{m-1}$. If $m_p < d_p$ then $\hat{w}_m = x_{m_p, (d_p - m_p)/2}$ and $m_p = m_x = i_m$. Then

$$\frac{d_p - m_p}{2} = \frac{m_u - m_x}{2} = \frac{i_m + 2j_m - i_m}{2} = j_m,$$

i.e. $\hat{w}_m = x_{i_m, j_m} = w_m$. For the case $m_p > d_p$, we can prove $\hat{w}_m = w_m$ in a similar way. Thus, $\hat{w}_1 ... \hat{w}_{m-1} \hat{w}_m = w_1 ... w_{m-1} w_m$. Then $w = \hat{w}$, i.e. $w^* = \hat{w}^*_{\alpha_{v_A w^*}}$. The next goal is to show that $A = \tilde{A}$.

1) To show that $A \subseteq \tilde{A}$: let $a \in A$. We have $A \subseteq A_w$ since $v_A w^* \in W_n$. Therefore, we have the following cases: $a \in \{a_m, ..., n\} = A_1$ or $a \in \{a_k, ..., (k+1)_u - 1\} = A_2$ for some $k \in \{1, ..., m-1\}$ or

$$a \in \{1 + 1_u - \min\{1_u, 1_x\}, ..., 1_u - 1\} = A_3.$$

If $a \in A_1$ and $m_p \neq d_p$ then $a \in \tilde{A}$ since (1.1) and (1.2), respectively. If $a \in A_1$ and $a \in \{d_p + 1, ..., n\}$ with $m_p = d_p$ then $a \in \tilde{A}$ since (1.3) and (2), respectively.

Suppose $a \in A_2$ with $a \in \{a_k, ..., d_{r_k+1} - 1\}$. If $2 < d_{r_k+1} - d_{r_k} < m_{r_k+1} - m_{r_k}$ then $w_k \in W_x$. Note that $a_k = k_u + 2 = d_{r_k} + 2$. Thus, $a \in \tilde{A}$ since (3.3). If $2 < m_{r_k+1} - m_{r_k} < d_{r_k+1} - d_{r_k}$ then $w_k \in W_u$.

Note

$$d_{r_k+1} - a_k = m_{r_k+1} - b_k, \quad b_k = k_x + 2,$$
$$a_k = a_k - b_k + b_k = d_{r_k+1} - m_{r_k+1} + k_x + 2 = d_{r_k+1} - m_{r_k+1} + m_{r_k} + 2.$$

Thus, $a \in \tilde{A}$ since (3.2).

Suppose $a \in A_3$. If $1 < d_1 \leq m_1$ and $a \in \{1, ..., d_1 - 1\}$ then $a \in \tilde{A}$ since (5). If $1 < m_1 < d_1$ and $a \in \{d_1 - m_1 + 1, ..., 1_u - 1\}$ then $a \in \tilde{A}$ since (6) (note that $1_u - 1_x = d_1 - m_1$).

Suppose $a \in A_1 \cup A_2 \cup A_3$ and there exists $s \in \{2, ..., p\}$ such that $d_s - d_{s-1} = m_s - m_{s-1} \geq 2$ with $a \in \{d_{s-1} + 1, ..., d_s - 1\}$. Then $a \in \tilde{A}$ since (3.1). By any of the above, we have $A \subseteq \tilde{A}$.

2) To show that $\tilde{A} \subseteq A$: let

$$A_1 = \{1 + 1_u - \min\{1_u, 1_x\}, ..., 1_u - 1\},$$
$$A_2 = \{a_1, ..., 2_u - 1\} \cup \{a_2, ..., 3_u - 1\} \cup ... \cup \{a_{m-1}, ..., m_u - 1\},$$
$$A_3 = \{a_m, ..., n\}.$$

Because $A \subseteq A_w$, we have $A \subseteq A_1 \cup A_2 \cup A_3$ and $A \cap \{d_1, ..., d_p\} = \emptyset$. This implies $A \subseteq A_1 \cup A_2 \cup A_3 \backslash \{d_1, ..., d_p\}$. Conversely, we have $A_1 \cup A_2 \cup A_3 \backslash \{d_1, ..., d_p\} \subseteq A$ by the definition of $\alpha_{v_A w^*}$. Thus, $A = A_1 \cup A_2 \cup A_3 \backslash \{d_1, ..., d_p\}$.

Let $a \in \tilde{A}$. By the definition of $\tilde{A}$, we can observe that $a \neq d_i$ for all $i \in \{1, ..., p\}$.

Suppose a is given by (1.1) or (1.2) or (1.3) or (2). Then $a \in A_3 \backslash \{d_1, ..., d_p\}$.

Suppose a is given by (3.1). Then $a \in A_1 \cup A_2 \cup A_3 \backslash \{d_1, ..., d_p\}$.

Suppose a is given by (3.2), i.e. $a \in \{d_s - m_s + m_{s-1} + 2, ..., d_s - 1\}$ for some $s \in \{2, ..., p\}$.

We have already shown that there is $k \in \{1, ..., m-1\}$ such that $d_s - m_s + m_{s-1} + 2 = a_k$. Then $a \in A_2 \backslash \{d_1, ..., d_p\}$.

Suppose a is given by (3.3). Then $a \in A_2 \backslash \{d_1, ..., d_p\}$.

Suppose a is given by (5). Then $a \in A_1 \backslash \{d_1, ..., d_p\}$.

Suppose a is given by (6). Then $a \in A_1 \backslash \{d_1, ..., d_p\}$ (note that $d_1 - m_1 = 1_u - 1_x$). Therefore, we have $a \in A$, i.e. $\tilde{A} \subseteq A$.

By 1) and 2), we get $A = \tilde{A}$. This implies $v_A w^* = v_{\tilde{A}} \hat{w}^* = w_{\alpha_{v_A w^*}}$. □

Lemma 18 establishes that f is surjective, which implies $|W_n| \leq |IOF_n^{par}|$. We will now adjust our alphabet and relations to meet the requirements of Theorem 1. As mentioned previously, $\overline{X}_n = \{\overline{s} : s \in X_n\}$ is a generating set for the monoid IOF_n^{par}. Building on the insights from Lemma 1, we can conclude that $\overline{X}_n$ satisfies all the relations from $\overline{R} = \{\overline{s}_1 \approx \overline{s}_2 : s_1 \approx s_2 \in R\}$.

Corollary 1 further shows that for any $w \in \overline{X}_n^*$, there exists a corresponding $w' \in \overline{W}_n$, for which $w \approx w'$ is a consequence of $\overline{R}$. This implies that $\overline{R} \subseteq \overline{X}_n^* \times \overline{X}_n^*$ and that $\overline{W}_n \subseteq \overline{X}_n^*$ meet the conditions 1–3 in Theorem 1. We now possess all the necessary items to conclude our main result.

Theorem 2. *$\langle \overline{X}_n \,|\, \overline{R} \rangle$ is a monoid presentation for IOF_n^{par}.*

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