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LATTICE UNIVERSALITY OF LOCALLY FINITE p-GROUPS

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Abstract: For an arbitrary prime p, we prove that every algebraic lattice is isomorphic to a complete sublattice in the subgroup lattice of a suitable locally finite p-group. In particular, every lattice is embeddable in the subgroup lattice of a locally finite p-group.

Keywords: Subgroup lattice, Algebraic lattice, Complete sublattice, Lattice-universal class of algebras, Locally finite p-group, Group valuation.

1. Introduction and formulation of results

A lattice is called *algebraic* if it is complete and each element is a join of compact elements. Important examples of algebraic lattices include subalgebra lattices of universal algebras, particularly subgroup lattices of groups.

If a lattice L is complete and a subset M of L has the property that $\bigvee S$, $\bigwedge S \in M$ for every nonempty subset $S \subseteq M$, then M is called a *complete sublattice* of L. It is well known that a complete sublattice of an algebraic lattice is itself algebraic. This implies that complete sublattices of subgroup lattices are algebraic as well. Conversely, we proved in [7] that every algebraic lattice can be represented as a complete sublattice of the subgroup lattice of a suitable locally finite 2-group. It should be noted that every lattice can be embedded in some algebraic lattice, namely, in the lattice of its ideals. It follows that every lattice is embeddable in the subgroup lattice of a locally finite 2-group. For a given class K of algebras, we say that K is *lattice-universal* if every lattice is embeddable in the subalgebra lattice of some algebra from K. In this sense, the class of all locally finite 2-groups is lattice-universal. It must be said here that the lattice universality of the class of all groups was first proved by Whitman in [12]. Other examples of lattice-universal classes of algebras can be found in [3–5, 10].

The main theorem of the present paper generalizes the key result of the author's paper [7] from the case $p = 2$ to the case of an arbitrary prime number p.

Theorem 1. *For an arbitrary prime p, let K be an abstract class of groups satisfying the following conditions*:

(1) *K contains a group of order p*;

(2) *K is closed under restricted direct products, semidirect products and direct limits over totally ordered sets.*

Then every algebraic lattice is isomorphic to a complete sublattice in the subgroup lattice of some group in K.

Since the class of all locally finite p-groups obviously satisfies conditions (1)–(2) of the theorem, from here we get the following corollary.

Corollary 1. *For an arbitrary prime p, every algebraic lattice is isomorphic to a complete sublattice in the subgroup lattice of a suitable locally finite p-group.*

As a consequence, we get the following statement.

Corollary 2. *For every prime p, the class of all locally finite p-groups is lattice-universal.*

The proof of Theorem 1 is given in the next section. Technically, it is based on the concept of so-called group valuations used by us in [7] when proving a specific case of this theorem for $p = 2$. Here we essentially apply the ideas and constructions of the mentioned paper.

Below we give some additional concepts and notation.

$\operatorname{Sub} G$ is the subgroup lattice of a group G.

$\langle X \rangle$ is the subgroup generated by a subset X of a given group.

The commutator $[u, v]$ of elements u and v of a group means $u^{-1}v^{-1}uv$ and u^v means $v^{-1}uv$.

$\prod\limits_{\lambda \in \Lambda} G_\lambda$ is the direct product of a set $\{G_\lambda \mid \lambda \in \Lambda\}$ of groups.

$\overline{\prod\limits_{\lambda \in \Lambda}} G_\lambda$ is the restricted direct product of a set $\{G_\lambda \mid \lambda \in \Lambda\}$ of groups; this group is a subgroup of the corresponding direct product, and it can be regarded as the group of all functions from $\operatorname{fun}(\Lambda, \bigcup\limits_{\lambda \in \Lambda} G_\lambda)$ with the property $f(\lambda) \in G_\lambda$ and with finite supports.

For given groups G and T, let us consider the direct product $\prod\limits_{h \in T} G^h$ of isomorphic copies G^h of the group G; here $G^1 = G$. This group is regarded as the group $\operatorname{fun}(T, G)$ of all functions from T to G and is denoted by G^T. The group T naturally acts on the group G^T in the following way: $f^t(h) = f(th)$ for all $f \in G^T$ and $t \in T$. With respect to this action, one can consider the semidirect product $T \leftthreetimes G^T$, which is denoted by $G \wr T$ and is called the *wreath product* of the group G by the group T. Here, for $t_1, t_2 \in T$ and $f_1, f_2 \in G^T$, we have

$$(t_1, f_1) \cdot (t_2, f_2) = (t_1 t_2,\ f_1^{t_2} f_2).$$

For a prime p, the group

$$\langle u, v \mid u^p = v^p = [u, v]^p = 1,\ [u, [u, v]] = [v, [u, v]] = 1 \rangle$$

is of the order p^3 and is isomorphic to the multiplicative *unitriangular group of matrices* of order 3 over the field of order p. So it will be denoted by $\mathbb{UT}_3(p)$. Here the generating elements u and v correspond to the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. For $p = 2$, this group is isomorphic to the 8-element *dihedral group*

$$\mathbb{D}_4 = \langle u, v \mid u^2 = v^2 = (uv)^4 = 1 \rangle.$$

Obviously, we have

$$\mathbb{UT}_3(p) \cong \langle u \rangle \leftthreetimes \langle v, [u, v] \rangle,$$

where the group $\langle u \rangle$ acts by conjugation on the normal subgroup $\langle v, [u, v] \rangle$ in $\mathbb{UT}_3(p)$ and

$$\langle v, [u, v] \rangle \cong \langle v \rangle \times \langle [u, v] \rangle.$$

$\mathrm{J}(P)$ is the *ideal lattice* of a given $\vee$-semilattice P with zero. Recall that every algebraic lattice is isomorphic to the ideal lattice $\mathrm{J}(P)$ of its $\vee$-semilattice P of compact elements.

The other definitions and notations used but undefined in the paper can be found in the books [1, 2, 9].

2. Concept of group valuation and the proof of the theorem

Let us give a key notion of the present paper. It was introduced by the author in [6] (see also [7, 8, 11]).

Definition 1. *Let P be a $\vee$-semilattice with zero. For a group G, we call a mapping $\delta\colon G \to P$ a **group valuation** if the following conditions hold*:

(1) $\delta(1) = 0$;

(2) $\delta(g^{-1}) = \delta(g)$ *for every* $g \in G$;

(3) $\delta(g_1 g_2) \le \delta(g_1) \vee \delta(g_2)$ *for every* $g_1, g_2 \in G$.

For an arbitrary ideal $I \in \mathrm{J}(P)$ and $a \in P$, let us put $O_\delta(I) = \delta^{-1}(I)$ and $O_\delta(a) = O_\delta(a{\downarrow})$, where $a{\downarrow}$ denotes the principal ideal generated by the element a. Obviously, $O_\delta(I)$ is a subgroup of G and $O_\delta(I) = \bigvee_{a \in I} O_\delta(a)$.

The following simple proposition explains the role of the notion of group valuation in our examination (see its proof, for example, in [7]).

Proposition 1. *Let a valuation $\delta\colon G \to P$ satisfy the following conditions*:

(1) *for every $a, b \in P$ and $g \in G$, $\delta(g) \le a \vee b$ implies $g \in \langle O_\delta(a), O_\delta(b)\rangle$*;

(2) *δ is a surjective mapping of G onto P.*

Then the mapping $O_\delta\colon \mathrm{J}(P) \to \operatorname{Sub} G$ is a complete embedding of the ideal lattice $\mathrm{J}(P)$ in the subgroup lattice $\operatorname{Sub} G$, and so the lattice $\mathrm{J}(P)$ is isomorphic to a complete sublattice of the corresponding subgroup lattice.

For groups G, G' and their valuations $\delta\colon G \to P$, $\delta'\colon G' \to P$, we say that the pair (G', δ') is an *extension* of the pair (G, δ) if G is a subgroup of G' and $\delta'|G = \delta$.

The following statement is key in proving our theorem.

Proposition 2. *Let G be a group and $\delta\colon G \to P$ a group valuation. Then, for arbitrary elements $a, b \in P$, there exists an extension (G', δ') of the pair (G, δ) such that, if $\delta(g) \le a \vee b$ for an element $g \in G$, then in the group G' the membership $g \in \langle O_{\delta'}(a), O_{\delta'}(b)\rangle$ holds*; *moreover, for an arbitrary prime p, one may take the group $G \wr \mathbb{UT}_3(p)$ as G'.*

This statement was proved in [7] for $p = 2$, and in this case the role of the group $\mathbb{UT}_3(p)$ was played by the dihedral group $\mathbb{D}_4$ isomorphic to $\mathbb{UT}_3(2)$.

We will prove Proposition 2 at the end of the section. But now we will explain how the theorem is derived from it. This derivation practically iterates a similar derivation for $p = 2$ in our paper [7], but we will give it for the reader's convenience.

Let L be an arbitrary algebraic lattice and P its $\vee$-semilattice of compact elements. Thus, we have $L \cong \mathrm{J}(P)$. Let p be an arbitrary prime number and K an abstract class of groups satisfying conditions (1)–(2) of the Theorem 1. Now consider the set $\{\langle w_a\rangle \mid a \in P\}$ of groups of order p generated by the elements w_a indexed by elements from P. Let

$$G^* = \overline{\prod_{a \in P}} \langle w_a \rangle$$

be the corresponding restricted direct product of these groups. Then $G^* \in K$ and each non-identity element from G^* can be uniquely represented (up to the permutation of the factors) as a term of the form $w_{a_1}^{\epsilon_1} w_{a_2}^{\epsilon_2} \cdots w_{a_n}^{\epsilon_n}$ (here $n \geq 1$, $w_{a_i} \neq w_{a_j}$ for $i \neq j$ and $1 \leq \epsilon_i \leq p-1$). Now we define a mapping $\delta^* : G^* \longrightarrow P$ by the following rule: $\delta^*(1) = 0$ and, if $g = w_{a_1}^{\epsilon_1} w_{a_2}^{\epsilon_2} \cdots w_{a_n}^{\epsilon_n}$, then $\delta^*(g) = a_1 \vee a_2 \vee \cdots \vee a_n$. It is easy to see that the mapping δ^* is a group valuation.

Now let

$$\{(a_\gamma, b_\gamma) \in P \times P \mid 0 < \gamma < \chi\}$$

be the well-ordered set of all couples from P^2. We define by induction on γ a set

$$\{G_\gamma \mid 0 \leq \gamma < \chi\}$$

of groups from K and a set

$$\{\delta_\gamma \mid 0 \leq \gamma < \chi\}$$

of group valuations $\delta_\gamma \colon G_\gamma \to P$ in the following way.

Put $G_0 = G^*$ and $\delta_0 = \delta^*$. If the ordinal $\gamma > 0$ is not limit, then the pair $(G_\gamma, \delta_\gamma)$ is an extension of the pair $(G_{\gamma-1}, \delta_{\gamma-1})$ with the property $g \in \langle O_{\delta_\gamma}(a_\gamma), O_{\delta_\gamma}(b_\gamma)\rangle$ in G_γ for every $g \in G^*$ satisfying the condition $\delta_{\gamma-1}(g) = \delta^*(g) \leq a_\gamma \vee b_\gamma$. Such an extension exists by Proposition 2, and in this case $G_\gamma = G_{\gamma-1} \wr \mathbb{UT}_3(p)$. If the ordinal γ is limit, then put

$$G_\gamma = \bigcup(G_\zeta \mid \zeta < \gamma) \quad \text{and} \quad \delta_\gamma = \bigcup(\delta_\zeta \mid \zeta < \gamma).$$

Further, put

$$G^{(1)} = \bigcup(G_\gamma \mid \gamma < \chi) \quad \text{and} \quad \delta^{(1)} = \bigcup(\delta_\gamma \mid \gamma < \chi).$$

By construction, the mapping $\delta^{(1)} \colon G^{(1)} \to P$ is a valuation and the pair $(G^{(1)}, \delta^{(1)})$ is an extension of the pair (G^*, δ^*) such that, for any elements $g \in G^*$ and $a, b \in P$ with $\delta^*(g) \leq a \vee b$, the membership $g \in \langle O_{\delta^{(1)}(a)}, O_{\delta^{(1)}(b)}\rangle$ holds in $G^{(1)}$. In addition, we have $\mathbb{UT}_3(p) \in K$, since K contains all p-element groups and, by condition (2) of the theorem, is closed under direct and semidirect products of any two of its groups and $\mathbb{UT}_3(p)$ as mentioned above is a semidirect product of the p-element group $\langle u\rangle$ with the direct product of the p-element groups $\langle v\rangle$ and $\langle [u, v]\rangle$. Therefore, $G^{(1)} \in K$ because, by the same condition, the class K is closed also under direct limits over totally ordered sets.

Next we construct the following increasing under inclusion chains of groups $G^{(n)} \in K$ and valuations $\delta^{(n)} \colon G^{(n)} \to P$ $(n \in \mathbb{N})$:

$$G^* = G^{(0)} \subset G^{(1)} \subset G^{(2)} \subset \ldots, \qquad \delta^* = \delta^{(0)} \subset \delta^{(1)} \subset \delta^{(2)} \subset \ldots$$

such that, for every $n > 1$, $\delta^{(n)}|G^{(n-1)} = \delta^{(n-1)}$ and, for every $g \in G^{(n-1)}$ and $a, b \in P$, if $\delta^{(n-1)}(g) \leq a \vee b$, then $g \in \langle O_{\delta^{(n)}}(a), O_{\delta^{(n)}}(b)\rangle$ in $G^{(n)}$; here the pair $(G^{(n)}, \delta^{(n)})$ is constructed from the pair $(G^{(n-1)}, \delta^{(n-1)})$ as it was done in the first step corresponding to $n = 1$.

Put

$$G = \bigcup(G^{(n)} \mid n \in \mathbb{N}) \quad \text{and} \quad \delta = \bigcup(\delta^{(n)} \mid n \in \mathbb{N}).$$

Then, by construction, we have $G \in K$ and the mapping $\delta \colon G \to P$ is a valuation with the property: for all $g \in G$ and $a, b \in P$, the inequality $\delta(g) \leq a \vee b$ implies $g \in \langle O_\delta(a), O_\delta(b)\rangle$, i.e., δ satisfies condition (1) of Proposition 1. In addition, δ^* is surjective and $\delta|G^* = \delta^*$, whence we deduce that the valuation δ is surjective as well, i.e., it satisfies condition (2) of Proposition 1. Therefore, by this proposition, there exists a complete embedding of the ideal lattice $\mathrm{J}(P)$ in the subgroup lattice $\mathrm{Sub}\, G$ of the constructed group $G \in K$. This means that the algebraic lattice L, for which P is the $\vee$-semilattice of compact elements, is isomorphic to a complete sublattice of $\mathrm{Sub}\, G$. The derivation is over. $\square$

The following two constructions of the semidirect product and direct product of group valuations were introduced by the author in [7].

If a group T acts on a group H, and if $\hat{\delta}\colon T \to P$ and $\tilde{\delta}\colon H \to P$ are group valuations such that

$$\tilde{\delta}(h^t) \leq \hat{\delta}(t) \vee \tilde{\delta}(h)$$

for any $h \in H$ and $t \in T$, then the mapping $\hat{\delta} \leftthreetimes \tilde{\delta}\colon T \leftthreetimes H \to P$ defined by

$$(\hat{\delta} \leftthreetimes \tilde{\delta})(t, h) = \hat{\delta}(t) \vee \tilde{\delta}(h)$$

is evidently a group valuation. It extends both $\tilde{\delta}\colon H \to P$ and $\hat{\delta}\colon T \to P$ under the canonical isomorphic embeddings $h \mapsto (1, h)$ and $t \mapsto (t, 1)$ of H and T respectively into $T \leftthreetimes H$. Thus the pair $(T \leftthreetimes H, \hat{\delta} \leftthreetimes \tilde{\delta})$ is an extension both of the pair $(H, \tilde{\delta})$ and of the pair $(T, \hat{\delta})$.

Definition 2. *The valuation $\hat{\delta} \leftthreetimes \tilde{\delta}$ is called the **semidirect product of** $\hat{\delta}$ **and** $\tilde{\delta}$.*

Definition 3. *If $\delta_i\colon G_i \to P$, $i = 1, \ldots, n$, are group valuations, then the mapping $\delta_1 \times \cdots \times \delta_n\colon G_1 \times \cdots \times G_n \to P$ defined by*

$$(\delta_1 \times \cdots \times \delta_n)(g_1, \ldots, g_n) = \delta_1(g_1) \vee \cdots \vee \delta_n(g_n)$$

*is evidently a group valuation extending each δ_i under the canonical embedding of G_i into $G_1 \times \cdots \times G_n$. It is called **the direct product of the valuations** δ_i**'s**.*

Let G be a group, $\delta\colon G \to P$ be a valuation, and let $a, b \in G$. Consider the group

$$\mathbb{UT}_3(p) = \big\langle u, v \mid u^p = v^p = [u, v]^p = 1,\ [u, [u, v]] = [v, [u, v]] = 1 \big\rangle.$$

Each of its elements can be uniquely written in the form $u^\alpha v^\beta [u, v]^\gamma$, where $\alpha, \beta, \gamma \in \{0, 1, \ldots, p-1\}$. It is easy to check that the product of two such terms is

$$u^{\alpha_1} v^{\beta_1} [u, v]^{\gamma_1} \cdot\ u^{\alpha_2} v^{\beta_2} [u, v]^{\gamma_2} = u^{\alpha_1+\alpha_2} v^{\beta_1+\beta_2} [u, v]^{\gamma_1+\gamma_2-\beta_1\alpha_2},$$

where exponents are added and multiplied modulo p.

Now we define a mapping $\hat{\delta}\colon \mathbb{UT}_3(p) \to P$ by the rule:

$$\hat{\delta}(1) = 0, \quad \hat{\delta}(u^\alpha) = a \quad \text{if} \quad 1 \leq \alpha \leq p-1, \quad \hat{\delta}(v^\beta) = b \quad \text{if} \quad 1 \leq \beta \leq p-1,$$
$$\hat{\delta}(t) = a \vee b \quad \text{for all other} \quad t \in \mathbb{UT}_3(p).$$

It is easy to see that the mapping $\hat{\delta}$ is a group valuation.

Next we construct an extension $\delta'\colon G \wr \mathbb{UT}_3(p) \to P$ of a group valuation δ desired in Proposition 2 from the group valuation $\hat{\delta}\colon \mathbb{UT}_3(p) \to P$ and from an additional valuation $\tilde{\delta}\colon G^{\mathbb{UT}_3(p)} \to P$. The valuation $\tilde{\delta}$ will be the direct product (see Definition 3) of group valuations $\delta_t\colon G \to P$ defined below for every $t \in \mathbb{UT}_3(p)$. Their constructions generalize the corresponding constructions for $p = 2$ given in [7].

We set $\delta_t(1) = 0$ for any $t \in \mathbb{UT}_3(p)$. For a non-identity element $g \in G$, we set

(1) $\delta_1(g) = \delta(g)$,

(2) $\delta_{u^\alpha}(g) = a \vee \delta(g) \quad \text{if} \quad 1 \leq \alpha \leq p-1$,

(3) $\delta_{v^\beta}(g) = b \vee \delta(g) \quad \text{if} \quad 1 \leq \beta \leq p-1$,

(4) $\delta_{[u,v]^\gamma}(g) = \begin{cases} 0 & \text{if} \quad \delta(g) \le a \vee b, \\ a \vee b \vee \delta(g) & \text{otherwise} \end{cases} \quad \text{if} \quad 1 \le \gamma \le p-1,$

(5) $\delta_{u^\alpha[u,v]^\gamma}(g) = \begin{cases} a & \text{if} \quad \delta(g) \le a \vee b, \\ a \vee b \vee \delta(g) & \text{otherwise} \end{cases} \quad \text{if} \quad 1 \le \alpha \le p-1 \quad \text{and} \quad 1 \le \gamma \le p-1,$

(6) $\delta_{v^\beta[u,v]^\gamma}(g) = \begin{cases} b & \text{if} \quad \delta(g) \le a \vee b, \\ a \vee b \vee \delta(g) & \text{otherwise} \end{cases} \quad \text{if} \quad 1 \le \beta \le p-1 \quad \text{and} \quad 1 \le \gamma \le p-1,$

(7) $\delta_{u^\alpha v^\beta[u,v]^\gamma}(g) = a \vee b \vee \delta(g) \quad \text{if} \quad 1 \le \alpha \le p-1, \quad 1 \le \beta \le p-1 \quad \text{and} \quad 0 \le \gamma \le p-1.$

Checking that δ_t is a group valuation for every $t \in \mathbb{UT}_3(p)$ is very simple and completely identical to checking a similar statement in [7].

Lemma 1. *For any $w, t \in \mathbb{UT}_3(p)$ and $g \in G$, the inequality $\delta_{tw}(g) \le \hat{\delta}(t) \vee \delta_w(g)$ holds.*

P r o o f. From the definitions of the group valuations δ_t, it directly follows that the equality $a \vee b \vee \delta_w(g) = a \vee b \vee \delta(g)$ holds for every $w \in \mathbb{UT}_3(p)$ and $g \in G$. Therefore, if $\hat{\delta}(t) = a \vee b$, then we have

$$\delta_{tw}(g) \le a \vee b \vee \delta(g) = a \vee b \vee \delta_w(g) = \hat{\delta}(t) \vee \delta_w(g).$$

The case $\hat{\delta}(t) \ne a \vee b$ is true only if

$$t \in \{1, u^\alpha, v^\beta \mid 1 \le \alpha \le p-1,\ 1 \le \beta \le p-1\}.$$

In the case $t = 1$ the inequality $\delta_{tw}(g) \le \hat{\delta}(t) \vee \delta_w(g)$ is evident.

Let $t = u^\alpha$, where $1 \le \alpha \le p-1$. Then $\hat{\delta}(t) = a$. Here, if $b \le \delta_w(g)$, then again

$$\delta_{tw}(g) \le a \vee b \vee \delta(g) = a \vee b \vee \delta_w(g) = \hat{\delta}(t) \vee \delta_w(g).$$

If $b \not\le \delta_w(g)$, then the following four cases are possible:

(1) $w = 1$,

(2) $w = u^{\alpha'}$, where $1 \le \alpha' \le p-1$ and $\delta_w(g) = a \vee \delta(g)$,

(3) $w = [u,v]^\gamma$, where $1 \le \gamma \le p-1$, $\delta(g) \le a \vee b$ and $\delta_w(g) = 0$,

(4) $w = u^{\alpha'}[u,v]^\gamma$, where $1 \le \alpha' \le p-1$, $1 \le \gamma \le p-1$, $\delta(g) \le a \vee b$ and $\delta_w(g) = a$.

In case (1), we have $\delta_{tw}(g) = \delta_{u^\alpha}(g) = a \vee \delta(g) = a \vee \delta_1(g) = \hat{\delta}(t) \vee \delta_w(g)$.

In case (2), we have $\delta_{tw}(g) = \delta_{u^{\alpha+\alpha'}}(g) \le a \vee \delta(g) = a \vee \delta_w(g) = \hat{\delta}(t) \vee \delta_w(g)$.

In case (3), we have $\delta_{tw}(g) = \delta_{u^\alpha[u,v]^\gamma}(g) = a \le a \vee \delta_w(g) = \hat{\delta}(t) \vee \delta_w(g)$.

In case (4), we have $\delta_{tw}(g) = \delta_{u^{\alpha+\alpha'}[u,v]^\gamma}(g) \le a = \hat{\delta}(t) \vee \delta_w(g)$.

The case $t = v^\beta$, where $1 \le \beta \le p-1$, is symmetric to the previous one and can be checked by a parallel argument. □

Next, as noted above, we set $\tilde{\delta}\colon G^{\mathbb{UT}_3(p)} \to P$ as the direct product of the valuations δ_t, $t \in \mathbb{UT}_3(p)$. This is a group valuation extending $\delta_1 = \delta$ and

$$\tilde{\delta}(f) = \bigvee_{t \in \mathbb{UT}_3(p)} \delta_t(f(t))$$

for any $f\colon \mathbb{UT}_3(p) \to G$.

We then define $\delta'\colon G \wr \mathbb{UT}_3(p) \to P$ as the semidirect product of $\hat{\delta}\colon \mathbb{UT}_3(p) \to P$ and $\tilde{\delta}\colon G^{\mathbb{UT}_3(p)} \to P$ (see Definition 2). This is an extension of $\tilde{\delta}$, hence also of δ. To prove that the semidirect product $\delta' = \hat{\delta} \leftthreetimes \tilde{\delta}$ is indeed a group valuation, we have to check that

$$\tilde{\delta}(f^t) \leq \hat{\delta}(t) \vee \tilde{\delta}(f)$$

for any $f \in G^{\mathbb{UT}_3(p)}$ and $t \in \mathbb{UT}_3(p)$. But this is easily accomplished using Lemma 1:

$$\begin{aligned}
\tilde{\delta}(f^t) &= \bigvee_{w \in \mathbb{UT}_3(p)} \delta_w(f^t(w)) = \bigvee_{w \in \mathbb{UT}_3(p)} \delta_w(f(tw)) = \bigvee_{t^{-1}w \in \mathbb{UT}_3(p)} \delta_{t^{-1}w}(f(tt^{-1}w)) \\
&= \bigvee_{t^{-1}w \in \mathbb{UT}_3(p)} \delta_{t^{-1}w}(f(w)) \leq \bigvee_{t^{-1}w \in \mathbb{UT}_3(p)} \left(\hat{\delta}(t^{-1}) \vee \delta_w(f(w))\right) \\
&= \hat{\delta}(t^{-1}) \vee \left(\bigvee_{t^{-1}w \in \mathbb{UT}_3(p)} \delta_w(f(w)) \right) = \hat{\delta}(t) \vee \left(\bigvee_{w \in \mathbb{UT}_3(p)} \delta_w(f(w)) \right) \\
&= \hat{\delta}(t) \vee \tilde{\delta}(f).
\end{aligned}$$

Thus, we proved that the mapping $\delta'\colon G \wr \mathbb{UT}_3(p) \to P$ is a group valuation and the pair $(G \wr \mathbb{UT}_3(p), \delta')$ is an extension of the pair (G, δ). Now it remains to note that this extension has the required property, i.e., if $\delta(g) \leq a \vee b$ for an element $g \in G$, then, in the group $G \wr \mathbb{UT}_3(p)$, the membership $g \in \langle O_{\delta'}(a), O_{\delta'}(b) \rangle$ holds. Indeed, by construction, $\delta'(u) = \hat{\delta}(u) = a$ and $\delta'(v) = \hat{\delta}(v) = b$. In addition, the element $g \in G$ as an element of the group $G \wr \mathbb{UT}_3(p)$ means a function

$$f(w) \in G^{\mathbb{UT}_3(p)}$$

such that

$$f(w) = \begin{cases} g, & \text{if} \quad w = 1, \\ 1, & \text{if} \quad w \neq 1. \end{cases}$$

Therefore, for an arbitrary $t \in \mathbb{UT}_3(p)$, we have $g^t = f^t(w) = f(tw)$ and

$$\begin{aligned}
\delta'(g^t) = \tilde{\delta}(f^t) &= \bigvee_{w \in \mathbb{UT}_3(p)} \delta_w(f^t(w)) = \bigvee_{w \in \mathbb{UT}_3(p)} \delta_w(f(tw)) \\
&= \delta_{t^{-1}}(f(1)) \vee \left(\bigvee_{w \neq t^{-1}} \delta_w(f(tw)) \right) = \delta_{t^{-1}}(g) \vee \left(\bigvee_{w \neq t^{-1}} \delta_w(1) \right) = \delta_{t^{-1}}(g).
\end{aligned}$$

From here we obtain that $\delta'(g^{[v,u]}) = \delta_{[u,v]}(g) = 0$ for every element $g \in G$ such that $\delta(g) \leq a \vee b$. This implies that

$$g = [v,u] \cdot g^{[v,u]} \cdot [u,v] \in \langle O_{\delta'}(a), O_{\delta'}(b), O_{\delta'}(0) \rangle = \langle O_{\delta'}(a), O_{\delta'}(b) \rangle.$$

The proof of Proposition 2, and with it the Theorem 1, is complete. □

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