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A CHARACTERIZATION OF MEIXNER ORTHOGONAL POLYNOMIALS VIA A CERTAIN TRANSFERT OPERATOR

Emna Abassi

Faculté des Sciences de Tunis, Université de Tunis El Manar,
Rommana 1068, Tunisia

emna.abassi@fst.utm.tn

Lotfi Khériji

Institut Préparatoire aux Etudes d'Ingénieur El Manar,
Université de Tunis El Manar,
Rommana 1068, Tunisia

kheriji@yahoo.fr

Abstract: Here we consider a certain transfert operator $\mathrm{M}_{(c,\omega)} = I_{\mathcal{P}} - c\,\tau_\omega$, $\omega \neq 0$, $c \in \mathbb{R} - \{0, 1\}$, and we prove the following statement: up to an affine transformation, the only orthogonal sequence that remains orthogonal after application of this transfert operator is the Meixner polynomials of the first kind.

Keywords: Orthogonal polynomials, Regular form, Meixner polynomials, Divided-difference operator, Transfert operator, Hahn property.

1. Introduction and preliminaries

Let $\mathcal{O}$ be a linear operator acting on the space of polynomials as a lowering operator (the derivative [4, 18, 19], the q-derivative [4, 12, 14, 15], the divided-difference [1], the Dunkl [6, 8, 9, 11, 13], the q-Dunkl [5, 7, 13], other [17, 21]), a transfert operator (see [20]) or a raising operator (see [2, 3, 17]). Many researchers in this vast field cited above had the concern to characterize the $\mathcal{O}$-classical polynomial sequences that is those which fulfill the so-called Hahn property: the sequences $\{P_n\}_{n\geq 0}$ and $\{\mathcal{O}P_n\}_{n\geq 0}$ are orthogonal.

By the way, in [20], the authors characterized the $I_{(q,\omega)}$-classical orthogonal polynomials where $I_{(q,\omega)}$ is a transfert operator acting on the space of polynomials $\mathcal{P}$ and defined by [20]

$$I_{(q,\omega)} := I_{\mathcal{P}} + \omega h_q, \quad \omega \in \mathbb{C} \setminus \{0\}, \quad q \in \mathbb{C}_\omega := \big\{z \in \mathbb{C},\ z \neq 0,\ z^{n+1} \neq 1,\ 1 + \omega z^n \neq 0,\ n \in \mathbb{N}\big\},$$

with $I_{\mathcal{P}}$ being the identity operator in $\mathcal{P}$ and $(h_q f)(x) = f(qx)$, $f \in \mathcal{P}$ (homothety). Therefore, our goal is to consider the following transfert operator $\mathrm{M}_{(c,\omega)}$ acting on $\mathcal{P}$ and defined by

$$\mathrm{M}_{(c,\omega)} = I_{\mathcal{P}} - c\,\tau_\omega, \quad \omega \neq 0, \quad c \in \mathbb{R} - \{0, 1\}, \tag{1.1}$$

where

$$(\tau_\omega f)(x) = f(x - \omega), \quad f \in \mathcal{P},$$

(translation) and to characterize all sequences of orthogonal polynomials $\{P_n\}_{n\geq 0}$ having the Hahn property; the resulting up an affine transformation (that is to say up a composition of a homothety and a translation; see (1.4) below), is the Meixner polynomials of the first kind (see Theorem 2 below). Indeed, in Section 2, firstly we deal with the $\mathrm{M}_{(c,\omega)}$-character by presenting some characterizations of it (see Theorem 1), secondly, we establish the system verified by the elements of second-order recurrence relation for the sequences $\{P_n\}_{n\geq 0}$ and $\{\mathrm{M}_{(c,\omega)}P_n\}_{n\geq 0}$ and thirdly we solve it to deduce the desired result (Theorem 2). Moreover, the divided-difference equation fulfilled by its canonical form and the second order linear divided-difference equation satisfied by any Meixner polynomial are highlighted.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by

$$(u)_n := \langle u, x^n\rangle, \quad n \geq 0$$

the moments of u. The form u is called regular if we can associate with it a sequence of monic polynomials $\{P_n\}_{n\geq 0}$ with $\deg P_n = n,\ n \geq 0$ ((MPS) in short) [18] such that

$$\langle u, P_m P_n\rangle = r_n \delta_{n,m}, \quad n, m \geq 0; \quad r_n \neq 0, \quad n \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is then called orthogonal with respect to u ((MOPS) in short). In this case, the (MOPS) $\{P_n\}_{n\geq 0}$ fulfils the standard recurrence relation ((TTRR) in short) [10, 18]

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0\ , \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \end{cases} \tag{1.2}$$

where

$$\beta_n = \frac{\langle u, xP_n^2\rangle}{r_n}, \quad \gamma_{n+1} = \frac{r_{n+1}}{r_n} \neq 0, \quad n \geq 0.$$

Moreover, the regular form u will be supposed normalized that is to say $(u)_0 = 1$.

For any form u, any polynomial g and $a, \omega \in \mathbb{C}\backslash\{0\}$, $b \in \mathbb{C}$, we let $\tau_b u$, $h_a u$, gu, $Du = u'$, $D_\omega u$ be the forms defined by duality [18] namely

$$\langle \tau_b u, f\rangle = \langle u, \tau_{-b} f\rangle, \quad \langle h_a u, f\rangle = \langle u, h_a f\rangle, \quad \langle gu, f\rangle = \langle u, gf\rangle,$$
$$\langle u', f\rangle = -\langle u, f'\rangle, \quad \langle D_\omega u, f\rangle = -\langle u, D_{-\omega} f\rangle$$

where

$$(\tau_{-b} f)(x) = f(x+b), \quad (h_a f)(x) = f(ax), \quad (D_{-\omega} f)(x) = \frac{f(x) - f(x-\omega)}{\omega}, \quad f \in \mathcal{P},$$

and due to the well known formulas [1, 18] we have

$$\tau_b(fu) = (\tau_b f)(\tau_b u), \quad h_a(fu) = (h_{a^{-1}} f)\,(h_a u), \quad u \in \mathcal{P}', \quad f \in \mathcal{P}. \tag{1.3}$$

Let δ_b be the Dirac mass at b defined by

$$\langle \delta_b, f\rangle = f(b), \quad b \in \mathbb{C}, \quad f \in \mathcal{P}.$$

In addition, let $\{\widehat{P}_n\}_{n\geq 0}$ be the (MPS) defined by

$$\widehat{P}_n(x) = a^{-n} P_n(ax+b), \quad n \geq 0, \quad a \neq 0, \quad b \in \mathbb{C}.$$

If $\{P_n\}_{n\geq 0}$ is a (MOPS) associated with u, then $\{\widehat{P}_n\}_{n\geq 0}$ is a (MOPS) associated with

$$\widehat{u} = \big(h_{a^{-1}} \circ \tau_{-b}\big) u$$

and fulfilling the (TTRR) in (1.2) ($\beta_n \hookleftarrow \widehat{\beta}_n$, $\gamma_{n+1} \hookleftarrow \widehat{\gamma}_{n+1}$, $n \geq 0$) with [18]

$$\widehat{\beta}_n = \frac{\beta_n - b}{a}, \quad \widehat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0. \tag{1.4}$$

Let now $\{P_n\}_{n\geq 0}$ be a (MPS) and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ defined by

$$\langle u_n, P_m \rangle = \delta_{n,m}, \quad n, m \geq 0.$$

Let us recall some results [18].

Lemma 1 [18]. *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent*

(i) $\langle u, P_{m-1} \rangle \neq 0, \quad \langle u, P_n \rangle = 0, \quad n \geq m,$
(ii) $\exists \lambda_\nu \in \mathbb{C}\,, \quad 0 \leq \nu \leq m-1, \quad \lambda_{m-1} \neq 0,$

such that

$$u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu.$$

As a consequence,

- the dual sequence $\{\widehat{u}_n\}_{n\geq 0}$ of $\{\widehat{P}_n\}_{n\geq 0}$ is given by

$$\widehat{u}_n = a^n \big(h_{a^{-1}} \circ \tau_{-b}\big) u_n, \quad n \geq 0,$$

- when $\{P_n\}_{n\geq 0}$ be a (MOPS) then $u = u_0$. In this case, we have

$$u_n = r_n^{-1} P_n u_0, \quad n \geq 0$$

and reciprocally. Lastly, when u_0 is regular and Φ is a polynomial such that $\Phi u_0 = 0$, then $\Phi = 0$.

The monic Meixner polynomials $\{M_n(.;\alpha,c)\}_{n\geq 0}$ of the first kind are given by [10, 16]

$$M_n(x;\alpha,c) = (\alpha+1)_n \left(\frac{c}{c-1}\right)^n {}_2F_1\left(\begin{matrix} -n, -x \\ \alpha+1 \end{matrix} \Big| 1 - \frac{1}{c} \right), \quad n \geq 0,$$

they are orthogonal with respect to the discrete weight

$$\rho(x) = \frac{c^x (\alpha+1)_x}{x!}, \quad x \in \mathbb{N}$$

for $\alpha > -1$, $0 < c < 1$. Here, the Pochhammer symbol $(z)_n$ takes the form

$$(z)_0 = 1, \quad (z)_n = \prod_{k=1}^{n} (z + k - 1), \quad n \geq 1,$$

and ${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1\left(\begin{matrix} p, q \\ r \end{matrix} \Big| s \right) = \sum_{k=0}^{\infty} \frac{(p)_k (q)_k}{(r)_k} \frac{s^k}{k!}.$$

By describing exhaustively the $D_{-\omega}$-classical orthogonal polynomials in [1], the authors rediscover the (MOPS) of Meixner $\{M_n(.;\alpha,c)\}_{n\geq 0}$ orthogonal with respect to the D_{-1}-classical Meixner form $\mathcal{M}(\alpha,c)$ for $\alpha \neq -n-1$, $n \geq 0$, $c \in \mathbb{C} - \{0,1\}$ and the positive definite case occurring for $\alpha+1 > 0$, $c \in (0,\infty) - \{1\}$; they establish successively the (TTRR) elements, the divided-difference equation, the modified moments, the discrete representation and the second order linear divided-difference equation (see the following),

$$\begin{cases} \beta_n = \dfrac{c}{1-c}(\alpha+1) + \dfrac{1+c}{1-c}n, \quad \gamma_{n+1} = \dfrac{c}{(1-c)^2}(n+1)(n+\alpha+1), \quad n \geq 0, \\ D_{-1}\big((x+\alpha+1)\mathcal{M}(\alpha,c)\big) - \big((1-c^{-1})x+\alpha+1\big)\mathcal{M}(\alpha,c) = 0, \\ \big(\mathcal{M}(\alpha,c)\big)_n^{\phi} = \left(\dfrac{c}{1-c}\right)^n \dfrac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)}, \quad n \geq 0, \quad c \in \mathbb{C}-\{0,1\}, \quad \alpha+1 \in \mathbb{C}-(-\mathbb{N}), \\ \mathcal{M}(\alpha,c) = (1-c)^{\alpha+1} \displaystyle\sum_{k\geq 0} \dfrac{\Gamma(\alpha+1+k)}{\Gamma(\alpha+1)} \dfrac{c^{-k}}{k!}\delta_k, \quad 0<|c|<1, \quad \alpha \neq -n-1, \quad n \geq 0, \\ (x+\alpha+1)(D_{-1}\circ D_1 M_{n+1})(x;\alpha,c) + \big((1-c^{-1})x+\alpha+1\big)(D_1 M_{n+1})(x;\alpha,c) \\ \qquad\qquad -(n+1)(1-c^{-1})M_{n+1}(x;\alpha,c) = 0, \quad n \geq 0. \end{cases} \tag{1.5}$$

2. Main result

2.1. The $\mathrm{M}_{(c,\omega)}$-classical character

First of all, let $\omega \neq 0$ and $c \in \mathbb{R}-\{0,1\}$. By virtue of (1.1) we have

$$(\mathrm{M}_{(c,\omega)}f)(x) = f(x) - cf(x-\omega), \quad f \in \mathcal{P}. \tag{2.1}$$

Particularly,

$$(\mathrm{M}_{(c,\omega)}1)(x) = 1-c, \quad (\mathrm{M}_{(c,\omega)}\xi^n)(x) = (1-c)x^n + \text{lower degree terms}, \quad n \geq 1. \tag{2.2}$$

When $c=1$, $\mathrm{M}_{(1,\omega)}$ is not a transfert operator but a lowering one since $\mathrm{M}_{(1,\omega)} = \omega D_{-\omega}$. From (1.1), we have

$$\mathrm{M}_{(c,\omega)} = I_{\mathcal{P}} - c\,\tau_\omega.$$

The transposed ${}^t\mathrm{M}_{(c,\omega)}$ of $\mathrm{M}_{(c,\omega)}$ is

$${}^t\mathrm{M}_{(c,\omega)} = I_{\mathcal{P}'} - c\,\tau_{-\omega} = \mathrm{M}_{(c,-\omega)},$$

leaving out a light abuse of notation without consequence.

Thus,

$$\langle \mathrm{M}_{(c,-\omega)}u, f\rangle = \langle u, \mathrm{M}_{(c,\omega)}f\rangle, \; u \in \mathcal{P}', \; f \in \mathcal{P}.$$

Particularly, by virtue of (2.2) we get

$$(M_{(c,-\omega)}u)_0 = 1-c, \quad (M_{(c,-\omega)}u)_n = (1-c)(u)_n - c\sum_{k=0}^{n-1}\binom{n}{k}(-\omega)^{n-k}(u)_k, \quad n \geq 1.$$

Lemma 2. *The following formulas hold*

$$\mathrm{M}_{(c,\omega)}(fg)(x) = f(x)(\mathrm{M}_{(1,\omega)}g)(x) + (\tau_\omega g)(x)(\mathrm{M}_{(c,\omega)}f)(x), \quad f,g \in \mathcal{P}, \tag{2.3}$$

$$\mathrm{M}_{(c,-\omega)}(fu) = (\tau_{-\omega}f)(\mathrm{M}_{(c,-\omega)}u) + (\mathrm{M}_{(1,-\omega)}f)u, \quad u \in \mathcal{P}', \quad f \in \mathcal{P}, \tag{2.4}$$

$$h_a \circ \mathrm{M}_{(c,\omega)} = \mathrm{M}_{(c,a^{-1}\omega)} \circ h_a \;\; in\; \mathcal{P}, \quad h_a \circ \mathrm{M}_{(c,-\omega)} = \mathrm{M}_{(c,-a\omega)} \circ h_a \;\; in\; \mathcal{P}', \quad a \in \mathbb{C}-\{0\}, \tag{2.5}$$

$$\tau_b \circ \mathrm{M}_{(c,\omega)} = \mathrm{M}_{(c,\omega)} \circ \tau_b \;\; in\; \mathcal{P}, \quad \tau_b \circ \mathrm{M}_{(c,-\omega)} = \mathrm{M}_{(c,-\omega)} \circ \tau_b \;\; in\; \mathcal{P}', \quad b \in \mathbb{C}. \tag{2.6}$$

P r o o f. The proof is straightforward since definitions and duality. ☐

Now consider a (MPS) $\{P_n\}_{n\geq 0}$. On account of (2.2), let us define the (MPS) $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ by

$$P_n^{[1]}(x;c,\omega) = \frac{(\mathrm{M}_{(c,\omega)}P_n)(x)}{1-c}, \quad \omega \neq 0, \quad c \in \mathbb{R}-\{0,1\}, \quad n \geq 0. \tag{2.7}$$

Denoting by $\{u_n^{[1]}(c,\omega)\}_{n\geq 0}$ the dual sequence of $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$, we have the result

Lemma 3. *The following formula holds*

$$\mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)) = (1-c)u_n, \quad n \geq 0. \tag{2.8}$$

P r o o f. Indeed, from the definition it follows

$$\langle u_n^{[1]}(c,\omega), P_m^{[1]}(x;c,\omega)\rangle = \delta_{n,m}, \quad n,m \geq 0,$$

so we have

$$\langle (\mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)), P_m\rangle = (1-c)\delta_{n,m}, \quad n,m \geq 0,$$

therefore,

$$\langle \mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)), P_m\rangle = 0, \quad m \geq n+1, \quad n \geq 0;$$
$$\langle \mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)), P_n\rangle = 1-c, \quad n \geq 0.$$

By virtue of Lemma 1, we get

$$\mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)) = \sum_{\nu=0}^{n} \lambda_{n,\nu} u_\nu, \quad n \geq 0.$$

But,

$$\langle \mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)), P_\mu\rangle = \lambda_{n,\mu}, \quad 0 \leq \mu \leq n,$$

with $\lambda_{n,\mu} = 0,\ 0 \leq \mu < n$ and $\lambda_{n,n} = 1-c$. The formula (2.8) is then established. ☐

Definition 1. *The (MPS) $\{P_n\}_{n\geq 0}$ is called $\mathrm{M}_{(c,\omega)}$-classical if $\{P_n\}_{n\geq 0}$ and $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ are orthogonal.*

Remark 1. When the (MPS) $\{P_n\}_{n\geq 0}$ is orthogonal, it satisfies the (TTRR) (1.2). When the (MPS) $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ is orthogonal, it satisfies the (TTRR) (1.2) with the notations ($\beta_n \hookleftarrow \beta_n^{[1]}$, $\gamma_{n+1} \hookleftarrow \gamma_{n+1}^{[1]}$, $n \geq 0$).

Theorem 1. *For any (MOPS) $\{P_n\}_{n\geq 0}$, the following assertions are equivalent.*

a) *The sequence $\{P_n\}_{n\geq 0}$ is $\mathrm{M}_{(c,\omega)}$-classical.*
b) *There exist a polynomial ϕ monic, $\deg \phi \leq 1$ and a constant $K \neq 0$ such that*

$$\mathrm{M}_{(c,-\omega)}(\phi u_0) - K^{-1}(1-c)u_0 = 0, \tag{2.9}$$
$$1 - c - K\phi'(0)\,\omega\, n \neq 0, \quad n \geq 0. \tag{2.10}$$

c) *There exist a polynomial ϕ monic,* $\deg\phi \leq 1$, *a constant* $K \neq 0$ *and a sequence of complex numbers* $\{\lambda_n\}_{n\geq 0}$, $\lambda_n \neq 0$, $n \geq 0$, *such that*

$$\begin{aligned}&\big(K\phi(x) - 1 + c\big)(\mathrm{M}_{(c,-\omega)} \circ \mathrm{M}_{(c,\omega)}P_n)(x)\\ &+(c-1)(K\phi(x)-1)(\mathrm{M}_{(c,\omega)}P_n)(x) = \lambda_n P_n(x), \quad n \geq 0.\end{aligned} \tag{2.11}$$

P r o o f. a) $\Rightarrow$ b), a) $\Rightarrow$ c).

From (2.8) and the regularity of u_0 and $u_0^{[1]}(c,\omega)$, we have

$$\mathrm{M}_{(c,-\omega)}\big(P_n^{[1]}(.;c,\omega)u_0^{[1]}(c,\omega)\big) = \zeta_n\, P_n u_0, \quad n \geq 0,$$

with

$$\zeta_n = (1-c)\,\frac{\langle u_0^{[1]}(c,\omega), (P_n^{[1]}(.;c,\omega))^2\rangle}{\langle u_0, P_n^2\rangle}, \quad n \geq 0.$$

By (2.4), we get

$$(\tau_{-\omega}P_n^{[1]}(.;c,\omega))\mathrm{M}_{(c,-\omega)}(u_0^{[1]}(c,\omega)) + (\mathrm{M}_{(1,-\omega)}P_n^{[1]}(.;c,\omega))u_0^{[1]}(c,\omega) = \zeta_n\, P_n u_0, \quad n \geq 0.$$

In accordance with the definition of $\mathrm{M}_{(c,-\omega)}$, one may write

$$\mathrm{M}_{(c,-\omega)}(u_0^{[1]}(c,\omega)) = u_0^{[1]}(c,\omega) - c(\tau_{-\omega}u_0^{[1]}(c,\omega)),$$

which yields

$$P_n^{[1]}(.;c,\omega)u_0^{[1]}(c,\omega) - c(\tau_{-\omega}P_n^{[1]}(.;c,\omega))(\tau_{-\omega}u_0^{[1]}(c,\omega)) = \zeta_n\, P_n u_0, \quad n \geq 0. \tag{2.12}$$

Taking $n=0$ in (2.12) leads to

$$u_0^{[1]}(c,\omega) - c(\tau_{-\omega}u_0^{[1]}(c,\omega)) = (1-c)u_0. \tag{2.13}$$

Injecting (2.13) in (2.12) gives

$$\big\{P_n^{[1]}(.;c,\omega) - (\tau_{-\omega}P_n^{[1]}(.;c,\omega))\big\}u_0^{[1]}(c,\omega) = \big\{\zeta_n\, P_n - (1-c)(\tau_{-\omega}P_n^{[1]}(.;c,\omega))\big\}u_0, \quad n \geq 0. \tag{2.14}$$

Now, taking $n=1$ in (2.14), we obtain

$$u_0^{[1]}(c,\omega) = K\phi(x)u_0, \tag{2.15}$$

where K be a normalization constant since ϕ monic and

$$K\phi(x) = \frac{1-c}{\omega}\Big\{(1-\frac{\gamma_1^{[1]}}{\gamma_1})x + \omega + \frac{\gamma_1^{[1]}}{\gamma_1}\beta_0 - \beta_0^{[1]}\Big\}.$$

Applying the operator $\tau_{-\omega}$ to (2.15), we get

$$(\tau_{-\omega}u_0^{[1]}(c,\omega)) = K(\tau_{-\omega}\phi)(x)(\tau_{-\omega}u_0). \tag{2.16}$$

Replacing (2.16) and (2.15) in (2.13) leads to the desired result (2.9). By virtue of (2.15), the formula in (2.14) becomes

$$\Big\{K\phi\Big(P_n^{[1]}(.;c,\omega) - (\tau_{-\omega}P_n^{[1]}(.;c,\omega))\Big) + (1-c)(\tau_{-\omega}P_n^{[1]}(.;c,\omega)) - \zeta_n\, P_n\Big\}u_0 = 0, \quad n \geq 0.$$

Therefore,

$$K\phi\Big(P_n^{[1]}(.;c,\omega)-(\tau_{-\omega}P_n^{[1]}(.;c,\omega))\Big)+(1-c)(\tau_{-\omega}P_n^{[1]}(.;c,\omega))-\zeta_n\,P_n=0,\quad n\geq 0,$$

thanks to the regularity of u_0. Moreover, from (2.1) with the change $\omega\leftarrow-\omega$, we may write

$$(\tau_{-\omega}P_n^{[1]}(.;c,\omega))=c^{-1}\Big(P_n^{[1]}(.;c,\omega)-(\mathrm{M}_{(c,-\omega)}P_n^{[1]}(.;c,\omega))\Big),\quad n\geq 0.$$

Consequently, the last equation becomes

$$\begin{aligned}\big(K\phi(x)-1+c\big)(\mathrm{M}_{(c,-\omega)}\circ\mathrm{M}_{(c,\omega)}P_n)(x)+(c-1)(K\phi(x)-1)(\mathrm{M}_{(c,\omega)}P_n)(x)\\ =c(1-c)\zeta_nP_n(x),\quad n\geq 0.\end{aligned} \tag{2.17}$$

Writing into (2.17)

$$\begin{cases}\phi(x)=\phi'(0)x+\phi(0),\\ (\mathrm{M}_{(c,\omega)}P_n)(x)=P_n(x)-cP_n(x-\omega),\\ (\mathrm{M}_{(c,-\omega)}\circ\mathrm{M}_{(c,\omega)}P_n)(x)=(1+c^2)P_n(x)-c\big(P_n(x-\omega)+P_n(x+\omega)\big),\\ P_n(x)=\displaystyle\sum_{k=0}^{n}a_{n,k}x^k,\quad a_{n,n}=1,\quad n\geq 0,\end{cases}$$

and by comparing the degrees we obtain

$$1-c-K\phi'(0)\,\omega\,n=\zeta_n\neq 0,\quad n\geq 0.$$

Hence (2.10) and a) ⇒ b).
Finally, (2.17) is (2.11) with $\lambda_n=c(1-c)\zeta_n\neq 0$, $n\geq 0$. We have also proved that a) ⇒ c).

b) ⇒ a) Let us suppose that there exist a polynomial ϕ monic, $\deg\phi\leq 1$ and a constant $K\neq 0$ such that (2.9)–(2.10) are valid. From (2.9), we have

$$0=\langle\mathrm{M}_{(c,-\omega)}(\phi u_0)-K^{-1}(1-c)u_0,1\rangle=(1-c)\big(\langle u_0,\phi\rangle-K^{-1}\big).$$

Thus,

$$K^{-1}=\langle u_0,\phi\rangle=\phi'(0)\beta_0+\phi(0)=\phi(\beta_0).$$

Necessarily, $\phi(\beta_0)\neq 0$. Let $v=K\phi u_0$. We are going to prove that the (MPS) $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ is orthogonal with respect to v. We have successively

$$\langle v,P_0^{[1]}(.;c,\omega)\rangle=K\langle u_0,\phi\rangle=1, \tag{2.18}$$

for all $n\geq 1$,

$$\begin{aligned}\langle v,P_n^{[1]}(.;c,\omega)\rangle&=\frac{K}{1-c}\langle\phi u_0,\mathrm{M}_{(c,\omega)}P_n\rangle=\frac{K}{1-c}\langle\mathrm{M}_{(c,-\omega)}(\phi u_0),P_n\rangle\\ &\underset{(2.9)}{=}\frac{K}{1-c}\langle K^{-1}(1-c)u_0,P_n\rangle=0,\end{aligned}$$

and for $m \geq 1,\ n \geq 0$,

$$\begin{aligned}\langle v, x^m P_n^{[1]}(.;c,\omega)\rangle &= \frac{K}{1-c}\langle \phi u_0, x^m(P_n(x) - cP_n(x-\omega))\rangle \\ &= \frac{K}{1-c}\langle \phi u_0, x^m P_n(x)\rangle - \frac{Kc}{1-c}\langle \phi u_0, \tau_\omega\big((\xi+\omega)^m P_n(\xi)\big)(x)\rangle \\ &= \frac{K}{1-c}\langle \phi u_0, x^m P_n(x)\rangle - \frac{K}{1-c}\langle c\tau_{-\omega}(\phi u_0), (x+\omega)^m P_n(x)\rangle \\ &\underset{c\tau_{-\omega}(\phi u_0)=(\phi - K^{-1}(1-c))u_0}{=} \frac{K}{1-c}\langle \phi u_0, (x^m - (x+\omega)^m)P_n(x)\rangle + \langle u_0, (x+\omega)^m P_n(x)\rangle,\end{aligned}$$

or equivalently, for $m \geq 1,\ n \geq 0$,

$$\langle v, x^m P_n^{[1]}(.;c,\omega)\rangle = -\frac{K\phi'(0)}{1-c}\sum_{k=1}^{m}\binom{m}{k-1}\omega^{m-k+1}\langle u_0, x^k P_n(x)\rangle$$
$$-\frac{K\phi(0)}{1-c}\sum_{k=0}^{m-1}\binom{m}{k}\omega^{m-k}\langle u_0, x^k P_n(x)\rangle + \sum_{k=0}^{m}\binom{m}{k}\omega^{m-k}\langle u_0, x^k P_n(x)\rangle$$

from which thanks to the orthogonality of $\{P_n\}_{n\geq 0}$ and (2.10) we get

$$\begin{cases} \langle v, x^m P_n^{[1]}(.;c,\omega)\rangle = 0, \quad 1 \leq m \leq n-1, \quad n \geq 2, \\ \langle v, x^n P_n^{[1]}(.;c,\omega)\rangle = \left(1 - \dfrac{K\phi'(0)}{1-c}\, n\,\omega\right)\langle u_0, P_n^2\rangle \neq 0, \quad n \geq 1. \end{cases} \tag{2.19}$$

By the identities in (2.18)–(2.19), we see that $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ is orthogonal with respect to v. We then obtain the desired result.

c) $\Rightarrow$ b) Comparing the degrees in (2.11), we can deduce (2.10). Making $n = 0$ into (2.11), we obtain

$$\lambda_0 = c(1-c)^2. \tag{2.20}$$

Moreover, from definitions, (2.11) may be written as

$$\phi\big((\mathrm{M}_{(c,\omega)}P_n) - (\tau_{-\omega}\circ \mathrm{M}_{(c,\omega)}P_n)\big) + K^{-1}(1-c)(\tau_{-\omega}\circ \mathrm{M}_{(c,\omega)}P_n) = c^{-1}K^{-1}\lambda_n P_n, \quad n \geq 0,$$

then,

$$\langle u_0, \phi\big((\mathrm{M}_{(c,\omega)}P_n) - (\tau_{-\omega}\circ \mathrm{M}_{(c,\omega)}P_n)\big) + K^{-1}(1-c)(\tau_{-\omega}\circ \mathrm{M}_{(c,\omega)}P_n)\rangle = c^{-1}K^{-1}\lambda_n\langle u_0, P_n\rangle, \quad n \geq 0.$$

Equivalently,

$$\langle \mathrm{M}_{(c,-\omega)}(\phi u_0) - (\mathrm{M}_{(c,-\omega)}\circ\tau_\omega)(\phi u_0) + K^{-1}(1-c)(\mathrm{M}_{(c,-\omega)}\circ\tau_\omega u_0), P_n\rangle = c^{-1}K^{-1}\lambda_n\langle u_0, P_n\rangle, \quad n \geq 0.$$

By virtue of Lemma 1 and (2.20), we get

$$\mathrm{M}_{(c,-\omega)}(\phi u_0) - (\mathrm{M}_{(c,-\omega)}\circ\tau_\omega)(\phi u_0) + K^{-1}(1-c)(\mathrm{M}_{(c,-\omega)}\circ\tau_\omega u_0) - K^{-1}(1-c)^2 u_0 = 0.$$

A similar expression is

$$\begin{aligned}\mathrm{M}_{(c,-\omega)}(\phi u_0) - K^{-1}(1-c)u_0 &= (\mathrm{M}_{(c,-\omega)}\circ\tau_\omega)(\phi u_0) \\ &\quad - K^{-1}(1-c)(\mathrm{M}_{(c,-\omega)}\circ\tau_\omega u_0) - K^{-1}(1-c)cu_0.\end{aligned} \tag{2.21}$$

But, by (2.6) and definition of the operator $(\mathrm{M}_{(c,-\omega)}$, we have for the right side of (2.21),

$$\begin{aligned}(\mathrm{M}_{(c,-\omega)}\circ\tau_\omega)(\phi u_0)-K^{-1}(1-c)(\mathrm{M}_{(c,-\omega)}\circ\tau_\omega u_0)-K^{-1}(1-c)cu_0\\ =\tau_\omega\big(\mathrm{M}_{(c,-\omega)}(\phi u_0)\big)-K^{-1}(1-c)\tau_\omega\big((\mathrm{M}_{(c,-\omega)}u_0)+c\tau_{-\omega}u_0\big)\\ =\tau_\omega\big(\mathrm{M}_{(c,-\omega)}(\phi u_0)-K^{-1}(1-c)u_0\big).\end{aligned}$$

Therefore, (2.21) becomes

$$\mathrm{M}_{(1,\omega)}\big(\mathrm{M}_{(c,-\omega)}(\phi u_0)-K^{-1}(1-c)u_0\big)=0.$$

From the fact that the operator $\mathrm{M}_{(1,\omega)}$ is injective in $\mathcal{P}'$ we get (2.9). □

Lemma 4. *If u_0 satisfies (2.9), then $\widehat{u}_0=\big(h_{a^{-1}}\circ\tau_{-b}\big)u_0$ fulfills the equation*

$$\mathrm{M}_{(c,-\omega a^{-1})}\big(a^{-\deg\phi}\phi(ax+b)\widehat{u}_0\big)-a^{-\deg\phi}K^{-1}(1-c)\widehat{u}_0=0.$$

P r o o f. We need the following formulas which are easy to prove from (1.3)

$$g(\tau_b u)=\tau_b\big((\tau_{-b}g)u\big);\quad g(h_a u)=h_a\big((h_a g)u\big),\quad g\in\mathcal{P},\quad u\in\mathcal{P}'. \tag{2.22}$$

Now, with $u_0=\big(\tau_b\circ(h_a\big)\widehat{u}_0$, we have

$$-K^{-1}(1-c)u_0=\big(\tau_b\circ(h_a\big)\big(-K^{-1}(1-c)\widehat{u}_0\big).$$

Further,

$$\begin{aligned}\mathrm{M}_{(c,-\omega)}(\phi u_0)&=\mathrm{M}_{(c,-\omega)}\big(\phi(\tau_b(h_a\widehat{u}_0))\big)\underset{(2.22)}{=}\mathrm{M}_{(c,-\omega)}\big(\tau_b((\tau_{-b}\phi)(h_a\widehat{u}_0))\big)\\ &\underset{(2.6)}{=}(\tau_b\circ\mathrm{M}_{(c,-\omega)})\big((\tau_{-b}\phi)(h_a\widehat{u}_0)\big)\underset{(2.22)}{=}(\tau_b\circ\mathrm{M}_{(c,-\omega)})\big(h_a((h_a\circ\tau_{-b}\phi)\widehat{u}_0)\big)\\ &\underset{(2.5)}{=}\big(\tau_b\circ h_a\circ\mathrm{M}_{(c,-\omega a^{-1})}\big)\big((h_a\circ\tau_{-b}\phi)\widehat{u}_0\big).\end{aligned}$$

Consequently, equation (2.9) becomes

$$\tau_b\circ h_a\Big(\mathrm{M}_{(c,-\omega a^{-1})}\big(\phi(ax+b)\big)\widehat{u}_0\big)-K^{-1}(1-c)\widehat{u}_0\Big)=0.$$

This leads to the desired equality. □

2.2. Determination of all $\mathrm{M}_{(c,\omega)}$-classical (MOPS)s

Lemma 5. *Let $\{P_n\}_{n\geq0}$ be a $\mathrm{M}_{(c,\omega)}$-classical (MOPS). The following equality holds*

$$\frac{c}{1-c}\,\omega\,P_{n+1}(x-\omega)=(\beta_{n+1}-\beta^{[1]}_{n+1})P^{[1]}_{n+1}(x;c,\omega)+(\gamma_{n+1}-\gamma^{[1]}_{n+1})P^{[1]}_n(x;c,\omega),\quad n\geq0. \tag{2.23}$$

P r o o f. From the (TTRR) (1.2) we have

$$P_{n+2}(x)=(x-\beta_{n+1})P_{n+1}(x)-\gamma_{n+1}P_n(x),\quad n\geq0. \tag{2.24}$$

Applying the transfert operator to (2.24), using (2.3) and (2.7) we obtain

$$\begin{aligned}(1-c)P^{[1]}_{n+2}(x;c,\omega)=(1-c)(x-\beta_{n+1})P^{[1]}_{n+1}(x;c,\omega)+c\,\omega P_{n+1}(x-\omega)\\ -\gamma_{n+1}(1-c)P^{[1]}_n(x;c,\omega),\quad n\geq0.\end{aligned} \tag{2.25}$$

But from the (TTRR) of $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$, one may write

$$xP_n^{[1]}(.;c,\omega) = P_{n+2}^{[1]}(.;c,\omega) + \beta_{n+1}^{[1]} P_{n+1}^{[1]}(.;c,\omega) + \gamma_{n+1}^{[1]} P_n^{[1]}(.;c,\omega), \quad n \geq 0. \tag{2.26}$$

Now, injecting (2.26) in (2.25) leads to the desired result (2.23). □

Proposition 1. *The coefficients* β_n, γ_{n+1}, $\beta_n^{[1]}$, $\gamma_{n+1}^{[1]}$ *satisfy the following system*

$$\beta_n - \beta_n^{[1]} = \omega\,\frac{c}{1-c}, \quad n \geq 0, \tag{2.27}$$

$$\gamma_{n+1} - \gamma_{n+1}^{[1]} = -\omega^2\,\frac{c}{(1-c)^2}\,(n+1), \quad n \geq 0, \tag{2.28}$$

$$\beta_{n+1} - \beta_n = \omega\,\frac{1+c}{1-c}, \quad n \geq 0, \tag{2.29}$$

$$\gamma_n^{[1]} = \frac{n}{n+1}\,\gamma_{n+1}, \quad n \geq 1. \tag{2.30}$$

P r o o f. Firstly, the higher degree test in (2.23) yields

$$\beta_{n+1} - \beta_{n+1}^{[1]} = \omega\,\frac{c}{1-c}, \ n \geq 0. \tag{2.31}$$

Secondly, $n = 0$ in (2.23) gives

$$\gamma_1 - \gamma_1^{[1]} = -\omega\,\frac{c}{1-c}\,(\omega + \beta_0 - \beta_0^{[1]}). \tag{2.32}$$

Thirdly, applying the transfert operator $\mathrm{M}_{(c,\omega)}$ to

$$P_1(x) = x - \beta_0$$

and by virtue of (2.7) and (2.31)–(2.32) we get (2.27) and

$$\gamma_1 - \gamma_1^{[1]} = -\omega^2\,\frac{c}{(1-c)^2}. \tag{2.33}$$

Thanks to (2.27), the formula in (2.23) becomes

$$c\,\omega\,P_{n+1}(x-\omega) = c\,\omega\,P_{n+1}^{[1]}(x;c,\omega) + (1-c)(\gamma_{n+1} - \gamma_{n+1}^{[1]})P_n^{[1]}(x;c,\omega), \quad n \geq 0. \tag{2.34}$$

Moreover, multiplication of (2.24) by $c\omega$ with the change $x \leftarrow x - \omega$ yields

$$c\,\omega P_{n+2}(x-\omega) = (x - \omega - \beta_{n+1})c\,\omega P_{n+1}(x-\omega) - \gamma_{n+1}c\,\omega P_n(x-\omega), \quad n \geq 0. \tag{2.35}$$

Replacing (2.34) for the index n, $n+1$, $n+2$ in (2.35), using (2.26) for the index n, $n+1$, the formula in (2.27) and the fact that $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ is a basis , we obtain successively

$$(\gamma_{n+2}^{[1]} - \gamma_{n+2}) - (\gamma_{n+1}^{[1]} - \gamma_{n+1}) = \omega^2 \frac{c}{(1-c)^2}, \quad n \geq 0, \tag{2.36}$$

$$(\gamma_{n+1}^{[1]} - \gamma_{n+1})\Big\{(1-c)(\beta_n - \beta_{n+1}) + (1+c)\omega\Big\} = 0, \tag{2.37}$$

$$(\gamma_{n+1}^{[1]} - \gamma_{n+1})\gamma_n^{[1]} = (\gamma_n^{[1]} - \gamma_n)\gamma_{n+1}, \quad n \geq 1. \tag{2.38}$$

Summing on (2.36) and taking into account (2.33) lead to (2.28) and (2.37) yields (2.29). Lastly, (2.30) is a direct consequence of (2.38) and (2.28).

Now, we are able to solve the system (2.27)–(2.30).
Summing on (2.29) leads to

$$\beta_n = \beta_0 + \omega\,\frac{1+c}{1-c}\,n, \quad n \geq 0. \tag{2.39}$$

Injecting (2.39) in (2.27) yields

$$\beta_n^{[1]} = \beta_0 - \omega\,\frac{c}{1-c} + \omega\,\frac{1+c}{1-c}\,n, \quad n \geq 0. \tag{2.40}$$

Also, injecting (2.30) in (2.28) gives

$$\frac{\gamma_{n+2}}{n+2} - \frac{\gamma_{n+1}}{n+1} = \omega^2\frac{c}{(1-c)^2}, \quad n \geq 0.$$

Summing the previous equality leads to

$$\gamma_{n+1} = (n+1)\left(\gamma_1 + \omega^2\frac{c}{(1-c)^2}\,n\right), \quad n \geq 0. \tag{2.41}$$

After replacing (2.41) in (2.30) we deduce the following

$$\gamma_{n+1}^{[1]} = (n+1)\left(\gamma_1 + \omega^2\frac{c}{(1-c)^2}\,(n+1)\right), \quad n \geq 0. \tag{2.42}$$

□

Corollary 1. *Let $\{P_n\}_{n\geq 0}$ be a $\mathrm{M}_{(c,\omega)}$-classical (MOPS). The following statements hold.*

1) *The recurrence elements of $\{P_n\}_{n\geq 0}$ are*

$$\begin{cases} \beta_n = \omega\left(\dfrac{\beta_0}{\omega} + \dfrac{1+c}{1-c}\,n\right), \quad n \geq 0, \\ \gamma_{n+1} = \omega^2\dfrac{c}{(1-c)^2}(n+1)\left(n + \dfrac{(1-c)^2}{c}\,\dfrac{\gamma_1}{\omega^2}\right), \quad n \geq 0. \end{cases} \tag{2.43}$$

2) *The recurrence elements of $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ are*

$$\begin{cases} \beta_n^{[1]} = \omega\left(\dfrac{\beta_0}{\omega} - \dfrac{c}{1-c} + \dfrac{1+c}{1-c}\,n\right), \quad n \geq 0, \\ \gamma_{n+1}^{[1]} = \omega^2\dfrac{c}{(1-c)^2}(n+1)\left(n + 1 + \dfrac{(1-c)^2}{c}\,\dfrac{\gamma_1}{\omega^2}\right), \quad n \geq 0. \end{cases} \tag{2.44}$$

P r o o f. The formula (2.43) is a consequence of (2.39) and (2.41). Also, (2.44) is a direct result from (2.40) and (2.42).

Theorem 2. *Up to an affine transformation, the only $\mathrm{M}_{(c,1)}$-classical (MOPS) is the Meixner's one of the first kind.*

P r o o f. The classification of the canonical situations depends on the fact that $\beta_0 \neq 0$ or $\beta_0 = 0$.

$\beta_0 \neq 0$. For (2.43)–(2.44), put

$$\omega\,\beta_0 = (1-c)\gamma_1$$

and

$$\frac{(1-c)^2}{c}\,\frac{\gamma_1}{\omega^2} = \alpha + 1.$$

Then,

$$\frac{\beta_0}{\omega} = \frac{c}{1-c}\,(\alpha+1).$$

Now, for (2.43), choosing $a=\omega,\ b=0$ in (1.4) and thanks to (2.5)–(2.6) this yields

$$\begin{cases} \widehat{\beta}_n = \dfrac{c}{1-c}(\alpha+1) + \dfrac{1+c}{1-c}\,n, \quad n\geq 0,\\ \widehat{\gamma}_{n+1} = \dfrac{c}{(1-c)^2}\,(n+1)(n+\alpha+1), \quad n\geq 0. \end{cases}$$

Therefore (see (1.5)),

$$\widehat{P}_n = M_n(.;\alpha,c), \quad n\geq 0,$$

with $\alpha\neq -n-1,\ n\geq 0$. Next, for (2.44), choosing

$$a=\omega, \quad b=-\frac{2\omega\,c}{1-c}$$

in (1.4) and thanks to (2.5)–(2.6) this yields

$$\begin{cases} \widehat{\beta}^{[1]}_n = \dfrac{c}{1-c}(\alpha+2) + \dfrac{1+c}{1-c}\,n, \quad n\geq 0,\\ \widehat{\gamma}^{[1]}_{n+1} = \dfrac{c}{(1-c)^2}\,(n+1)(n+\alpha+2), \quad n\geq 0. \end{cases}$$

Thus,

$$\widehat{P}^{[1]}_n = M_n(.;\alpha+1,c), \quad n\geq 0,$$

with $\alpha\neq -n-2,\ n\geq 0$.

$\beta_0=0$. In this case, (2.43)–(2.44) become successively,

$$\begin{cases} \beta_n = \omega\dfrac{1+c}{1-c}\,n, \quad n\geq 0,\\ \gamma_{n+1} = \omega^2\dfrac{c}{(1-c)^2}(n+1)\left(n+\dfrac{(1-c)^2}{c}\,\dfrac{\gamma_1}{\omega^2}\right), \quad n\geq 0, \end{cases} \tag{2.45}$$

$$\begin{cases} \beta^{[1]}_n = \omega\left(-\dfrac{c}{1-c}+\dfrac{1+c}{1-c}\,n\right), \quad n\geq 0,\\ \gamma^{[1]}_{n+1} = \omega^2\dfrac{c}{(1-c)^2}(n+1)\left(n+1+\dfrac{(1-c)^2}{c}\,\dfrac{\gamma_1}{\omega^2}\right), \quad n\geq 0. \end{cases} \tag{2.46}$$

For (2.45), putting

$$\frac{(1-c)^2}{c}\,\frac{\gamma_1}{\omega^2} = \alpha+1,$$

and choosing in (1.4)

$$a=\omega, \quad b=-\frac{\omega\,c}{1-c}(\alpha+1),$$

we obtain

$$\begin{cases} \widehat{\beta}_n = \dfrac{c}{1-c}(\alpha+1) + \dfrac{1+c}{1-c}\,n, \quad n\geq 0,\\ \widehat{\gamma}_{n+1} = \dfrac{c}{(1-c)^2}\,(n+1)(n+\alpha+1), \quad n\geq 0. \end{cases}$$

Consequently,

$$\widehat{P}_n = M_n(.;\alpha,c), \quad n\geq 0,$$

with $\alpha \neq -n-1,\ n \geq 0$. For (2.46), putting

$$\frac{(1-c)^2}{c}\,\frac{\gamma_1}{\omega^2} = \alpha + 1$$

and choosing in (1.4)

$$a = \omega, \quad b = -\frac{\omega\, c}{1-c}(\alpha+3),$$

we get

$$\begin{cases} \widehat{\beta}^{[1]}_n = \dfrac{c}{1-c}(\alpha+2) + \dfrac{1+c}{1-c}\,n, \quad n \geq 0, \\ \widehat{\gamma}^{[1]}_{n+1} = \dfrac{c}{(1-c)^2}\,(n+1)(n+\alpha+2), \quad n \geq 0. \end{cases}$$

Equivalently,

$$\widehat{P}^{[1]}_n = M_n(.;\alpha+1, c), \quad n \geq 0,$$

with $\alpha \neq -n-2,\ n \geq 0$.

The theorem is then proved. □

Remark 2. On account of Theorem 1, Theorem 2 and after some easy calculations we get for the divided-difference equation (2.9) fulfilled by the Meixner form $\mathcal{M}(\alpha, c)$,

$$\mathrm{M}_{(c,-1)}\Big(\Big(x - \frac{1+c}{1-c}\,(\alpha+1)\Big)\mathcal{M}(\alpha,c)\Big) + (\alpha+1)\mathcal{M}(\alpha,c) = 0,$$

and also for the second order linear divided-difference equation (2.11) satisfied by any Meixner polynomial $M_n(.;\alpha,c)$, for all $n \geq 0$,

$$\Big(-\frac{1-c}{\alpha+1}x + 2c\Big)(\mathrm{M}_{(c,-1)} \circ \mathrm{M}_{(c,1)} M_n)(x;\alpha,c) + (1-c)\Big(\frac{1-c}{\alpha+1}x - c\Big)(\mathrm{M}_{(c,1)} M_n)(x;\alpha,c)$$
$$= c(1-c)^2\,\frac{n+\alpha+1}{\alpha+1} M_n(x;\alpha,c).$$

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