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AROUND THE ERDÖS–GALLAI CRITERION

Vitaly A. Baransky[†], **Tatiana A. Senchonok**[††]

Ural Federal University,
51 Lenina av., Ekaterinburg, 620075, Russian Federation

[†]Vitaly.Baransky@urfu.ru [††]Tatiana.Senchonok@urfu.ru

Abstract: By an (integer) partition we mean a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers that contains a finite number of non-zero components. A partition λ is said to be graphic if there exists a graph G such that $\lambda = \operatorname{dpt} G$, where we denote by $\operatorname{dpt} G$ the degree partition of G composed of the degrees of its vertices, taken in non-increasing order and added with zeros. In this paper, we propose to consider another criterion for a partition to be graphic, the ht-criterion, which, in essence, is a convenient and natural reformulation of the well-known Erdös–Gallai criterion for a sequence to be graphical. The ht-criterion fits well into the general study of lattices of integer partitions and is convenient for applications. The paper shows the equivalence of the Gale–Ryser criterion on the realizability of a pair of partitions by bipartite graphs, the ht-criterion and the Erdös–Gallai criterion. New proofs of the Gale–Ryser criterion and the Erdös–Gallai criterion are given. It is also proved that for any graphical partition there exists a realization that is obtained from some splitable graph in a natural way. A number of information of an overview nature is also given on the results previously obtained by the authors which are close in subject matter to those considered in this paper.

Keywords: Integer partition, Threshold graph, Bipartite graph, Bipartite-threshold graph, Ferrers diagram.

1. Introduction

Everywhere by a graph we mean a simple graph, i.e. a graph without any loops and multiple edges.

An *integer partition*, or simply *partition*, is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers that contains a finite number of non-zero components (see [1]). Let $\operatorname{sum}\lambda$ denote the sum of all components of the partition λ and call it the *weight* of the partition λ. It is often said that a partition of λ is a partition of a non-negative integer $n = \operatorname{sum}\lambda$. The *length* $\ell(\lambda)$ of a partition λ is the number of its non-zero components. For convenience, the partition λ will often be written as $\lambda = (\lambda_1, \dots, \lambda_t)$, where $t \geq \ell(\lambda)$, i. e. we will omit the zeros by starting from some zero component without forgetting that the sequence is infinite.

The theory of partitions is one of the actively developing areas of contemporary combinatorics, the foundations of which were laid by L. Euler as early as the 18^{th} century. For some information about the achievements of this theory in the 19^{th} and 20^{th} centuries, see [1].

A partition λ is said to be *graphic* if there is a graph G such that $\lambda = \operatorname{dpt} G$, where we denote by $\operatorname{dpt} G$ the *degree partition* composed by the degrees of vertices taken in non-increasing order with added zeros. In this case, the graph G is called a *realization* of the partition λ, and λ is said to be realized by the graph G. It is clear that adding or removing isolated vertices does not change the degree partition of the graph.

A finite sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and n is a natural number will be called an *n-sequence*. Such an n-sequence is called *graphic* if there is a simple graph G on n vertices such that $\deg(v_1) = \lambda_1, \dots, \deg(v_n) = \lambda_n$, where $v_1, \dots, v_n$ is the sequence of all its vertices; and the graph G is called a *realization* of the n-sequence λ, and λ is said to be *realized* by the graph G.

Obviously, an n-sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is graphic if and only if the partition $(\lambda_1, \lambda_2, \ldots, \lambda_n, 0, 0, \ldots)$ obtained from λ by adding zeros, is graphic.

It should be noted that in [4] an algorithm was constructed for generating all graphic n-sequences which does not generate any non-graphic sequences during calculations.

We call an n-sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *proper* (*proper n-sequence*) if
1) $n - 1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$;
2) the sum λ of all components of the sequence λ is even.

Obviously, any graphic n-sequence is proper.

The first criterion for an n-sequence to be graphic was found by Erdös and Gallai [14].

Theorem 1 [14, Erdös and Gallai]. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a proper n-sequence. Then λ is a graphic n-sequence if and only if it is satisfied the inequality*

$$\sum_{i=1}^{k} \lambda_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, \lambda_i\}$$

for any $k = 1, \ldots, n$.

It is an easy matter to prove that the condition "$k = 1, \ldots, n$" can be replaced by the condition "$k = 1, \ldots, r(\lambda)$", where $r(\lambda) = \max\{i | \lambda_i \geq i\}$ is the rank of the n-sequence λ.

The paper [17] considers all seven graphic criteria known by that time: Erdös–Gallai, Ryser, Berge, Fulkerson–Hoffman–McAndrew, Bollobas, Grünbaum, Hässelbarth. It is shown, how they deduced from each other, and a new, more elegant proof of the Erdös–Gallai criterion is given.

In this paper, we propose for consideration (in our terminology) another graphic criterion, the ht-criterion (see Theorem 2), which has the simplest and most natural form. Moreover, as will be seen below, this criterion fits well into the general study of partition lattices.

It should be noted that considerations close to the ht-criterion can be found in [15].

As can be seen below, the ht-criterion can be in essence considered as a reformulation of the Erdös–Gallai criterion which is convenient for applications.

In § 2, by fairly simple reasoning, we establish the equivalence of Theorem 2 on the ht-criterion and the Erdös–Gallai Theorem 1.

§ 3 will provide a transparent proof of the Gale–Ryser theorem on the realization of two partitions by a bipartite graph, that does not use the partition graphicity criteria.

In § 4, with the Gale–Ryser theorem and without any partition graphicity criteria, we prove Theorem 2 on the ht-criterion and, therefore, obtain a new natural proof of the Erdös–Gallai theorem. From the proof of Theorem 2 we also extract Theorem 4 and Theorem 5 on the existence of a special kind of realizations for arbitrary partitions, and this result is one of the main ones in this paper.

§ 5 will give another proof of the Gale–Ryser theorem, in which the ht-criterion is used. As a result, we will show how the Gale–Ryser theorem, the ht-criterion Theorem 2 and the Erdös–Gallai Theorem 1 can be derived from each other.

At the end of paragraphs 4 and 5, we give a brief review of the previously obtained results of authors which are close in subject matter to those considered in this paper.

2. On the ht-criterion

Let us first give the necessary definitions.

We denote by IPL the set of all partitions of all natural numbers with added zero partition, and by $IPL(m)$ for a non-negative integer m we denote the set of all its partitions. On the sets IPL and $IPL(m)$, consider the dominance relation $\trianglelefteq$ [13], by setting $\lambda \trianglelefteq \mu$ if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$$

for any $i = 1, 2, \ldots$, i. e. the prefix partial sums of the partition λ do not exceed the corresponding prefix partial sums of the partition μ.

The partition can be conveniently depicted as a *Ferrers diagram*, which can be thought of as a set of square boxes of the same size (see an example in Fig. 1, which shows the partition $(6, 5, 4, 4, 3, 2, 1, 1)$ of the number 26, the length of this partition is 8). We will use Cartesian notation for Ferrers diagrams aligned to the bottom-left corner of the 1^{st} quadrant. Components correspond to columns and decrease in size from left to right. The coordinates for boxes resemble the standard Cartesian coordinates for the Euclidean plane.

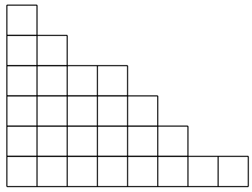

Figure 1. The Ferrers diagram of the partition (6, 5, 4, 4, 3, 2, 1, 1).

Let us define two types of elementary transformations (see [2–5]) of the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $n = \ell(\lambda) + 1$.

Let there be natural numbers $i, j \in \{1, \ldots, n\}$ such that $i < j \leq \ell(\lambda) + 1$ and

1) $\lambda_i - 1 \geq \lambda_{i+1}$,

2) $\lambda_{j-1} \geq \lambda_j + 1$,

3) $\lambda_i = \lambda_j + \delta$, where $\delta \geq 2$.

We will say that the partition $\eta = (\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_j + 1, \ldots, \lambda_n)$ is obtained from the partition $\lambda = (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_n)$ by an *elementary transformation of the first type* (or through *box movement*). It should be noted that η differs from λ on exactly two components with numbers i and j.

For the Ferrers diagram, such a transformation means moving the top box from the i-th column to the right to the top of the j-th column. The conditions 1), 2) and 3) guarantee that after such a move, a partition will again be obtained. It should be noted that a box can also be thrown to the zero component with the number $\ell(\lambda) + 1$.

The fact that η is obtained from λ by moving a box will be briefly written in the form $\lambda \rightharpoondown \eta$. It should be noted that an elementary transformation of the first type preserves the weight of the partition, while the length of the partition can be preserved or lifted by 1.

We now define elementary transformations of the second type for the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$.

Let $\lambda_i - 1 \geq \lambda_{i+1}$, where $i \leq \ell(\lambda)$. A transformation that replaces λ by

$$\eta = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots)$$

will be called an *elementary transformation of the second type* (or a *box removal*).

As in the previous case, we will briefly write $\lambda \rightharpoondown \eta$, i. e., the notation $\lambda \rightharpoondown \eta$ means that η obtained from λ by an elementary transformation of the first or second type. It should be noted that box removal reduces the weight of the partition by exactly 1, while the length of the partition can be preserved or lowered by 1.

On the set IPL and on sets of the form $IPL(m)$, we define the relation $\leq$ by setting $\eta \leq \lambda$ if η can be obtained from λ by sequentially applying a finite number (possibly a zero one) of elementary transformations of the stated types.

Of course, in the case of $IPL(m)$ we are forced to use only elementary transformations of the first type, which do not change the weight of the partitions. It was proved in [3] and [5] that the relation $\leq$ on each of the considered sets coincides with the dominance relation $\trianglelefteq$, and each of these sets is a lattice.

It is essential to note that the use of elementary transformations is often more convenient than the use of inequalities appearing in the definition of the dominance relation.

It should be noted that the IPL lattice is a disjoint union of lattices $IPL(m)$, where m ranges over non-negative integers corresponding to some natural transitive system of embeddings [5].

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition. We determine the *rank* $r(\lambda)$ of the partition λ by setting $r(\lambda) = \max\{i | \lambda_i \geq i\}$. Obviously, the rank $r = r(\lambda)$ of a partition λ is equal to the number of boxes on the main diagonal of the Ferrers diagram of this partition. The maximum square made up of boxes and symmetrical about the main diagonal is called the *Durfey square* of the partition λ (see Fig. 2).

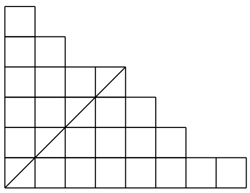

Figure 2. The main diagonal in the Durfey square.

Fig. 2 shows the partition $\lambda = (6, 5, 4, 4, 3, 2, 1, 1)$. Here $r(\lambda) = 4$ and the Durfey square consists of $16 = 4 \cdot 4$ boxes. Any row and any column of a Durfey square consists of $r = r(\lambda)$ boxes.

For each partition λ, we will consider an conjugate partition λ^* whose components are equal to the number of boxes in the corresponding rows of the Ferrers diagram of the partition λ. It is clear that the Ferrers diagram of the partition λ^* can be obtained from the Ferrers diagram of the partition λ by mirror symmetry with respect to the main diagonal. For Fig. 2, $\lambda^* = (8, 6, 5, 4, 2, 1)$ is satisfied. Of course, the equality $r(\lambda^*) = r(\lambda)$ is true.

It should be noted (see [3]) that for any $m \in \mathbb{N}$ the mapping $\lambda \to \lambda^*$ is an involutive anti-automorphism of the lattice $IPL(m)$ such that $(\lambda^*)^* = \lambda$ and the condition $\gamma_1 \leq \gamma_2$ implies the condition $\gamma_1^* \geq \gamma_2^*$.

Let $\xi, \eta \in IPL(m)$ and f be an elementary transformation of the first type $\xi \rightharpoondown \eta$, transforming ξ into η. It is plain to see (see Fig. 3) that the inverse transformation f^* to the transformation f is an elementary transformation of the first type $\eta^* \rightharpoondown \xi^*$, transforming η^* into ξ^*. Ferrers diagrams stated in Fig. 3 can also be considered as Ferrers diagrams of the corresponding conjugate partitions, only then they need to be considered lying “on their side”, i. e., the components should be read in rows.

Similarly, if f is an elementary transformation of the second type $\xi \rightharpoondown \eta$ (box removal) that transforms ξ into η, then the inverse transformation f^* of f is *a box insertion* that transforms η^*

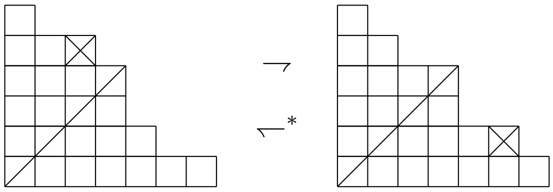

Figure 3. The elementary transformathion of the first type and the inverse transformation.

into ξ^*, i. e., $\eta^* \rightharpoondown \xi^*$ (it is convenient to use the same symbol $\rightharpoondown$ to indicate box insertion).

We now define the head and tail of the partition $\lambda = (\lambda_1, \lambda_2, \dots)$, the rank of which is equal to r.

As the *head* $\mathrm{hd}\,(\lambda)$, we take the partition that is obtained from the partition λ by reducing the first r components by the same number $r-1$ and zeroing all components with numbers $r+1, r+2, \dots$ (for an example, see the diagram in Fig. 4).

As the *tail* $\mathrm{tl}\,(\lambda)$ we take a partition for which the Ferrers diagram of the conjugate partition is obtained from the Ferrers diagram of the partition λ by deleting the first r columns, i. e. the Ferrers diagram of the partition $\mathrm{tl}\,(\lambda)^*$ is located to the right of the Durfey square (see Fig. 4).

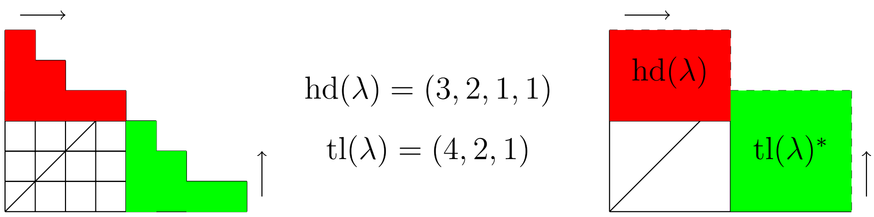

Figure 4. The head and the tail of the partition (6, 5, 4, 4, 3, 2, 1, 1).

The arrows in Fig. 4 indicate the directions in which the components of the partitions $\mathrm{hd}(\lambda)$ and $\mathrm{tl}(\lambda)$ are read. It is clear that the upper row of the Durfey square enters the Ferrers diagram of the partition $\mathrm{hd}(\lambda)$, the partition $\mathrm{hd}(\lambda)$ is "read" by column from left to right, and the length of the partition $\mathrm{hd}(\lambda)$ is equal to r. The partition $\mathrm{tl}(\lambda)$ is "read" by row from bottom to top and the length of the partition $\mathrm{tl}(\lambda)^*$ is equal to $\ell(\lambda) - r(\lambda)$, and the length of the partition $\mathrm{tl}(\lambda)$ is equal to $\mathrm{tl}(\lambda)^*_1$ — the value the first component of the partition $\mathrm{tl}(\lambda)^*$, hence $\ell(\mathrm{tl}(\lambda)) \le r(\lambda)$.

For n-sequences, the concepts of *rank*, *head*, and *tail* are introduced in exactly the same way.

In order to consider the ht-criterion for partitions to be graphic, we present two auxiliary lemmas.

Lemma 1. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be an n-sequence. Then for any $k = 1, \dots,\ \ r = r(\lambda)$, the condition*

$$\sum_{i=1}^{k} \lambda_i \le k(k-1) + \sum_{i=k+1}^{n} \min\{k, \lambda_i\} \tag{2.1}$$

is equivalent to the condition

$$\sum_{i=1}^{k} \mathrm{hd}\,(\lambda)_i \le \sum_{i=1}^{k} \mathrm{tl}\,(\lambda)_i,$$

where $\mathrm{hd}\,(\lambda)_i$ and $\mathrm{tl}\,(\lambda)_i$ are the i-components, respectively, of the head and tail of the partition λ for any $i = 1, \dots, k$.

P r o o f. Note first that for $k = 1, \ldots, r$ the sum

$$\sum_{i=k+1}^{n} \min\{k, \lambda_i\}$$

is equal to the number of boxes of the Ferrers diagram of the sequence λ located in the strip standing at the intersection of rows with numbers from 1 to k and columns with numbers from $k+1$ to n (see the shaded area in Fig. 5).

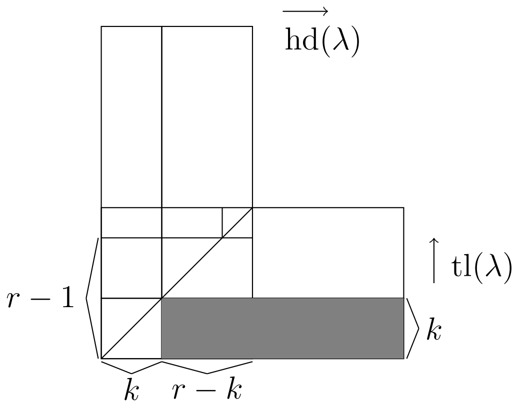

Figure 5. The Ferrers diagram of the sequence.

Let us rewrite inequality (2.1) in the equivalent form

$$\sum_{i=1}^{k} \lambda_i - k(k-1) - k(r-k) \leq \sum_{i=k+1}^{n} \min\{k, \lambda_i\} - k(r-k),$$

after transformations, the resulting inequality is equivalent to the inequality

$$\sum_{i=1}^{k} \lambda_i - k(r-1) \leq \sum_{i=k+1}^{n} \min\{k, \lambda_i\} - k(r-k).$$

It is plain to see that

$$\sum_{i=1}^{k} \lambda_i - k(r-1) = \sum_{i=1}^{k} \mathrm{hd}\,(\lambda)_i \quad \text{and} \quad \sum_{i=k+1}^{n} \min\{k, \lambda_i\} - k(r-k) = \sum_{i=1}^{k} \mathrm{tl}\,(\lambda)_i.$$

Therefore, inequality (2.1) is equivalent to the inequality

$$\sum_{i=1}^{k} \mathrm{hd}\,(\lambda)_i \leq \sum_{i=1}^{k} \mathrm{tl}\,(\lambda)_i.$$

□

Lemma 2. *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *be an* n*-sequence and* $\mathrm{hd}\,(\lambda) \leq \mathrm{tl}\,(\lambda)$*. Then* $\lambda_1 \leq n-1$.

P r o o f. Let $r = r(\lambda)$. The condition $\mathrm{hd}\,(\lambda) \le \mathrm{tl}\,(\lambda)$ implies $\mathrm{hd}\,(\lambda)_1 \le \mathrm{tl}\,(\lambda)_1$, so

$$\lambda_1 - (r-1) \le \ell(\lambda) - r.$$

Therefore,

$$\lambda_1 \le \ell(\lambda) - 1 \le n - 1.$$

□

Since $\ell(\mathrm{hd}\,(\lambda)) = r(\lambda)$ and $\ell(\mathrm{tl}\,(\lambda)) \le r(\lambda)$ for any partition λ, due to Lemmas 1 and 2 the statement of the Erdös–Gallay theorem is equivalent to the following statement.

Theorem 2. *Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be an arbitrary nonzero partition of even weight. Then λ is a graphic partition if and only if*

$$\mathrm{hd}\,(\lambda) \le \mathrm{tl}\,(\lambda).$$

The criterion for a partition to be graphic specified in Theorem 2 will be called the *ht-criterion*.

3. About the Gale–Ryser theorem

Our next main goal is to prove Theorem 2 without using the Erdös–Gallai theorem and other criteria for graphic partitions. To do this, we first give a direct, transparent proof of the well-known Gale–Ryser theorem on the representation of a pair of partitions by a bipartite graph, in which we will not use any of the criteria for the graphicity of partitions.

For a bipartite graph $H = (V_1, E, V_2)$, where V_1 and V_2 are its parts, we denote by $\mathrm{dpt}_H(V_1)$ and $\mathrm{dpt}_H(V_2)$ the degree partitions corresponding to its parts, i. e. partitions composed of the degrees of the vertices of the corresponding parts in non-increasing order and added with zeros.

Theorem 3 [16, Gale D., Ryser H.J. (1957)]. *Let $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ be nonzero partitions. Then there is a bipartite graph $H = (V_1, E, V_2)$ such that $\mathrm{dpt}_H(V_1) = \alpha$ and $\mathrm{dpt}_H(V_2) = \beta$ if and only if $\mathrm{sum}(\alpha) = \mathrm{sum}(\beta)$ and $\alpha \le \beta^*$.*

We need the following

Lemma 3. *Let $G = (V, E)$ be a graph, $V = \{v_1, v_2, \dots, v_n\}$ and $\lambda = (\lambda_1, \lambda_2, \dots)$ is a degree partition corresponding to the graph G such that $\lambda_i = \deg(v_i)$ for any $i = 1, \dots, \ell(\lambda)$ and $\lambda_j = 0$ for any $j > \ell(\lambda)$. Then $\mathrm{hd}\,(\lambda) \le tl(\lambda)$.*

P r o o f. By virtue of Lemma 1, it suffices to check the validity of inequality (2.1) for any $k = 1, \dots, r = r(\lambda)$. For $k = 1, \dots, r$, we estimate the sum

$$\sum_{i=1}^{k} \lambda_i.$$

Let us first estimate the part of this sum contributed by edges from $G(\{v_1, v_2, \dots, v_k\})$. Obviously, $\deg_G(v_i) \le k-1$ for any $i = 1, \dots, k$. Therefore, this part of the sum does not exceed $k(k-1)$.

Let us now estimate the contribution to the sum made by edges of the form $v_j v_i$, where $1 \le j \le k$ and $k+1 \le i \le n$. For a given i such that $k+1 \le i \le n$, the number of such edges does not exceed k and does not exceed λ_i, i. e., does not exceed $\min\{k, \lambda_i\}$.

Consequently, inequality (2.1) is satisfied for any $k = 1, \dots, r = r(\lambda)$, and therefore, by virtue of Lemma 1, the inequality $\mathrm{hd}\,(\lambda) \leq \mathrm{tl}\,(\lambda)$ also holds. $\square$

A graph G is said to be *splitable* if the set of its vertices can be represented as a disjoint union of a clique V_1 and a coclique V_2 (i. e., $V_1 \cap V_2 = \emptyset$, V_1 generates a complete subgraph $K(V_1)$, and V_2 generates a zero subgraph $O(V_2)$ with an empty set of edges). For such a graph G, the set of all edges can be represented as a disjoint union of the set of all edges of the complete subgraph $K(V_1)$ and the set E of all its edges connecting vertices from V_1 with vertices from V_2. Therefore, it is convenient to represent a splitable graph G in the form $G = (K(V_1), E, O(V_2))$. We will simply write $G = (K(V_1), E, V_2)$.

The following lemma proves the necessity of the conditions in the Gale–Ryser theorem.

Lemma 4. *Let $H = (V_1, E, V_2)$ be an arbitrary bipartite graph and $\mathrm{dpt}\,(V_1) = \alpha$, $\mathrm{dpt}\,(V_2) = \beta$ be the degree partitions of its parts. Then*

1) $\mathrm{sum}(\alpha) = \mathrm{sum}(\beta) = m$, *where* $m = |E|$;

2) $\alpha \leq \beta^*$.

(It should be noted that the condition $\alpha \leq \beta^*$ is equivalent to the condition $\beta \leq \alpha^*$, since the transformation $\gamma \to \gamma^*$ is an involutive antiautomorphism of the lattice $IPL(m)$.)

P r o o f. 1) It is obvious.

2) Without loss of generality, we will assume that H does not have any isolated vertices. Let $V_1 = \{u_1, u_2, \dots, u_p\}$ and $V_2 = \{v_1, v_2, \dots, v_q\}$, where $\alpha_i = \deg(u_i)$ for any $i = 1, \dots, p$ and $\beta_j = \deg(v_j)$ for any $j = 1, \dots, q$.

Let us embed the graph H into a splitable graph $H^+ = (K(V_1), E, V_2)$ by adding to H all possible edges connecting pairs of different vertices from V_1. In the graph H^+, the set V_1 is a clique and the set V_2 is a coclique. Then

$$\alpha_1 + (p-1) \geq \alpha_2 + (p-1) \geq \dots \geq \alpha_p + (p-1) \geq p \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_q,$$

i. e.

$$\mathrm{dpt}\,(H^+) = (\alpha_1 + (p-1), \alpha_2 + (p-1), \dots, \alpha_p + (p-1), \beta_1, \beta_2, \dots, \beta_q).$$

Let $\mathrm{dpt}\,(H^+) = \lambda$. Then

$$r(\lambda) = p, \quad \mathrm{hd}\,(\lambda) = (\alpha_1, \alpha_2, \dots, \alpha_p) = \alpha, \quad \mathrm{tl}^*(\lambda) = (\beta_1, \beta_2, \dots, \beta_q) = \beta.$$

By virtue of Lemma 3, we have $\mathrm{hd}\,(\lambda) \leq \mathrm{tl}\,(\lambda)$. Since $\mathrm{tl}\,(\lambda) = \beta^*$, we obtain $\alpha \leq \beta^*$. $\square$

To prove the sufficiency of the Gale–Ryser theorem conditions, we need additional definitions and two lemmas.

Let (x, v, y) be a triple of different vertices of the graph $G = (V, E)$ such that $xv \in E$ and $vy \notin E$. We call such a triple 1) *lifting* if $\deg(x) \leq \deg(y)$, 2) *lowering* if $\deg(x) \geq 2 + \deg(y)$, and 3) *preserving* if $\deg(x) = 1 + \deg(y)$.

A transformation φ of a graph G such that $\varphi(G) = G - xv + vy$, i. e., first the edge xv is removed from G and then the edge vy is added (see Fig. 6), is called the *rotation of the edge* (in the graph G around vertex v) corresponding to the triple (x, v, y).

The rotation of an edge in the graph $\varphi(G)$ corresponding to the triple (y, v, x) is called the *reverse rotation of an edge* for the rotation φ.

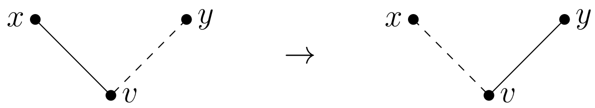

Figure 6. The rotation of an edge.

The rotation of an edge in G corresponding to a triple (x, v, y) is called 1) *lifting* if the triple (x, v, y) is lifting, 2) *lowering* if the triple (x, v, y) is lowering, and 3) *preserving* if the triple (x, v, y) is preserving.

It should be noted that the cases when $\deg(x) = 1$ or $\deg(y) = 0$ will be considered admissible, i. e. after the rotation of an edge, an isolated vertex may appear or the rotation of an edge will occur in the graph G with the addition of a new isolated vertex.

It should be noted that the rotation of an edge in the graph G is lowering if and only if the reverse rotation of the edge is lifting.

If the graph G_2 obtained from the graph G_1 by rotating an edge, then we write $G_1 \to G_2$.

Let $\mathrm{dpt}\,(G)$ be the degree partition corresponding to the graph G and φ be the rotation of the edge in the graph G corresponding to the triple (x, v, y), where $xv \in E$ and $vy \notin E$. Then the following assertions are correct.

1. If φ is a lifting rotation of an edge, then $\mathrm{dpt}\,(G) < \mathrm{dpt}\,(\varphi(G))$, moreover, $\mathrm{dpt}\,(G)$ is obtained from $\mathrm{dpt}\,(\varphi(G))$ with one elementary transformation of the first type, and G is obtained from $\varphi(G)$ with the reverse (lowering) rotation of an edge.

2. If φ is the lowering rotation of an edge, then $\mathrm{dpt}\,(G) > \mathrm{dpt}\,(\varphi(G))$, moreover, $\mathrm{dpt}\,(\varphi(G))$ is obtained from $\mathrm{dpt}\,(G)$ with one elementary transformation of the first type, and G is obtained from $\varphi(G)$ with the reverse (lifting) rotation of an edge.

3. If φ is the preserving rotation of an edge, then $\mathrm{dpt}\,(G) = \mathrm{dpt}\,(\varphi(G))$, and G is obtained from $\varphi(G)$ with the reverse (preserving) rotation of an edge.

Let (x, v, y) be a triple of distinct vertices of a bipartite graph $H = (V_1, E, V_2)$ such that $xv \in E$ and $vy \notin E$. If $x, y \in V_1$ and $v \in V_2$, then we call the triple V_1-*triple*, but if $x, y \in V_2$ and $v \in V_1$, then the triple will be called a V_2-*triple*. We will say that V_1-triples correspond to V_1-*rotations of edges*, and V_2-triples correspond to V_2-*rotations of edges*.

Lemma 5. 1. *Let $H_1 = (V_1, E_1, V_2)$ and $H_2 = (V_1, E_2, V_2)$ be bipartite graphs, and the graph H_2 is obtained from the graph H_1 by the lowering V_1-rotation of the edge $H_1 \to H_2$. Then $\mathrm{dpt}_{H_2}(V_1)$ is obtained from $\mathrm{dpt}_{H_1}(V_1)$ with an elementary transformation of the first type, i. e., $\mathrm{dpt}_{H_1}(V_1) \rightharpoonup \mathrm{dpt}_{H_2}(V_1)$, and $\mathrm{dpt}_{H_2}(V_2) = \mathrm{dpt}_{H_1}(V_2)$.*

2. *Let $H_1 = (V_1, E_1, V_2)$ be a bipartite graph and the partition μ be obtained from the partition $\mathrm{dpt}_{H_1}(V_1)$ with an elementary transformation of the first type, i. e., $\mathrm{dpt}_{H_2}(V_1) \rightharpoonup \mu$. Then there exists a bipartite graph $H_2 = (V_1, E_2, V_2)$ that is obtained from the graph H_1 by means of a lowering V_1-rotation of an edge $H_1 \to H_2$ and for which $\mu = \mathrm{dpt}_{H_2}(V_2)$ and $\mathrm{dpt}_{H_2}(V_2) = \mathrm{dpt}_{H_1}(V_2)$.*

P r o o f. Assertion 1 is obvious. Let us prove the assertion 2. Let $\mathrm{dpt}_{H_1}(V_1) = (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_t)$, where $\lambda_i \geq 2 + \lambda_j$, $1 \leq i < j \leq t$ and $\mu = (\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_j + 1, \ldots, \lambda_t)$. Let for vertices $x, y \in V_1$, $\deg_{H_1}(x) = \lambda_i$ and $\deg_{H_1}(y) = \lambda_j$. Since $\lambda_i > \lambda_j$, there is a vertex $v \in V_2$ such that $xv \in E_1$ and $vy \notin E_1$. Let φ be a lowering V_1-rotation of an edge corresponding to the triple (x, v, y) in the graph H_1. Then $\mu = \mathrm{dpt}_{H_2}(V_1)$ in the graph $H_2 = \varphi(H_1)$. The equality $\mathrm{dpt}_{H_2}(V_2) = \mathrm{dpt}_{H_1}(V_2)$ is obvious, since the lowering V_1-rotation of an edge does not change the degrees of vertices in V_2. □

The following lemma guarantees the sufficiency of the Gale–Ryser theorem conditions.

Lemma 6. *Let* $\alpha = (\alpha_1, \alpha_2, \dots)$ *and* $\beta = (\beta_1, \beta_2, \dots)$ *be nonzero partitions such that* $\mathrm{sum}\,(\alpha) = \mathrm{sum}\,(\beta)$ *and* $\alpha \le \beta^*$. *Then there is a bipartite graph* $H = (V_1, E, V_2)$ *such that* $\mathrm{dpt}\,(V_1) = \alpha$ *and* $\mathrm{dpt}\,(V_2) = \beta$.

P r o o f. Let $\ell(\alpha) = p$ and $\ell(\beta) = q$. Take two sets V_1 and V_2 such that $|V_1| = |V_2| = m$, where $m = \mathrm{sum}\,(\alpha) = \mathrm{sum}\,(\beta)$. It is clear that $p, q \le m$. Let $V_1 = \{u_1, u_2, \dots, u_m\}$ and $V_2 = \{v_1, v_2, \dots, v_m\}$.

First, we construct a bipartite graph $H_0 = (V_1, E_0, V_2)$. To do this, it suffices to specify the neighborhoods of its vertices $v_1, v_2, \dots, v_m$. Let

$$N_{H_0}(v_1) = \{u_1, u_2, \dots, u_{\beta_1}\}, N_{H_0}(v_2) = \{u_1, u_2, \dots, u_{\beta_2}\}, \dots, N_{H_0}(v_q) = \{u_1, u_2, \dots, u_{\beta_q}\}$$

and $N_{H_0}(v_i) = \emptyset$ if $i > q$ (for such i, $\beta_i = 0$). Neighborhoods of vertices $v_1, v_2, \dots, v_m$ form a system of nested subsets in V_1 and uniquely define the graph H_0.

Let us consider an $m \times m$ bipartite adjacency matrix A of the bipartite graph H_0, where the columns of the matrix A correspond to the vertices $v_1, v_2, \dots, v_m$ and are numbered from left to right, and the rows correspond to the vertices $u_1, u_2, \dots, u_m$ and are numbered from bottom to top. In matrix A, boxes containing 1's are concentrated in the lower left corner and form a Ferrers diagram for $\beta = (\beta_1, \beta_2, \dots)$, and by reading 1's row by row, we get a Ferrers diagram for β^*, i. e. $\mathrm{dpt}_{H_0}(V_1) = \beta^*$ and $\mathrm{dpt}_{H_0}(V_2) = \beta$.

Example 1. Let $\beta = (3, 2, 1, 1, 0, \dots)$. Then $\beta^* = (4, 2, 1, 0, \dots)$ and $m = 7$. Then the matrix A has the following form:

u_7	0	0	0	0	0	0	0
u_6	0	0	0	0	0	0	0
u_5	0	0	0	0	0	0	0
u_4	0	0	0	0	0	0	0
u_3	1	0	0	0	0	0	0
u_2	1	1	0	0	0	0	0
u_1	1	1	1	1	0	0	0
	v_1	v_2	v_3	v_4	v_5	v_6	v_7

Since $\beta^* \ge \alpha$ and the partitions β^* and α have the same weight m, in $IPL(m)$ there is a sequence of elementary transformations of the first type such that

$$\beta^* = \xi(0) \rightharpoondown \xi(1) \rightharpoondown \cdots \rightharpoondown \xi(t) = \alpha.$$

According to this sequence, by applying t times the assertion 2 of Lemma 5, we obtain, with lowering V_1-rotations of edges, a sequence of bipartite graphs:

$$H_0 = (V_1, E_0, V_2) \to H_1 = (V_1, E_1, V_2) \to \cdots \to H_t = (V_1, E_t, V_2)$$

such that $\mathrm{dpt}_{H_i}(V_1) = \xi_{(i)}$ and $\mathrm{dpt}_{H_i}(V_2) = \beta$ for any $i = 0, 1, \dots, t$. The graph $H_t = (V_1, E_t, V_2)$ is the sought one, since $\mathrm{dpt}_{H_t}(V_1) = \xi_{(t)} = \alpha$ and $\mathrm{dpt}_{H_t}(V_2) = \beta$. □

Gale–Ryser theorem proceeds from Lemmas 4 and 6.

4. Proof of Theorem 2 using the Gale–Ryser theorem

Now our goal is to prove Theorem 2 without using the Erdös–Gallai theorem and other criteria for partitions to be graphic. In addition, along the way, we will prove one of the main results of the paper, Theorem 5, on the existence for any nonzero partition λ of a realization that is obtained from some splitable graph with a certain sequence of lowering rotations of edges.

For this we need two auxiliary lemmas.

Lemma 7. 1. *Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be graphs and let the graph H_2 be obtained from the graph H_1 with the lowering rotation of an edge $H_1 \to H_2$. Then* $\operatorname{dpt}(H_2)$ *is obtained from* $\operatorname{dpt}(H_1)$ *with an elementary transformation of the first type* $\operatorname{dpt}(H_1) \rightharpoondown \operatorname{dpt}(H_2)$.

2. *Let $H_1 = (V_1, E_1)$ be a graph and let the partition μ be obtained from the partition* $\operatorname{dpt}(H_1)$ *with an elementary transformation of the first type* $\operatorname{dpt}(H_1) \rightharpoondown \mu$. *Then there exists a graph $H_2 = (V_2, E_2)$ which is obtained from the graph H_1 by means of a lowering rotation of the edge $H_1 \to H_2$ and for which $\mu = \operatorname{dpt}(H_2)$ is satisfied.*

P r o o f. Assertion 1 is obvious. Assertion 2 is proved similarly to assertion 2 of Lemma 5. □

Lemma 8. *For any partition λ of even weight, the number $C = \operatorname{sum}\operatorname{tl}(\lambda) - \operatorname{sum}\operatorname{hd}(\lambda)$ is even.*

P r o o f. Since

$$\operatorname{sum}\lambda = \operatorname{sum}\operatorname{hd}(\lambda) + r(r-1) + \operatorname{sum}\operatorname{tl}(\lambda),$$

where $r = r(\lambda)$, $\operatorname{sum}\operatorname{tl}(\lambda) + \operatorname{sum}\operatorname{hd}(\lambda)$ is even. It follows that the number $\operatorname{sum}\operatorname{tl}(\lambda) - \operatorname{sum}\operatorname{hd}(\lambda)$ is also even. □

The necessity of the condition of Theorem 2 is proved in Lemma 3.

Let us now give *a proof of the sufficiency of the condition* of Theorem 2, in which the Erdös–Gallai criterion and other criteria for the graphicity of partitions are not used, but the Gale–Ryser theorem is used.

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be an arbitrary nonzero partition of even weight, $\operatorname{hd}(\lambda) \le \operatorname{tl}(\lambda)$. Our goal is to prove the existence of a realization for the partition λ and to reveal a special kind of the realization that we obtain.

Let $r = r(\lambda)$ be the rank of the partition λ. It should be noted that $\ell(\operatorname{hd}(\lambda)) = r$ and $\ell(\operatorname{tl}(\lambda)) = (\operatorname{tl}^*(\lambda))_1 \le r$, where $(\operatorname{tl}^*(\lambda))_1$ is the first component of the partition $\operatorname{tl}^*(\lambda)$.

Since $\operatorname{hd}(\lambda) \le \operatorname{tl}(\lambda)$, there is a sequence of elementary transformations from $\operatorname{tl}(\lambda)$ to $\operatorname{hd}(\lambda)$, and both types of elementary transformations are admissible. Let us apply the algorithm [8] for constructing the shortest sequence of this type. For this, we take the component wise difference of the partitions

$$\operatorname{tl}(\lambda) - \operatorname{hd}(\lambda) = (\operatorname{tl}(\lambda)_1 - \operatorname{hd}(\lambda)_1, \operatorname{tl}(\lambda)_2 - \operatorname{hd}(\lambda)_2, \dots, \operatorname{tl}(\lambda)_r - \operatorname{hd}(\lambda)_r, 0, \dots).$$

Example 2. Assuming that $\lambda = (8, 7, 7, 7, 6, 6, 5, 3, 3, 2, 2)$. Then

$$r(\lambda) = 6, \quad \operatorname{hd}(\lambda) = (3, 2, 2, 2, 1, 1), \quad \operatorname{tl}^*(\lambda) = (5, 3, 3, 2, 2),$$
$$C = \operatorname{sum}\operatorname{tl}^*(\lambda) - \operatorname{sum}\operatorname{hd}(\lambda) = 15 - 11 = 4, \quad \operatorname{tl}(\lambda) = (5, 5, 3, 1, 1).$$

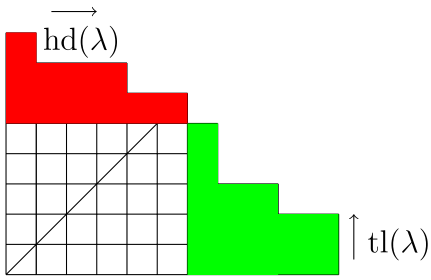

Figure 7. The head and the tail of the partition.

It should be noted that the conditions of Theorem 2 are satisfied since $\mathrm{sum}\,\lambda = 56$ and

$$\mathrm{hd}\,(\lambda) = (3,2,2,2,1,1) \leq (5,5,3,1,1) = \mathrm{tl}\,(\lambda),$$

since the prefix sums of the sequence $(3,2,2,2,1,1)$ do not exceed the corresponding prefix sums of the sequence $(5,5,3,1,1,0)$. Take the component-wise difference of the partitions $\mathrm{tl}\,(\lambda)$ and $\mathrm{hd}\,(\lambda)$:

$$\begin{array}{rcccccc} \mathrm{tl}\,(\lambda) = & (5, & 5, & 3, & 1, & 1, & 0) \\ \mathrm{hd}\,(\lambda) = & (3, & 2, & 2, & 2, & 1, & 1) \\ \hline \mathrm{tl}\,(\lambda) - \mathrm{hd}\,(\lambda) = & (+2 & +3 & +1 & -1 & 0 & -1) \end{array}$$

The partition $\mathrm{tl}\,(\lambda)$ over the partition $\mathrm{hd}\,(\lambda)$ in components with numbers 1, 2, and 3 has hills (see [8]) of heights 2, 3, and 1, respectively, and in components with numbers 4 and 6, it has pits (see [8]), each of which has depth 1. It should be noted that the sum of the heights of all hills is C plus the sum of the depths of all pits [8]. A pit is called admissible if adding 1 to it does not change the non-increasing order for the resulting partition (preserves the stepped form of the Ferrers diagram). Here the 6-pit (in the component with number 6), like the 4-pit, is admissible for the partition $\mathrm{tl}\,(\lambda)$. According to [8], if there is a pit, then there should be an admissible pit.

The algorithm for constructing some shortest sequence of elementary transformations from $\mathrm{tl}\,(\lambda)$ to $\mathrm{hd}\,(\lambda)$ [8] consists in sequentially moving a box into an admissible pit from the hill closest to it on the left or in removal the upper box from the last hill; be removal exactly C boxes. The admissible pits in the partition to be transformed will be chosen from right to left. The length of such a sequence is equal to the sum of the heights of all the hills of the partition $\mathrm{tl}\,(\lambda)$ over the partition $\mathrm{hd}\,(\lambda)$. Let us build such a sequence in our example. First we remove two boxes, then we fill two pits, and finally we remove two more boxes.

$$\begin{array}{rcccccc} \mathrm{tl}\,(\lambda) = & (5, & 5, & \underline{3}, & 1, & 1, & 0)^4; \\ & +2, & +3, & +1, & -1, & 0, & -1 \end{array} \qquad \begin{array}{cccccc} (5, & \underline{5}, & 2, & 1, & 1, & 0)^3; \\ +2, & +3, & 0, & -1, & 0, & -1 \end{array}$$

$$\begin{array}{cccccc} (5, & \underline{4}, & 2, & 1, & 1, & \underline{0})^2; \\ +2, & +2, & 0, & -1, & 0, & -1 \end{array} \qquad \begin{array}{cccccc} (5, & \underline{3}, & 2, & \underline{1}, & 1, & 1)^2; \\ +2, & +1, & 0, & -1, & 0, & 0 \end{array} \qquad \begin{array}{cccccc} (\underline{5}, & 2, & 2, & 2, & 1, & 1)^2; \\ +2, & 0, & 0, & 0, & 0, & 0 \end{array}$$

$$\begin{array}{cccccc} (\underline{4}, & 2, & 2, & 2, & 1, & 1)^1; \\ +1, & 0, & 0, & 0, & 0, & 0 \end{array} \qquad \begin{array}{cccccc} (3, & 2, & 2, & 2, & 1, & 1)^0 = \mathrm{hd}\,(\lambda) \\ 0, & 0, & 0, & 0, & 0, & 0 \end{array}$$

In this context, underlining at each step shows the choice of a hill for the subsequent elementary transformation of the second type or the choice of a hill and an admissible pit for the subsequent elementary transformation of the first type. In addition, at the top right of the current partition, we state the number of boxes that still need to be removed.

Since the partition λ has an even weight, by virtue of Lemma 8 the number C is even. Assuming that $s = 0.5C$. With the component-by-component difference of the partitions $\mathrm{tl}\,(\lambda)$ and $\mathrm{hd}\,(\lambda)$, by using the considered algorithm [8], we construct the shortest sequence of elementary transformations from $\mathrm{tl}\,(\lambda)$ to $\mathrm{hd}\,(\lambda)$, and at the beginning we remove s boxes and at the end we remove s more boxes:

$$\begin{aligned} \mathrm{tl}\,(\lambda) = \eta_{(0)} \rightharpoondown \eta_{(1)} \rightharpoondown \cdots \rightharpoondown \eta_{(s)} = \tau = \tau_{(0)} \rightharpoondown \cdots \rightharpoondown \tau_{(t)} = \xi \\ = \xi_{(0)} \rightharpoondown \xi_{(1)} \rightharpoondown \cdots \rightharpoondown \xi_{(s)} = \mathrm{hd}\,(\lambda). \end{aligned} \tag{4.2}$$

Since $\ell(\mathrm{tl}\,(\lambda)) \leq r$ and $\ell(\mathrm{hd}\,(\lambda)) = r$, in components with numbers greater than r in the difference $\mathrm{tl}\,(\lambda) - \mathrm{hd}\,(\lambda)$ all components are equal to 0, i. e. among them there are no hills or pits. Obviously, $\ell(\xi) \leq r$ is true, since in (4.2) when the pits are filled and the hills are removed, the lengths of the partitions cannot become larger than r. Hence, due to the equality $\ell(\mathrm{hd}\,(\lambda)) = r$ and the fact that in the sequence of transformations

$$\xi = \xi_{(0)} \rightharpoondown \xi_{(1)} \rightharpoondown \cdots \rightharpoondown \xi_{(s)} = \mathrm{hd}\,(\lambda)$$

only boxes are removed, we get the equality $\ell(\xi) = r$, i. e. for $\xi = (\xi_1, \xi_2, \ldots, \xi_r)$ $\xi_r \geq 1$.

Let us also consider a sequence of inverse transformations in reverse order from τ^* to $\mathrm{tl}^*(\lambda)$, which are box insertions:

$$\tau^* = \eta^*_{(s)} \rightharpoondown \eta^*_{(s-1)} \rightharpoondown \cdots \rightharpoondown \eta^*_{(0)} = \mathrm{tl}^*(\lambda).$$

Since in the sequence of transformations from τ^* to $\mathrm{tl}^*(\lambda)$ only block insertions occur and in the partition $\mathrm{tl}^*(\lambda)$ all components do not exceed r, in this sequence all components of all partitions do not exceed r and, in particular, $(\tau^*)_1 \leq r$.

Let us now take a pair of partitions: $\alpha = \xi$ and $\beta = \tau^*$. Then

$$\mathrm{sum}\,\alpha = \mathrm{sum}\,\tau = \mathrm{sum}\,\tau^* = \mathrm{sum}\,\beta,$$

since the transition from τ to ξ in (4.2) does not remove the boxes, and by virtue of (4.2) $\alpha = \xi \leq \tau = \beta^*$ also holds. Therefore, by virtue of the Gale–Ryser theorem, there is a bipartite graph $H = (V_1, R, V_2)$ such that $\mathrm{dpt}_H(V_1) = \alpha$ and $\mathrm{dpt}_H(V_2) = \beta$. We add V_1 to a complete subgraph by adding $1/2 \cdot r(r-1)$ edges. We obtain a splitable graph $H^+ = (K(V_1), R, V_2)$.

Since $r \geq (\tau^*)_1 = \beta_1$ and the partition $\xi = (\xi_1, \xi_2, \ldots, \xi_r)$ satisfies $\xi_r \geq 1$, we have

$$\xi_1 + (r-1) \geq \cdots \geq \xi_r + (r-1) \geq r \geq \beta_1 \geq \beta_2 \geq \ldots$$

Therefore, the degree partition corresponding to the graph H^+ has the form:

$$\mathrm{dpt}(H^+) = (\xi_1 + (r-1), \ldots, \xi_r + (r-1), \beta_1, \beta_2, \ldots, \beta_{\ell(\beta)}, 0, \ldots).$$

Let $\sigma_{(0)} = \mathrm{dpt}(H^+)$. Since $\beta = \tau^*$,

$$\sigma_{(0)} = \mathrm{dpt}(H^+) = (\xi_1 + (r-1), \ldots \xi_r + (r-1), (\tau^*)_1, (\tau^*)_2, \ldots, (\tau^*)_{\ell(\tau^*)}, 0, \ldots).$$

It is clear that $r(\sigma_{(0)}) = r$, $\mathrm{hd}\,(\sigma_{(0)}) = \xi = \xi_{(0)}$ and $\mathrm{tl}^*(\sigma_{(0)}) = \tau^* = \eta^*_{(s)}$.

It should be noted that for $s = 0$, due to (4.2) we have

$$\mathrm{hd}\,(\sigma_{(0)}) = \xi = \mathrm{hd}\,(\lambda), \quad \mathrm{tl}^*(\sigma_{(0)}) = \tau^* = \mathrm{tl}^*(\lambda)$$

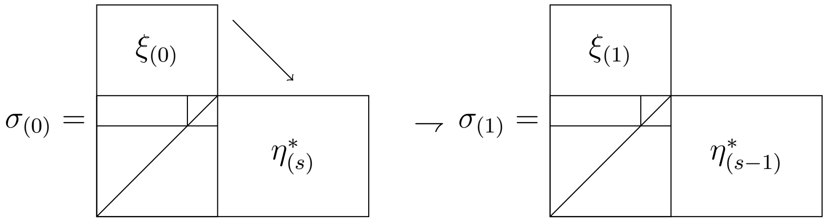

Figure 8. The elementary transformation of the first type obtained by removing and inserting of the box.

and $r(\mathrm{dpt}(H^+)) = r$, so $\mathrm{dpt}(H^+) = \lambda$, i. e. the splitable graph H^+ is a realization of the partition λ.

Thus, in the case $s = 0$, the existence of a splitable realization for λ has been proved.

In what follows, we will assume that $s > 0$.

Starting from the partition $\sigma_{(0)} = \mathrm{dpt}(H^+)$, we sequentially perform s elementary transformations of the first type in the partitions.

At step 1), we remove a box from the head of the partition $\sigma_{(0)}$ by removing the box $\xi_{(0)} \rightharpoondown \xi_{(1)}$ and insert this box into the partition $\eta^*_{(s)}$ by inserting the box $\eta^*_{(s)} \rightharpoondown \eta^*_{(s-1)}$.

As a result, we get a partition $\sigma_{(1)}$ such that $r(\sigma_{(1)}) = r$, $\mathrm{hd}\,(\sigma_{(1)}) = \xi_{(1)}$ and $\mathrm{tl}^*(\sigma_{(1)}) = \eta^*_{(s-1)}$, and $\eta_{(1)}$ is obtained from $\sigma_{(0)}$ with an elementary transformation of the first type $\sigma_{(0)} \rightharpoondown \sigma_{(1)}$ (see Fig. 8).

At step 2) remove a box from the head of the current partition $\sigma_{(1)}$ by removing the box $\xi_{(1)} \rightharpoondown \xi_{(2)}$ and insert this box into the partition $\eta^*_{(s-1)}$ by inserting the box $\eta^*_{(s-1)} \rightharpoondown \eta^*_{(s-2)}$. As a result, we get a partition $\sigma_{(2)}$ such that $r(\sigma_{(2)}) = r$, $\mathrm{hd}\,(\sigma_{(2)}) = \xi_{(2)}$ and $\mathrm{tl}^*(\sigma_{(2)}) = \eta^*_{(s-2)}$, and $\sigma_{(2)}$ is obtained from $\sigma_{(1)}$ with an elementary transformation of the first type $\sigma_{(1)} \rightharpoondown \sigma_{(2)}$.

We perform such steps s times.

At step s) we obtain a partition $\sigma_{(s)}$ such that $\mathrm{hd}\,(\sigma_{(s)}) = \xi_{(s)} = \mathrm{hd}\,(\lambda)$, $\mathrm{tl}^*(\sigma_{(s)}) = \eta^*_{(0)} = \mathrm{tl}^*(\lambda)$ and $r(\sigma_{(s)}) = r(\lambda)$. Therefore $\sigma_{(s)} = \lambda$ and

$$\mathrm{dpt}(H^+) = \sigma_{(0)} \rightharpoondown \sigma_{(1)} \rightharpoondown \sigma_{(2)} \rightharpoondown \cdots \rightharpoondown \sigma_{(s)} = \lambda.$$

Now, starting from the graph H^+, we apply Lemma 7 s times to this sequence, and we obtain graph G such that $\mathrm{dpt}(G) = \lambda$.

Thus, in the case $s > 0$, the sought realization of the partition λ is obtained from the splitable graph H^+ with the s lowering rotations of edges.

The theorem has been proved. □

It is plain to see that this proof also shows the validity of the two assertions as follows.

Theorem 4. *Let λ be a graphic partition. Then λ has a realization that is a splitable graph if and only if* $\mathrm{sum}\,\mathrm{hd}\,(\lambda) = \mathrm{sum}\,\mathrm{tl}\,(\lambda)$.

Theorem 5. *Let λ be a graphic partition and*

$$s = \frac{1}{2}[\mathrm{sum}\,\mathrm{tl}^*(\lambda)\; - \;\mathrm{sum}\,\mathrm{hd}\,(\lambda)].$$

Then the partition λ has a realization G that is obtained from some splitable graph H by s successive lowering rotations and, conversely, H is obtained from G by s successive lifting rotations of edges.

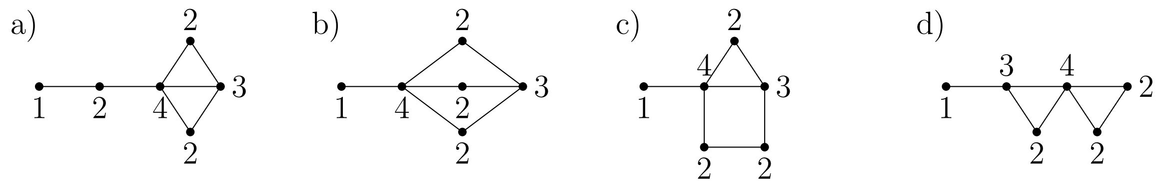

Figure 9. All realizations of the partition.

Example 3. Assuming that $\lambda = (4,3,2,2,2,1)$. Here $\mathrm{sum}\lambda = 14$, $r = 2$, $\mathrm{hd}\,(\lambda) = (3,2)$ and $\mathrm{tl}\,(\lambda)^* = (2,2,2,1)$, therefore $\mathrm{tl}\,(\lambda) = (4,3)$, $\mathrm{sum}\,\mathrm{tl}\,(\lambda) - \mathrm{sum}\,\mathrm{hd}\,(\lambda) = 7 - 5 = 2$ and $s = 1$. To get $\mathrm{hd}\,(\lambda)$ from $\mathrm{tl}\,(\lambda)$ we need to remove two boxes, hence, $\mathrm{hd}\,(\lambda) \leq \mathrm{tl}\,(\lambda)$.

Up to isomorphism and isolated vertices, there are 4 realization of the partition λ (see Fig. 9). According to the proof of Theorem 2, we have

$$\mathrm{tl}\,(\lambda) = (4,3) \rightharpoondown (4,2) = \tau = \xi \rightharpoondown (3,2) = \mathrm{hd}\,(\lambda).$$

Therefore, $\alpha = \xi = (4,2)$ and $\beta = \tau^* = (2,2,1,1)$. We sequentially construct a bipartite graph H, a splitable graph H^+, two vertices of which generate a clique and six vertices generate a coclique, then we perform one lowering rotation of an edge, we obtain the sought realization G of the partition $\lambda = (4,3,2,2,2,1)$ (see Fig. 10). Graph G is shown in Fig. 9a.

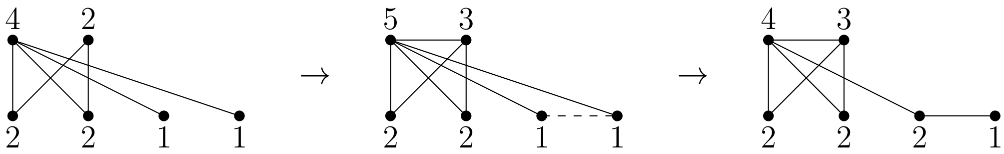

Figure 10. The graph G obtaining.

It is easy to check that in this example, each realization of the partition $\lambda = (4,3,2,2,2,1)$ can be obtained from a suitable splitable graph with one lowering rotation of an edge.

At the end of this section, we present a brief review of related results previously obtained by the authors.

It is worth reminding that any graphic partition has an even weight. The set of all graphic partitions of fixed weight $2m$ is an order ideal of the lattice $IPL(2m)$, i. e. it is closed under smaller partitions. A graphic partition λ of weight $2m$ will be called a maximal graphic partition if it is maximal in the set of all graphic partitions of the lattice $IPL(2m)$.

The graph G is called a *threshold* one (see [16]) if its set of vertices can be represented as a disjoint union of the clique V_1 and coclique V_2 (i. e. $V_1 \cup V_2 = \emptyset$, V_1 generates the complete subgraph $K(V_1)$ and V_2 is the zero subgraph $O(V_2)$ in G), and the set of neighbourhoods in G of vertices from V_2 forms a chain of subsets of the set V_1 with respect to the set-theoretic inclusion. It should be noted that the cases $V_1 = \emptyset$ or $V_2 = \emptyset$ are allowed, i. e. the complete and zero graphs are threshold. For the threshold graph G, the set of all edges can be represented as a disjoint union of the set of all edges of the complete subgraph $K(V_1)$ and the set E of all its edges connecting vertices from V_1 with vertices from V_2. Thus, the threshold graph can be represented as $G = (K(V_1), E, O(V_2))$. We will simply write $G = (K(V_1), E, V_2)$. A bipartite subgraph $H = (V_1, E, V_2)$ will be called its *sandwich subgraph*. In the trivial cases when $V_1 = \emptyset$, or $V_2 = \emptyset$, or V_2 consists of isolated vertices, the sandwich subgraph H is an empty subgraph in G.

The following statements are true, which were proved in [6, 7].

1. An arbitrary partition λ is a maximal graphic partition if and only if $\mathrm{hd}\,(\lambda) = \mathrm{tl}\,(\lambda)$.
2. A graph is threshold if and only if it does not contain any lifting triples of vertices.
3. The degree partition corresponding to the graph G is the maximum graphic partition if and only if the graph G is threshold.
4. Any graph can be reduced to a threshold form with a finite sequence of lifting rotations of edges.
5. For an arbitrary graphic partition λ, all of its realizations H, and only they are obtained from the threshold realizations G of maximal graphic partitions μ such that $\mu \geq \lambda$ and $\mathrm{sum}(\mu) = \mathrm{sum}(\lambda)$ with the finite sequences of lowering edge rotations from G to H.

Assume that μ and λ are two arbitrary non-zero partitions and $\mu \geq \lambda$. The *height* (μ, λ) of a partition μ over a partition λ is the number of transformations in the shortest sequence of elementary transformations transforming μ into λ.

For a given graphic partition λ, a maximal graphic partition μ such that $\mu \geq \lambda$ and $\mathrm{sum}\,(\mu) = \mathrm{sum}\,(\lambda)$ is said to be closest in height to a partition λ if it has the minimum possible height over λ among all such partitions.

The following assertion was proved in [9].

Assume that λ be an arbitrary graphic partition and μ be the maximum graphic partition closest to it in height. Then

1) either $r(\mu) = r(\lambda) - 1$ and $\ell(\mathrm{tl}(\lambda)) < r(\lambda)$, or $r(\mu) = r(\lambda)$;

2) $\mathrm{height}(\mu, \lambda) = \mathrm{height}(\mathrm{tl}(\lambda), \mathrm{hd}(\lambda)) - \dfrac{1}{2}[\mathrm{sum}(\mathrm{tl}(\lambda)) - \mathrm{sum}(\mathrm{hd}(\lambda))] = \dfrac{1}{2}\sum_{i=1}^{r} |\mathrm{tl}(\lambda)_i - \mathrm{hd}(\lambda)_i|$, where $r = r(\lambda)$.

An algorithm was found in [9] that constructs some maximum graphic partition μ closest to λ in height such that $r(\mu) = r(\lambda)$. In the case when $\ell(\mathrm{tl}(\mu)) < r(\mu)$, we also found an algorithm that constructs some maximum graphic partition μ closest to λ in height such that $r(\mu) = r(\lambda) - 1$.

Assuming that λ be an arbitrary non-zero graphic partition of weight $2m$ and there is maximal graphic partitions μ such that $\mu \geq \lambda$ and $r(\mu) = r$, where r is some natural number. Then the set of heads of all such maximal graphic partitions μ creates an interval in the lattice $IPL(m - 1/2 \cdot r(r-1))$. This result was obtained by our PhD-student V.V. Zuev (Ural Federal University). The full version of this result will be published in the article being prepared.

5. Proof of the Gale–Ryser theorem with the ht-criterion

Let us now give another rather simple proof of the Gale–Ryser theorem, in which the ht-criterion is used.

The necessity of the condition of the theorem is satisfied by virtue of Lemma 4.

Let us prove *the sufficiency* of the condition of the theorem. Let $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \beta)$ be non-zero partitions such that $\mathrm{sum}\,\alpha = \mathrm{sum}\,\beta$ and $\alpha \leq \beta^*$. Assume that $m = \mathrm{sum}\,\alpha = \mathrm{sum}\,\beta$, i. e. $\alpha, \beta \in IPL(m)$, $p = \ell(\alpha)$ and $q = \ell(\beta)$.

Since $\alpha_p + (p-1) \geq p \geq \beta_1$, the sequence as follows

$$\lambda = (\alpha_1 + (p-1), \dots, \alpha_p + (p-1), \beta_1, \beta_2, \dots, \beta_q, 0, \dots)$$

is a partition. Obviously, $r(\lambda) = p$ and

$$\mathrm{hd}\,(\lambda) = (\alpha_1, \alpha_2, \dots) = \alpha, \quad \mathrm{tl}^*(\lambda) = (\beta_1, \beta_2, \dots) = \beta,$$

thus, $\mathrm{tl}\,(\lambda) = \beta^*$.

It should be noted that

$$\mathrm{sum}\,\lambda = \mathrm{sum}\,\mathrm{hd}\,(\lambda) + p(p-1) + \mathrm{sum}\,\mathrm{tl}\,(\lambda) = \mathrm{sum}\,\alpha + p(p-1) + \mathrm{sum}\,\beta = 2m + p(p-1)$$

is even number and $\mathrm{hd}\,(\lambda) = \alpha \le \beta^* = \mathrm{tl}\,(\lambda)$. Therefore, by virtue of the ht-criterion, there is a graph H realizing the partition λ, i. e. a graph H such that $\mathrm{dpt}(H) = \lambda$. It is clear that $\ell(\lambda) = p+q$. Without loss of generality, we assume that H has no isolated vertices and $V_H = \{v_1, v2, \dots, v_{p+q}\}$.

Assume that $V_1 = \{v_1, v_2, \dots, v_p\}$, where $\deg_H v_i = \alpha_i + (p-1)$ for any $i = 1, 2, \dots, p$, and $V_2 = \{v_{p+1}, v_{p+2}, \dots, v_{p+q}\}$, where $\deg_H v_{p+j} = \beta_j$ for any $j = 1, 2, \dots, q$.

Let us remove all edges in H that connect pairs of different vertices from V_1. We obtain a graph G. For each $i = 1, 2, \dots, p$, the degree α_i of the vertex v_i will decrease by no more than $p-1$ when passing from H to G, so $\deg_G v_i = \alpha_i + \delta_i$, where δ_i will hold in G, where δ_i is a non-negative integer. Moreover, $\delta_i = 0$ is satisfied if, when passing from H to G, the degree of the vertex v_i decreases by $p-1$.

Case 1. Assume that the set $V_1 = \{v_1, v_2, \dots, v_p\}$ is not a clique in H.

Then there are vertices v_i from the set V_1 such that $\delta_i > 0$. Therefore,

$$\sum_{i=1}^{p} \deg_G v_i = \sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{p} \delta_i > \mathrm{sum}\,\alpha = m.$$

Therefore, in the graph G there are more than m edges going from V_1 to V_2. Since $\mathrm{sum}\,\beta$ is greater than or equal to the number of such edges, we get the $\mathrm{sum}\,\beta > m$, which contradicts $\mathrm{sum}\,\beta = m$.

Therefore, this case is impossible.

Case 2. Let the set $V_1 = \{v_1, v_2, \dots, v_p\}$ is a clique in H.

Then $\deg_G v_i = \alpha_i$ for any $i = 1, 2, \dots, p$. Since $\mathrm{sum}\,\alpha = m$, the number of edges of the graph G going from V_1 to V_2 is equal to m. By virtue of the equality $\mathrm{sum}\,\beta = m$, it follows that there are no edges in the graph G that connect pairs of different vertices from V_2. Therefore, V_1 and V_2 are two cocliques in G.

Since $\mathrm{dpt}_G(V_1) = \alpha$ and $\mathrm{dpt}_G(V_2) = \beta$, the graph G is the sought bipartite graph with parts V_1 and V_2.

The theorem has been proved. □

At the end of this section, we present a brief review of the results previously obtained by the authors and similar in subject matter to the Gale–Ryser theorem.

We first give the necessary definitions.

We say that a bipartite graph $H = (V_1, E, V_2)$ contains a *bipartite 4-pseudocycle* x_1, x_2, x_3, x_4, x_1, if $x_1, x_3 \in V_2$; $x_2, x_4 \in V_1$; $x_1x_2 \in E$; $x_2x_3 \notin E$; $x_3x_4 \in E$; $x_4x_1 \notin E$.

We call the bipartite graph $H = (V_1, E, V_2)$ a *bipartite-threshold graph* [10] if it does not have any lifting triples of both the first and second parts, i. e. such lifting triples (x, v, y), that $x, y \in V_1$, $v \in V_2$, or $x, y \in V_2$, $v \in V_1$.

It should be reminded [7] that a graph is a threshold one if and only if it does not contain any lifting triples of vertices. Therefore, bipartite-threshold graphs are analogues of threshold graphs for the class of bipartite graphs.

In [10], the properties of bipartite-threshold graphs were studied. The following assertion is true [10].

Let $H = (V_1, E, V_2)$ be a bipartite graph. Then the following conditions are equivalent

1. H is a sandwich subgraph of the threshold graph $G = (K(V_1), E, V_2)$.

2. H is a sandwich subgraph of the threshold graph $G = (K(V_2), E, V_1)$.

3. The neighborhoods in H of the vertices of each of the parts V_1 and V_2 are nested, i. e. they form chains with respect to the set-theoretical inclusion.

4. Neighborhoods in H of the vertices of the part V_1 are nested, i. e. it forms a chain with respect to the set-theoretical inclusion.

5. Neighborhoods in H of the vertices of V_2 are nested, i. e. it forms a chain with respect to the set-theoretical inclusion.

6. H is a bipartite-threshold graph, i. e. it does not contain lifting V_1-triples and lifting V_2-triples.

7. H does not contain lifting V_1-triples.

8. H does not contain lifting V_2-triples.

9. $\mathrm{dpt}_H(V_2) = \mathrm{dpt}_H(V_1)^*$.

10. $\mathrm{dpt}_H(V_1) = \mathrm{dpt}_H(V_2)^*$.

11. H has no bipartite 4-pseudocycles.

Assume that α and β be non-zero partitions such that $\operatorname{sum}\alpha = \operatorname{sum}\beta = m$ and $\alpha \leq \beta^*$. A bipartite graph $H = (V_1, E, V_2)$ such that $\mathrm{dpt}_H(V_1) = \alpha$ and $\mathrm{dpt}_H(V_2) = \beta$ will be called a *realization of a pair of partitions* (α, β). The class of all such bipartite graphs is denoted by $\mathrm{BG}\,(\alpha, \beta)$ (the *family of bipartite graphs corresponding to the pair* (α, β)).

For an arbitrary partition γ, we denote by $\mathrm{btg}\,(\gamma, \gamma^*)$ a bipartite threshold graph with parts V_1 and V_2 without any isolated vertices such that $\mathrm{dpt}_G(V_1) = \gamma$ and $\mathrm{dpt}_G(V_2) = \gamma^*$. It should be noted that this graph is unique up to isomorphism (see [12]).

Any bipartite graph $H = (V_1, E, V_2)$ from the family of graphs $\mathrm{BG}\,(\alpha, \beta)$ can be reduced with finite sequences of bipartite lifting rotations of edges to bipartite threshold graphs, each of which, up to isomorphism and isolated vertices, has the form $\mathrm{btg}\,(\gamma, \gamma^*)$ for a suitable partition γ, and the graph $H = (V_1, E, V_2)$ is obtained from such graphs $\mathrm{btg}\,(\gamma, \gamma^*)$ with reverse sequences of bipartite lowering edge rotations.

We denote by $\mathrm{BTG}_\uparrow\,(\alpha, \beta)$ the family of all bipartite threshold graphs that can be obtained from the graphs of the family $\mathrm{BG}\,(\alpha, \beta)$ with bipartite lifting rotations (the *family of bipartite threshold graphs over the pair* (α, β)).

Let a bipartite graph $H = (V_1, E_2, V_2)$ be obtained from a bipartite graph $G = (V_1, E_1, V_2)$ with a finite sequence of bipartite lifting edge rotations. The least number of bipartite lifting rotations of edges in the sequence taking G to H is denoted by $\mathrm{updist}\,(G, H)$ and is called the *upper distance* from G to H.

The following two theorems are valid [12].

1. The family of bipartite threshold graphs $\mathrm{BTG}_\uparrow\,(\alpha, \beta)$ up to isomorphisms and isolated vertices consists of graphs of the form $\mathrm{btg}\,(\gamma, \gamma^*)$, where $\alpha \leq \gamma \leq \beta^*$ (compare with the Zuev theorem about the interval of heads of maximal graphic partitions over the given partition given at the end of Section 4).

2. Let the bipartite threshold graph $H = (V_1, E_2, V_2) = \mathrm{btg}\,(\gamma, \gamma^*) \in \mathrm{BTG}_{\uparrow}\,(\alpha, \beta)$ be obtained from the graph $G = (V_1, E_1, V_2) \in \mathrm{BG}(\alpha, \beta)$ with a finite sequence of bipartite lifting rotations of edges. Then

$$\mathrm{updist}\,(G, H) \geq \mathrm{height}\,(\beta^*, \alpha) = \mathrm{height}\,(\alpha^*, \beta).$$

This estimate is achieved on the graphs $\mathrm{btg}\,(\beta^*, \beta)$ and $\mathrm{btg}\,(\alpha, \alpha^*)$, i. e. for $\gamma = \beta^*$ and for $\gamma = \alpha$.

It is clear that any bipartite graph is reduced by successive rotations of edges, each of which corresponds to a lifting triple of only the first part, to a bipartite-threshold graph.

Assume that the bipartite graph $H_2 = (V_1, E_2, V_2)$ can be obtained from the graph $H_1 = (V_1, E_1, V_2)$ with a finite sequence of V_1-rotations of the edges. Let $V_1\text{-dist}\,(H_1, H_2)$ denote the smallest number of V_1-rotations of edges in the sequence that maps H_1 to H_2 and call it as the V_1-distance from H_1 to H_2. In [11], with the Hungarian algorithm, a polynomial algorithm was constructed that transforms an arbitrary bipartite graph $H = (V_1, E, V_2)$ into a bipartite-threshold graph G with a finite sequence of the smallest possible length consisting of V_1-rotations of edges, i. e. the length equal to $V_1\text{-dist}\,(H, G)$.

In conclusion we make the following important remark. Let λ be an arbitrary nonzero partition. It corresponds to two partitions $\alpha = \mathrm{hd}\,(\lambda)$ and $\beta = \mathrm{tl}\,(\lambda)^*$. According to the ht-criterion, a partition λ is graphic if and only if its sum is even and $\alpha \leq \beta^*$. It is clear that the ht-criterion is essentially an analog of the Gale–Reiser criterion when passing from the class of bipartite graphs to the class of all graphs. There are many facts indicating a deep analogy between the properties of degree partitions in the class of all bipartite graphs and in the class of all graphs.

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