URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 28–43

DOI: 10.15826/umj.2024.1.003

ON A GROUP EXTENSION INVOLVING THE SPORADIC JANKO GROUP J_2

Ayoub B. M. Basheer

School of Mathematical and Computer Sciences, University of Limpopo (Turfloop),
P. Bag X1106, Sovenga 0727, South Africa

Mathematics Program, Faculty of Education and Arts, Sohar University,
Sohar, Oman

ayoubbasheer@gmail.com

Abstract: According to the electronic Atlas [23], the group J_2 has an absolutely irreducible module of dimension 6 over $\mathbb{F}_4$. Therefore, a split extension group having the form $4^6{:}J_2 := \overline{G}$ exists. In this paper, we consider this group. Our purpose is to determine its conjugacy classes and character table using the methods of the coset analysis together with Clifford–Fischer theory. We determine the inertia factors of $\overline{G}$ by analyzing the maximal subgroups of J_2 and the maximal of the maximal subgroups of J_2 together with other various information. It turns out that the character table of $\overline{G}$ is a 53×53 real-valued matrix, while Fischer matrices of the extension are all integer-valued matrices with sizes ranging from 1 to 8.

Keywords: Group extensions, Janko sporadic simple group, Inertia groups, Fischer matrices, Character table.

1. Introduction

Visiting the history of the classification of finite simple groups, one can see that it was only a century after the establishment of the last Mathieu group that Z. Janko could construct a new sporadic simple group in 1964. This simple group has been named in his honor, is denoted by J_1, and has order 175560. Then Janko predicted the existence of other sporadic simple groups; namely, J_2, J_3, and J_4, which later are all proved to exist. According to Wilson [22], the original construction of the second Janko group J_2 was due to Marshall Hall (and thus, in some other papers, this group is referred to as Hall–Janko group HJ but here we use the more familiar notation J_2). Hall constructed this group as a permutation group acting on 100 points. Starting with the group $U_3(3)$, the group J_2 appears as a maximal normal subgroup of index 2 of the automorphism group of a graph Γ associated with $U_3(3)$ (for further details on the vertices and how are they connected, see the description given on page 224 of [22]).

The group J_2 has order $604800 = 2^7 \times 3^3 \times 5^2 \times 7$. It has Schur multiplier and outer automorphism groups both isomorphic to $\mathbb{Z}_2$. From the Atlas of Wilson [23], one can see that the group J_2 has a 6-dimensional absolutely irreducible module over $\mathbb{F}_4$. Therefore, a split extension group of the form $4^6{:}J_2 := \overline{G}$ exists. The present paper focuses on the group $\overline{G}$. Our purpose is to determine its conjugacy classes and the inertia factors of this extension with the fusions of their conjugacy classes into the classes of J_2. We will also find the character tables of these inertia factors and, finally, the full character table of the extension $\overline{G}$ under consideration. The methods used here to achieve the previous purpose are the coset analysis technique and the theory of Clifford–Fischer matrices. The most interesting part of this paper is the process of determining the inertia factor groups, where there are three inertia factor groups; namely, $H_1 = J_2$, H_2, and H_3. The main technique used for determining the structures of H_2 and H_3 is the analysis of the maximal subgroups of J_2 and the maximal subgroups of these maximal subgroups. There are many possibilities for H_2 and H_3, and combining all of them leads to contradictions except for only one possibility where we find that $H_2 = 2^{2+4}{:}S_3$ and $H_3 = 2^2 \times A_5$. We use a method of the coset analysis together with Clifford–Fischer theory to construct the character tables of H_2 and H_3, but we organize the columns of the character tables of these inertia factors according to the centralizers sizes. This paper determines all Fischer matrices of $\overline{G}$; their sizes vary between 1 and 8. The character table of $\overline{G}$ is a 53×53 real-valued matrix, which will be divided into 63 parts corresponding to 3 inertia factors and 21 conjugacy classes of $G = J_2$.

If $\overline{G} = N{\cdot}G$ is a group extension (here, N is the kernel of the extension and G is isomorphic to $\overline{G}/N$), then the character table of G produced using the coset analysis and Clifford–Fischer theory is in a special format that cannot be obtained by the direct computations using GAP [18] or Magma [15]. Another interesting point is the interplay between the coset analysis and Clifford–Fischer theory. This can be seen at the size of each Fischer matrix, where it is equal to the number of $\overline{G}$-classes corresponding to $[g_i]_G$ obtained via the coset analysis technique. In other words, computations of the conjugacy classes of $\overline{G}$ using the coset analysis technique will determine the sizes of all Fischer matrices.

From the Atlas [23], we can see that J_2 has an absolutely irreducible module of dimension 6 over the field $\mathbb{F}_4$. With α being a generator of the field $\mathbb{F}_4$, the following two elements g_1 and g_2 are 6×6 matrices over $\mathbb{F}_4$ that generates J_2:

$$g_1 = \begin{pmatrix} \alpha^2 & \alpha^2 & 0 & 0 & 0 & 0 \\ 1 & \alpha^2 & 0 & 0 & 0 & 0 \\ 1 & 1 & \alpha^2 & \alpha^2 & 0 & 0 \\ \alpha & 1 & 1 & \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & \alpha^2 & \alpha^2 & 0 & \alpha \\ \alpha^2 & 1 & \alpha^2 & 0 & \alpha^2 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \alpha & 1 & \alpha^2 & 1 & \alpha^2 & \alpha^2 \\ \alpha & 1 & \alpha & 1 & 1 & \alpha \\ \alpha & \alpha & \alpha^2 & \alpha^2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \alpha^2 & 1 & \alpha^2 & \alpha^2 & \alpha & \alpha^2 \\ \alpha^2 & 1 & \alpha^2 & \alpha & \alpha^2 & \alpha \end{pmatrix},$$

where $o(g_1) = 2$, $o(g_2) = 3$, and $o(g_1 g_2) = 7$.

Using the above two generators of J_2 together with few GAP commands, we were able to construct our split extension group $\overline{G} = 4^6{:}J_2$ in terms of 7×7 matrices over $\mathbb{F}_4$. With α being a generator of the field $\mathbb{F}_4$, the following elements $\overline{g}_1$ and $\overline{g}_2$ generate the group $\overline{G}$:

$$\overline{g}_1 = \begin{pmatrix} 0 & 1 & \alpha & \alpha & \alpha^2 & 0 & 0 \\ \alpha & \alpha^2 & 0 & \alpha & \alpha & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 & \alpha^2 & \alpha^2 & 0 & 0 \\ 1 & \alpha & 0 & \alpha^2 & 1 & 1 & 0 \\ 1 & 0 & 1 & \alpha & 0 & 0 & 0 \\ \alpha^2 & \alpha & 1 & 0 & 0 & \alpha^2 & 0 \\ \alpha^2 & \alpha^2 & \alpha & 0 & \alpha^2 & 1 & 1 \end{pmatrix}, \quad \overline{g}_2 = \begin{pmatrix} \alpha & 0 & 1 & 1 & 0 & \alpha^2 & 0 \\ 1 & 1 & \alpha^2 & \alpha^2 & \alpha^2 & 1 & 0 \\ 1 & \alpha & 0 & \alpha & 0 & \alpha^2 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ \alpha^2 & 0 & 1 & 0 & 1 & 0 & 0 \\ \alpha & 1 & \alpha & \alpha & 0 & \alpha^2 & 0 \\ \alpha & 1 & \alpha^2 & 1 & \alpha & 1 & 1 \end{pmatrix},$$

where $o(\overline{g}_1) = 6$, $o(\overline{g}_2) = 12$, and $o(\overline{g}_1 \overline{g}_2) = 10$.

To make the computations easier, we used a few GAP commands to convert the matrix representation of $\overline{G}$ into permutation representation. We represented $\overline{G}$ in terms of the set $\{1, 2, \ldots, 4096\}$.

Using GAP, we see that the group $\overline{G}$ possesses only one proper normal subgroup of order 4096. This normal subgroup is an elementary abelian group isomorphic to N. In GAP, one can check for the complements of N in $\overline{G}$, where in our case we obtained four complements, all isomorphic to J_2, and each of these four complements, together with N, gives the split extension in consideration.

For the notation used in this paper and how Clifford–Fischer theory and the coset analysis techniques are used, we follow [1–14, 17].

2. Conjugacy classes of $\overline{G} = 4^6{:}J_2$

Here we compute the conjugacy classes of $\overline{G}$ using the coset analysis technique (see [2] by Basheer, [5, 6, 8] by Basheer and Moori, or [20] and [21] by Moori for more details) since we are interested in organizing the classes of $\overline{G}$ corresponding to the classes of J_2. Note that J_2 has 21 conjugacy classes (see the Atlas [16] or Atlas of Wilson [23]). Corresponding to these 21 classes of J_2, we obtained 53 classes in $\overline{G}$.

In Table 1, we list the conjugacy classes of $\overline{G}$, where in this table:

- k_i represents the number of orbits $Q_{i1}, Q_{i2}, \ldots, Q_{ik_i}$ for the action of N on the coset $N\overline{g}_i = Ng_i$, where g_i is a representative of a class of the complement J_2 of N in $\overline{G}$. In particular, the action of N on the identity coset N produces 4 096 orbits each consists of a single element. Therefore, for $\overline{G}$, we have $k_1 = 4\,096$.
- f_{ij} is the number of orbits fused under the action of $C_G(g_i)$ on $Q_1, Q_2, \ldots, Q_k$. In particular, the action of $C_G(1_G) = G = J_2$ on the orbits $Q_1, Q_2, \ldots, Q_k$ affords three orbits of lengths 1, 1 575, and 2 520 (with corresponding point stabilizers J_2, $2^{2+4}{:}S_3$, and $2^2 \times A_5$. Thus, $f_{11} = 1$, $f_{12} = 1\,575$, and $f_{13} = 2\,520$.
- m_{ij} are weights (attached to each class of $\overline{G}$). These weights are computed by the formula

$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N|\frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|},$$

  where N is the kernel of an extension $\overline{G}$ in consideration.

Table 1. The conjugacy classes of $\overline{G}$.

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_1 = 1A$	$k_1 = 4\,096$	$f_{11} = 1$	$m_{11} = 1$	g_{11}	1	1	2 477 260 800
		$f_{12} = 1\,575$	$m_{12} = 1\,575$	g_{12}	2	1 575	1 570 864
		$f_{13} = 2\,520$	$m_{13} = 2\,520$	g_{13}	2	2 520	983 040
$g_2 = 2A$	$k_2 = 256$	$f_{21} = 1$	$m_{21} = 16$	g_{21}	2	5 040	491 520
		$f_{22} = 15$	$m_{22} = 240$	g_{22}	2	75 600	32 768
		$f_{23} = 120$	$m_{23} = 1\,920$	g_{23}	4	604 800	4 096
		$f_{24} = 120$	$m_{24} = 1\,920$	g_{24}	4	604 800	4 096
$g_3 = 2B$	$k_3 = 64$	$f_{31} = 1$	$m_{31} = 64$	g_{31}	2	161 280	15 360
		$f_{32} = 1$	$m_{32} = 64$	g_{32}	4	161 280	15 360
		$f_{33} = 1$	$m_{33} = 64$	g_{33}	4	161 280	15 360
		$f_{34} = 1$	$m_{34} = 64$	g_{34}	4	161 280	15 360
		$f_{35} = 15$	$m_{35} = 960$	g_{35}	4	2 419 200	1 024
		$f_{36} = 15$	$m_{36} = 960$	g_{36}	4	2 419 200	1 024
		$f_{37} = 15$	$m_{37} = 960$	g_{37}	4	2 419 200	1 024
		$f_{38} = 15$	$m_{38} = 960$	g_{38}	4	2 419 200	1 024
$g_4 = 3A$	$k_4 = 1$	$f_{41} = 1$	$m_{41} = 4\,096$	g_{41}	3	2 293 760	1 080
$g_5 = 3B$	$k_5 = 16$	$f_{51} = 1$	$m_{51} = 256$	g_{51}	3	4 300 800	576
		$f_{52} = 4$	$m_{52} = 768$	g_{52}	6	12 902 400	192
		$f_{53} = 12$	$m_{53} = 3\,072$	g_{53}	6	51 609 600	48
		$f_{61} = 1$	$m_{61} = 256$	g_{61}	4	1 612 800	1 536

continued on the next page

Table 1 (continued from the previous page)

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_6 = 4A$	$k_6 = 16$	$f_{62} = 3$	$m_{62} = 768$	g_{62}	4	4 838 400	512
		$f_{63} = 3$	$m_{63} = 768$	g_{63}	4	4 838 400	512
		$f_{64} = 3$	$m_{64} = 768$	g_{64}	4	4 838 400	512
		$f_{65} = 6$	$m_{65} = 1\,536$	g_{65}	4	9 676 800	256
$g_7 = 5A$	$k_7 = 16$	$f_{71} = 1$	$m_{71} = 256$	g_{71}	5	516 096	4 800
		$f_{72} = 15$	$m_{72} = 3\,840$	g_{72}	10	7 741 440	320
$g_8 = 5B$	$k_8 = 16$	$f_{81} = 1$	$m_{81} = 256$	g_{81}	5	516 096	4 800
		$f_{82} = 15$	$m_{82} = 3\,840$	g_{82}	10	7 741 440	320
$g_9 = 5C$	$k_9 = 1$	$f_{91} = 1$	$m_{91} = 4\,096$	g_{91}	5	49 545 216	50
$g_{10} = 5D$	$k_{10} = 1$	$f_{10,1} = 1$	$m_{10,1} = 4\,096$	$g_{10,1}$	5	49 545 216	50
$g_{11} = 6A$	$k_{11} = 1$	$f_{11,1} = 1$	$m_{11,1} = 4\,096$	$g_{11,1}$	6	103 219 200	24
$g_{12} = 6B$	$k_{12} = 4$	$f_{12,1} = 1$	$m_{12,1} = 1\,024$	$g_{12,1}$	6	51 609 600	48
		$f_{12,2} = 1$	$m_{12,2} = 1\,024$	$g_{12,2}$	12	51 609 600	48
		$f_{12,3} = 1$	$m_{12,3} = 1\,024$	$g_{12,3}$	12	51 609 600	48
		$f_{12,4} = 1$	$m_{12,4} = 1\,024$	$g_{12,4}$	12	51 609 600	48
$g_{13} = 7A$	$k_{13} = 1$	$f_{13,1} = 1$	$m_{13,1} = 4\,096$	$g_{13,1}$	7	353 894 400	7
$g_{14} = 8A$	$k_{14} = 4$	$f_{14,1} = 1$	$m_{14,1} = 1\,024$	$g_{14,1}$	8	77 414 400	32
		$f_{14,2} = 1$	$m_{14,2} = 1\,024$	$g_{14,2}$	8	77 414 400	32
		$f_{14,3} = 1$	$m_{14,3} = 1\,024$	$g_{14,3}$	8	77 414 400	32
		$f_{14,4} = 1$	$m_{14,4} = 1\,024$	$g_{14,4}$	8	77 414 400	32
$g_{15} = 10A$	$k_{15} = 4$	$f_{15,1} = 1$	$m_{15,1} = 1\,024$	$g_{15,1}$	10	30 965 760	80
		$f_{15,2} = 1$	$m_{15,2} = 1\,024$	$g_{15,2}$	20	30 965 760	80
		$f_{15,3} = 1$	$m_{15,3} = 1\,024$	$g_{15,3}$	20	30 965 760	80
		$f_{15,4} = 1$	$m_{15,4} = 1\,024$	$g_{15,4}$	20	30 965 760	80
$g_{16} = 10B$	$k_{16} = 4$	$f_{16,1} = 1$	$m_{16,1} = 1024$	$g_{16,1}$	10	30 965 760	80
		$f_{16,2} = 1$	$m_{16,2} = 1\,024$	$g_{16,2}$	20	30 965 760	80
		$f_{16,3} = 1$	$m_{16,3} = 1\,024$	$g_{16,3}$	20	30 965 760	80
		$f_{16,4} = 1$	$m_{16,4} = 1\,024$	$g_{16,4}$	20	30 965 760	80
$g_{17} = 10C$	$k_{17} = 1$	$f_{17,1} = 1$	$m_{17,1} = 4\,096$	$g_{17,1}$	10	247 726 080	10
$g_{18} = 10D$	$k_{18} = 1$	$f_{18,1} = 1$	$m_{18,1} = 4\,096$	$g_{18,1}$	10	247 726 080	10
$g_{19} = 12A$	$k_{19} = 1$	$f_{19,1} = 1$	$m_{19,1} = 4\,096$	$g_{19,1}$	12	206 438 400	12
$g_{20} = 15A$	$k_{20} = 1$	$f_{20,1} = 1$	$m_{20,1} = 4\,096$	$g_{20,1}$	15	165 150 720	15
$g_{21} = 15B$	$k_{21} = 1$	$f_{21,1} = 1$	$m_{21,1} = 4\,096$	$g_{21,1}$	15	165 150 720	15

3. Inertia factor groups of $\overline{G} = 4^6{:}J_2$

We have seen in Section 2 that the action of $\overline{G}$ on N produced three orbits of lengths 1, 1 575, and 2 520. By a theorem of Brauer (see, for example, [2, Theorem 5.1.1]), it follows that the action of $\overline{G}$ on $\mathrm{Irr}(N)$ will also produce three orbits of lengths 1, r, and s, where

$$1 + r + s = |\mathrm{Irr}(N)| = 4\,096;$$

that is

$$r + s = 4\,095. \tag{3.1}$$

The values of r and s will be determined through deep investigation on the maximal subgroups of J_2 or on the maximal of the maximal subgroups of J_2 together with various information including sizes of the Fischer matrices, fusions of the conjugacy classes of some subgroups into the group J_2, and other information. In Table 2, we supply the maximal subgroups of J_2 (see the Atlas [16]). We will need these subgroups to determine H_2 and H_3.

Table 2. The maximal subgroups of $G = J_2$.

M_i	$\|M_i\|$	$[J_2 : M_i]$
$U_3(3)$	6 048	100
$(3^{\cdot}A_6){:}2$	2 160	280
$2_-^{1+4}{:}A_5$	1 920	315
$2^{2+4}{:}(3 \times S_3)$	1 152	525
$A_4 \times A_5$	720	840
$A_5 \times D_{10}$	600	1 008
$L_3(2){:}2$	336	1 800
$5^2{:}D_{12}$	300	2 016
A_5	60	10 080

First, since 1, r, and s are the lengths of the orbits on the action of $\overline{G}$ on N (which can be reduced to the action of G on N), it follows that $[G : H_1] = 1$, $[G : H_2] = r$, and $[G : H_3] = s$, where H_1, H_2, and H_3 are the inertia factors in $G = J_2$. It follows that $H_1 = G = J_2$ and $r, s \mid |G|$; that is $r, s|604\,800$. Now, 604 800 has 192 positive divisors, where 140 divisors are less than 4 095. Out of these 140 divisors, only four pairs (r, s) satisfy (3.1). These are the pairs

$$(r, s) \in \{(63, 4\,032), (315, 3\,780), (945, 3\,150), (1\,575, 2\,520)\}. \tag{3.2}$$

Here, we do not distinguish between the pairs (r, s) and (s, r) and therefore we exclude the other four pairs $(4\,032, 63), (3\,780, 315), (3\,150, 945)$, and $(2\,520, 1\,575)$ from our consideration and restrict ourselves only to those in (3.2). Another point we put in mind is that since $\overline{G}$ is a split extension of 4^6 by J_2 and 4^6 is an elementary abelian group, it follows that the three character tables of H_1, H_2, and H_3, which we will use to construct the character table of $\overline{G}$, are ordinary. From the Atlas and Table 1, we have $|\mathrm{Irr}(\overline{G})| = 53$ and $|\mathrm{Irr}(H_1)| = |\mathrm{Irr}(G)| = |\mathrm{Irr}(J_2)| = 21$. Since

$$\sum_{i=1}^{3} |\mathrm{Irr}(H_i)| = |\mathrm{Irr}(\overline{G})| = 53,$$

we have $|\mathrm{Irr}(H_1)| + |\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| = |\mathrm{Irr}(\overline{G})| = 53$, that is

$$|\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| = 32. \tag{3.3}$$

Our next task is to show that $(r, s) = (1\,575, 2\,520)$ and the action of $\overline{G}$ on $\mathrm{Irr}(N)$ is dual to the action of $\overline{G}$ on classes of N. This will be achieved by excluding the other possible pairs by getting a contradiction to some fact in each case.

Proposition 1. $(r, s) \neq (63, 4\,032)$.

P r o o f. To obtain a contradiction, suppose that $(r, s) = (63, 4\,032)$, i.e., $r = 63$ and $s = 4\,032$ (or $[J_2 : H_2] = 63$ and $[J_2 : H_3] = 4\,032$) and consequently $|H_2| = 9\,600$ and $|H_3| = 150$. Since $|H_2| = 9\,600$ and the maximal subgroups of J_2 are given in Table 2, it follows that $|H_2|$ is bigger than the size of any maximal subgroup of J_2, a contradiction. Thus, (r, s) cannot be $(63, 4\,032)$. □

Proposition 2. $(r, s) \neq (315, 3\,780)$.

P r o o f. To obtain a contradiction, suppose that $(r, s) = (315, 3\,780)$, i.e., $r = 315$ and $s = 3\,780$ (or $[J_2 : H_2] = 315$ and $[J_2 : H_3] = 3\,780$) and, consequently, $|H_2| = 3\,840$ and $|H_3| = 320$. Since $|H_2| = 3\,840$ and the maximal subgroups of J_2 are given in Table 2, it follows that H_2 does not sit in any of the maximal subgroups of J_2, a contradiction. Thus, (r, s) cannot be $(315, 3\,780)$. □

Proposition 3. $(r, s) \neq (945, 3\,150)$.

P r o o f. To obtain a contradiction, suppose that $(r, s) = (945, 3150)$, i.e., $r = 945$ and $s = 3150$ (or $[J_2 : H_2] = 945$ and $[J_2 : H_3] = 3\,150$) and, consequently, $|H_2| = 1\,280$ and $|H_3| = 384$. Since $|H_2| = 1\,280$ and the maximal subgroups of J_2 are given in Table 2, we see that H_2 is not among the maximal subgroups of J_2 and does not sit in any of them. This contradiction proves that (r, s) cannot be $(945, 3\,150)$. □

Proposition 4. *The action of J_2 on $\mathrm{Irr}(4^6)$ is dual to the action of J_2 on the conjugacy classes of $N = 4^6$.*

P r o o f. We have seen in Section 2 that the action of J_2 on the conjugacy classes of $N = 4^6$ produced 3 orbits of lengths 1, 1 575, and 2 520. From (3.1), we have $r+s = 4\,095$, where r and s are the lengths of the second the third orbits on the action of J_2 on $\mathrm{Irr}(4^6)$. Further, by (3.2), we have $(r, s) \in \{(63, 4\,032), (315, 3\,780), (945, 3\,150), (1\,575, 2\,520)\}$. We also proved in Propositions 1, 2, and 3 that $(r, s) \notin \{(63, 4\,032), (315, 3\,780), (945, 3\,150)\}$. Therefore, $(r, s) = (1\,575, 2\,520)$ and the action of J_2 on $\mathrm{Irr}(4^6)$ is dual to the action of J_2 on the conjugacy classes of $N = 4^6$, as claimed. □

Proposition 5. *The inertia factor groups have the forms $2^{2+4}{:}S_3$ and $2^2 \times A_5$.*

P r o o f. From Proposition 4, we can see that the orbit lengths on the action of J_2 on $\mathrm{Irr}(4^6)$ are 1, 1 575, and 2 520. It follows that $[G : H_1] = 1$, $[G : H_2] = 1\,575$ and $[G : H_3] = 2\,520$ and, consequently, $H_1 = G = J_2$, $|H_2| = 384$, and $|H_3| = 240$. By (3.3), we also have $|\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| = 32$. Now we investigate the maximal subgroups of J_2 to locate H_2 and H_3. Since $|H_2| = 384$ and the maximal subgroups of J_2 are given in Table 2, it follows that H_2 is either an index 5 subgroup of $2_-^{1+4}{:}A_5$ or an index 3 subgroup of $2^{2+4}{:}(3 \times S_3)$. If $H_2 \leq 2_-^{1+4}{:}A_5$ is such that $[2_-^{1+4}{:}A_5 : H_2] = 5$, then H_2 must be a maximal subgroup in it since the index is a prime number. Now, $2_-^{1+4}{:}A_5$ has 4 maximal subgroups of orders 384, 320, 192, and 120. The maximal subgroup of order 384 has the structure $2^{2+4}{:}6$ and 19 ordinary irreducible characters. Also, if $H_2 \leq 2^{2+4}{:}(3 \times S_3)$ is such that $[2^{2+4}{:}(3 \times S_3) : H_2] = 3$, then H_2 must be a maximal subgroup in it since the index is a prime number. Now, $2^{2+4}{:}(3 \times S_3)$ has 4 maximal subgroups of orders 576, 384 (twice), and 72. The two maximal subgroups of order 384 have structures $2^{2+4}{:}S_3$ and $2^{1+4}{:}A_4$, where $|\mathrm{Irr}(2^{2+4}{:}S_3)| = 12$ and $|\mathrm{Irr}(2^{1+4}{:}A_4)| = 15$. Thus, we have

$$\begin{gathered} H_2 \in \{2^{2+4}{:}6, 2^{2+4}{:}S_3, 2^{1+4}{:}A_4\}, \\ |\mathrm{Irr}(2^{2+4}{:}6)| = 19, \quad |\mathrm{Irr}(2^{2+4}{:}S_3)| = 12, \quad |\mathrm{Irr}(2^{1+4}{:}A_4)| = 15. \end{gathered} \tag{3.4}$$

Next, consider H_3. Since $|H_3| = 240$ and the maximal subgroups of J_2 are given in Table 2, we deduce that H_3 is either

- an index 9 subgroup of $(3^{\cdot}A_6){:}2$,
- an index 8 subgroup of $2_-^{1+4}{:}A_5$, or
- an index 3 subgroup of $A_4 \times A_5$.

Consider each of these cases. Using GAP, one can see that the group $(3^{\cdot}A_6){:}2$ has four maximal subgroups of orders 1 080, 216, 60, and 48. Therefore, H_3 cannot be a subgroup of $(3^{\cdot}A_6){:}2$ since $[(3^{\cdot}A_6){:}2 : H_3] = 9$, which is impossible. Next, consider the case where H_3 is an index 8 subgroup of $2_-^{1+4}{:}A_5$. Checking the order of all maximal subgroups of $2_-^{1+4}{:}A_5$, which can be done using GAP, shows that $2_-^{1+4}{:}A_5$ has four maximal subgroups of orders 384, 320, 192, and 120. Therefore, $H_3 \not\leq 2_-^{1+4}{:}A_5$. Finally, we turn to the last case where we consider H_3 to be a subgroup of $A_4 \times A_5$ of index 3. The group $A_4 \times A_5$ has five maximal subgroups of orders 240, 180, 144, 120, and 72. The maximal subgroup of order 240 has the structure $2^2 \times A_5$ and 20 ordinary irreducible characters. We deduce that H_3 has the structure $2^2 \times A_5$ and $|\mathrm{Irr}(H_3)| = 20$. Using this together with (3.4), we conclude that $(H_2, H_3) = (2^{2+4}{:}S_3, 2^2 \times A_5)$ is the required pair of inertia factor groups since it consists of (3.3), and all other possibilities are exhausted and each lead to a contradiction, except $(H_2, H_3) = (2^{2+4}{:}S_3, 2^2 \times A_5)$. Hence, we have the result. □

Next, we construct the character tables of H_1, H_2, and H_3 and determine the fusions of the conjugacy classes of these groups into the classes of $H_1 = G = J_2$. The character table of the simple Janko group J_2 can be found at the Atlas. As subgroups of $G = J_2$ that generated by g_1 and g_2 given in Section 1, and α being a generator of $\mathbb{F}_4$, the two inertia factor groups $H_2 = 2^{2+4}{:}S_3$ and $H_3 = 2^2 \times A_5$ are generated as follows: $H_2 = \langle \alpha_1, \alpha_2 \rangle$ and $H_3 = \langle \beta_1, \beta_2 \rangle$, where

$$\alpha_1 = \begin{pmatrix} 1 & 1 & \alpha & \alpha & \alpha & 0 \\ \alpha^2 & \alpha^2 & 1 & \alpha & \alpha & \alpha^2 \\ \alpha^2 & \alpha & \alpha^2 & \alpha^2 & \alpha^2 & \alpha^2 \\ 1 & \alpha & 1 & 1 & 1 & 0 \\ \alpha^2 & \alpha & \alpha & \alpha^2 & \alpha & \alpha^2 \\ \alpha^2 & \alpha & \alpha & \alpha^2 & \alpha^2 & \alpha \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} \alpha & 1 & \alpha & 1 & \alpha^2 & \alpha \\ 0 & 1 & 0 & 0 & 0 & \alpha \\ \alpha & 0 & 1 & 0 & \alpha & 1 \\ \alpha & \alpha & 1 & \alpha^2 & \alpha^2 & \alpha \\ \alpha & \alpha^2 & \alpha & \alpha & \alpha & \alpha^2 \\ \alpha & 1 & \alpha^2 & \alpha^2 & 0 & \alpha^2 \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} 1 & \alpha & \alpha^2 & 0 & 1 & 1 \\ 0 & 0 & \alpha^2 & 1 & 1 & 1 \\ 0 & \alpha^2 & 0 & 1 & \alpha^2 & \alpha^2 \\ \alpha & \alpha & 1 & 1 & 1 & 1 \\ 1 & 0 & \alpha^2 & \alpha^2 & 0 & 0 \\ 0 & 1 & \alpha^2 & 0 & \alpha & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} \alpha & 0 & \alpha^2 & 0 & 1 & \alpha^2 \\ 1 & \alpha^2 & \alpha & \alpha & \alpha & 0 \\ \alpha^2 & \alpha & 1 & 0 & 1 & 1 \\ 0 & \alpha & \alpha^2 & \alpha & 0 & \alpha^2 \\ 0 & \alpha & \alpha^2 & 1 & 1 & 0 \\ 0 & 1 & \alpha & \alpha & 0 & \alpha^2 \end{pmatrix}.$$

We recursively use Clifford–Fischer theory to construct the character table of H_2. The action of S_3 on the set $\mathrm{Irr}(2^{2+4})$ produced 6 orbits of lengths 1, 3, 3, 3, 3, and 6 with the corresponding inertia factor groups S_3, $\mathbb{Z}_2$ (four times), and the identity group. Also, H_3 is the direct product of the elementary abelian group 2^2 by A_5. Thus, the character table of H_3 is easy to construct since we know the character tables of both 2^2 and A_5. In this paper, we list the full character tables of H_2 and H_3 and organize the columns of the character tables according to the orders and the sizes of the centralizers.

Recall that H_2 and H_3 are not maximal subgroups of J_2, but they are maximal of some maximal subgroups of J_2 (H_2 is a maximal subgroup of $2^{2+4}{:}(3 \times S_3)$ while H_3 is a maximal subgroup of $A_4 \times A_5$). We determined the fusions of the conjugacy classes of H_2 and H_3 into the classes J_2 using the permutation characters of J_2 on $2^{2+4}{:}(3 \times S_3)$ and $A_4 \times A_5$; the permutation characters of $2^{2+4}{:}(3 \times S_3)$ and $A_4 \times A_5$ on H_2 and H_3, respectively, together with the sizes of centralizers. The following proposition plays a great role in determining the fusions; its proof can be found in [2].

Proposition 6. *Let $K_1 \leq K_2 \leq K_3$, and let ψ be a class function on K_1. Then, $(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3} = \psi\uparrow_{K_1}^{K_3}$. More generally, if $K_1 \leq K_2 \leq \cdots \leq K_n$ is a nested sequence of subgroups of K_n and ψ is a class function on K_1, then $(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3}\cdots\uparrow_{K_{n-1}}^{K_n} = \psi\uparrow_{K_1}^{K_n}$.*

P r o o f. See Proposition 3.5.6 of [2]. □

We supply the full character tables of the inertia factor groups H_2 and H_3 together with the fusions of their conjugacy classes into the classes of J_2 in Tables 3 and 4.

Table 3. The character table of $H_2 = 2^{2+4}{:}S_3$.

$[g]_{H_2}$	$1a$	$2a$	$2b$	$2c$	$3a$	$4a$	$4b$	$4c$	$4d$	$8a$	$8b$	$8c$
$\lvert C_{H_2}(g)\rvert$	384	128	16	16	3	32	32	32	16	8	8	8
$\hookrightarrow J_2$	$1A$	$2A$	$2B$	$2A$	$3B$	$4A$	$4A$	$4A$	$4A$	$8A$	$8A$	$8A$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	1	1	1	1	-1	-1	-1	-1
χ_3	2	2	2	0	-1	2	2	2	0	0	0	0
χ_4	3	3	-1	-1	0	3	-1	-1	-1	-1	1	1
χ_5	3	3	-1	-1	0	-1	3	-1	-1	1	1	-1
χ_6	3	3	-1	1	0	-1	3	-1	1	-1	-1	1
χ_7	3	3	-1	1	0	3	-1	-1	1	1	-1	-1
χ_8	3	3	-1	-1	0	-1	-1	3	-1	1	-1	1
χ_9	3	3	-1	1	0	-1	-1	3	1	-1	1	-1
χ_{10}	6	6	2	0	0	-2	-2	-2	0	0	0	0
χ_{11}	12	-4	0	-2	0	0	0	0	2	0	0	0
χ_{12}	12	-4	0	2	0	0	0	0	-2	0	0	0

4. Fischer matrices of $\overline{G} = 4^6{:}J_2$

We now calculate the Fischer matrices of $\overline{G} = 4^6{:}J_2$. Following Section 3 of [5], we label the top and bottom of the columns of the Fischer matrix $\mathcal{F}_i$ corresponding to g_i by the sizes of the centralizers of g_{ij}, $1 \leq j \leq c(g_i)$, in $\overline{G}$ and m_{ij}, respectively.

The rows of $\mathcal{F}_i$ are partitioned into parts $\mathcal{F}_{ik}$, $1 \leq k \leq t$, corresponding to the inertia factors $H_1, H_2, \ldots, H_t$, where each $\mathcal{F}_{ik}$ consists of $c(g_{ik})$ rows corresponding to the α_k^{-1}-regular classes (those are the H_k-classes that fuse to the class $[g_i]_G$). Thus, each row of $\mathcal{F}_i$ is labeled by a pair (k, m), where $1 \leq k \leq t$ and $1 \leq m \leq c(g_{ik})$. We list the values of $|C_{\overline{G}}(g_{ij})|$ and m_{ij}, $1 \leq i \leq 27$, $1 \leq j \leq c(g_i)$, in Table 1. The fusions of classes of H_2 and H_3 into classes of G are given in Tables 3 and 4, respectively. Since the size of the Fischer matrix $\mathcal{F}_i$ is $c(g_i)$, it follows from Table 1 that the sizes of the Fischer matrices of $\overline{G} = 4^6{:}J_2$ range between 1 and 8 for every $i \in \{1, 2, \ldots, 21\}$.

The Fisher matrices have interesting arithmetic properties (see Proposition 3.6 in [5]). We used these properties to calculate some entries of these matrices and construct systems of algebraic equations. We solved these systems of equations using the symbolic mathematical package Maxima [19] and, hence, computed all of the Fisher matrices $\overline{G}$ that we list below.

$\mathcal{F}_1$

g_1		g_{11}	g_{12}	g_{13}
$o(g_{1j})$		1	2	2
$\lvert C_{\overline{G}}(g_{1j})\rvert$		2 477 260 800	1 570 864	983 040
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$			
$(1,1)$	604 800	1	1	1
$(2,1)$	384	1 575	39	-25
$(3,1)$	240	2 520	-40	24
m_{1j}		1	1 575	2 520

$\mathcal{F}_2$

g_2		g_{21}	g_{22}	g_{23}	g_{24}
$o(g_{2j})$		2	2	4	4
$\lvert C_{\overline{G}}(g_{2j})\rvert$		491 520	32 768	4 096	4 096
(k,m)	$\lvert C_{H_k}(g_{2km})\rvert$				
$(1,1)$	1 920	1	1	1	1
$(2,1)$	128	15	15	-1	-1
$(2,2)$	16	120	-8	-8	8
$(3,1)$	16	120	-8	8	-8
m_{2j}		16	240	1 920	1 920

$\mathcal{F}_3$

g_3		g_{31}	g_{32}	g_{33}	g_{34}	g_{35}	g_{36}	g_{37}	g_{38}
$o(g_{3j})$		2	4	4	4	4	4	4	4
$\lvert C_{\overline{G}}(g_{3j})\rvert$		15 360	15 360	15 360	15 360	1 024	1 024	1 024	1 024
(k,m)	$\lvert C_{H_k}(g_{3km})\rvert$								
$(1,1)$	240	1	1	1	1	1	1	1	1
$(2,1)$	16	15	15	15	15	-1	-1	-1	-1
$(3,1)$	240	1	-1	-1	1	-1	-1	1	1
$(3,2)$	240	1	-1	1	-1	-1	1	1	-1
$(3,3)$	240	1	1	-1	-1	-1	1	-1	1
$(3,4)$	16	15	-15	15	-15	1	-1	1	-1
$(3,5)$	16	15	-15	-15	15	1	-1	-1	1
$(3,6)$	16	15	15	-15	-15	1	1	-1	-1
m_{3j}		64	64	64	64	960	960	960	960

$\mathcal{F}_4$

g_4		g_{41}
$o(g_{4j})$		3
$\lvert C_{\overline{G}}(g_{4j})\rvert$		1 080
(k,m)	$\lvert C_{H_k}(g_{4km})\rvert$	
$(1,1)$	1 080	1
m_{4j}		4 096

$\mathcal{F}_5$

g_5		g_{51}	g_{52}	g_{53}
$o(g_{5j})$		3	6	6
$\lvert C_{\overline{G}}(g_{5j})\rvert$		576	192	48
(k,m)	$\lvert C_{H_k}(g_{5km})\rvert$			
$(1,1)$	36	1	1	1
$(2,1)$	3	12	-4	0
$(3,1)$	12	3	3	-1
m_{5j}		256	768	3 072

$\mathcal{F}_6$

g_6		g_{61}	g_{62}	g_{63}	g_{64}	g_{65}
$o(g_{6j})$		4	4	4	4	4
$\lvert C_{\overline{G}}(g_{6j})\rvert$		1 536	512	512	512	256
(k,m)	$\lvert C_{H_k}(g_{6km})\rvert$					
$(1,1)$	96	1	1	1	1	1
$(2,1)$	32	3	−1	−1	3	−1
$(2,2)$	32	3	3	−1	−1	−1
$(2,3)$	32	3	−1	3	−1	−1
$(2,4)$	16	6	−2	−2	−2	2
m_{6j}		256	768	768	768	1 536

$\mathcal{F}_7$

g_7		g_{71}	g_{72}
$o(g_{7j})$		5	10
$\lvert C_{\overline{G}}(g_{7j})\rvert$		4 800	320
(k,m)	$\lvert C_{H_k}(g_{7km})\rvert$		
$(1,1)$	300	1	1
$(3,1)$	20	15	−1
m_{7j}		256	3 840

$\mathcal{F}_8$

g_8		g_{81}	g_{82}
$o(g_{8j})$		5	10
$\lvert C_{\overline{G}}(g_{8j})\rvert$		4 800	320
(k,m)	$\lvert C_{H_k}(g_{8km})\rvert$		
$(1,1)$	300	1	1
$(3,1)$	20	15	−1
m_{8j}		256	3 840

$\mathcal{F}_9$

g_9		g_{91}
$o(g_{9j})$		5
$\lvert C_{\overline{G}}(g_{9j})\rvert$		50
(k,m)	$\lvert C_{H_k}(g_{9km})\rvert$	
$(1,1)$	50	1
m_{9j}		4 096

$\mathcal{F}_{10}$

g_{10}		$g_{10,1}$
$o(g_{10j})$		5
$\lvert C_{\overline{G}}(g_{10j})\rvert$		50
(k,m)	$\lvert C_{H_k}(g_{10km})\rvert$	
$(1,1)$	50	1
m_{10j}		4 096

$\mathcal{F}_{11}$

g_{11}		$g_{11,1}$
$o(g_{11j})$		6
$\lvert C_{\overline{G}}(g_{11j})\rvert$		24
(k,m)	$\lvert C_{H_k}(g_{11km})\rvert$	
$(1,1)$	24	1
m_{11j}		4 096

$\mathcal{F}_{12}$

g_{12}		$g_{12,1}$	$g_{12,2}$	$g_{12,3}$	$g_{12,4}$
$o(g_{12j})$		6	12	12	12
$\lvert C_{\overline{G}}(g_{12j})\rvert$		48	48	48	48
(k,m)	$\lvert C_{H_k}(g_{12km})\rvert$				
$(1,1)$	12	1	1	1	1
$(3,1)$	12	1	−1	1	−1
$(3,2)$	12	1	1	−1	−1
$(3,3)$	12	1	−1	−1	1
m_{12j}		1 024	1 024	1 024	1 024

$\mathcal{F}_{13}$

g_{13}		$g_{13,1}$
$o(g_{13j})$		7
$\lvert C_{\overline{G}}(g_{13j})\rvert$		7
(k,m)	$\lvert C_{H_k}(g_{13km})\rvert$	
$(1,1)$	7	1
m_{13j}		4096

$\mathcal{F}_{14}$

g_{14}		$g_{14,1}$	$g_{14,2}$	$g_{14,3}$	$g_{14,4}$
$o(g_{14j})$		8	8	8	8
$\lvert C_{\overline{G}}(g_{14j})\rvert$		32	32	32	32
(k,m)	$\lvert C_{H_k}(g_{14km})\rvert$				
$(1,1)$	8	1	1	1	1
$(2,1)$	12	1	−1	1	−1
$(2,2)$	12	1	1	−1	−1
$(2,3)$	12	1	−1	−1	1
m_{14j}		1024	1024	1024	1024

$\mathcal{F}_{15}$

g_{15}		$g_{15,1}$	$g_{15,2}$	$g_{15,3}$	$g_{15,4}$
$o(g_{15j})$		10	20	20	20
$\|C_{\overline{G}}(g_{15j})\|$		80	80	80	80
(k,m)	$\|C_{H_k}(g_{15km})\|$				
$(1,1)$	8	1	1	1	1
$(3,1)$	12	1	-1	1	-1
$(3,2)$	12	1	1	-1	-1
$(3,3)$	12	1	-1	-1	1
m_{15j}		1 024	1 024	1 024	1 024

$\mathcal{F}_{16}$

g_{16}		$g_{16,1}$	$g_{16,2}$	$g_{16,3}$	$g_{16,4}$
$o(g_{16j})$		10	20	20	20
$\|C_{\overline{G}}(g_{16j})\|$		80	80	80	80
(k,m)	$\|C_{H_k}(g_{16km})\|$				
$(1,1)$	8	1	1	1	1
$(3,1)$	12	1	-1	1	-1
$(3,2)$	12	1	1	-1	-1
$(3,3)$	12	1	-1	-1	1
m_{16j}		1 024	1 024	1 024	1 024

$\mathcal{F}_{17}$

g_{17}		$g_{17,1}$
$o(g_{17j})$		10
$\|C_{\overline{G}}(g_{17j})\|$		10
(k,m)	$\|C_{H_k}(g_{17km})\|$	
$(1,1)$	10	1
m_{17j}		4 096

$\mathcal{F}_{18}$

g_{18}		$g_{18,1}$
$o(g_{18j})$		10
$\|C_{\overline{G}}(g_{18j})\|$		10
(k,m)	$\|C_{H_k}(g_{18km})\|$	
$(1,1)$	10	1
m_{18j}		4 096

$\mathcal{F}_{19}$

g_{19}		$g_{19,1}$
$o(g_{19j})$		12
$\|C_{\overline{G}}(g_{19j})\|$		12
(k,m)	$\|C_{H_k}(g_{19km})\|$	
$(1,1)$	12	1
m_{19j}		4 096

$\mathcal{F}_{20}$

g_{20}		$g_{20,1}$
$o(g_{20j})$		15
$\|C_{\overline{G}}(g_{20j})\|$		15
(k,m)	$\|C_{H_k}(g_{20km})\|$	
$(1,1)$	15	1
m_{20j}		4 096

$\mathcal{F}_{21}$

g_{21}		$g_{21,1}$
$o(g_{21j})$		15
$\|C_{\overline{G}}(g_{21j})\|$		15
(k,m)	$\|C_{H_k}(g_{21km})\|$	
$(1,1)$	15	1
m_{21j}		4 096

5. Character table of $\overline{G} = 4^6{:}J_2$

In Sections 2, 3, and 4, we have determined:

- the conjugacy classes of $\overline{G} = 4^6{:}J_2$ (Table 1);
- the inertia factors H_1, H_2, and H_3;
- the character tables of all inertia factor groups of G (the Atlas together with Tables 3 and 4); in these two tables, we also supplied the fusions of the classes of the inertia factors H_2 and H_3 into classes of G;
- the Fischer matrices of $\overline{G}$ (see Section 4).

Following [2, 5], without any difficulties, one can construct the full character table of $\overline{G}$ in the format of Clifford–Fischer theory. The table will be composed of 63 parts corresponding to 21 cosets and three inertia factor groups. The full character table of $\overline{G}$ is a 53×53 $\mathbb{R}$-valued matrix, and we give it in the format of Clifford–Fischer theory in Table 5. We conclude by remarking that the accuracy of this character table has been tested using GAP.

Acknowledgements

The author would like to thank Professor J. Moori, from whom he studied group theory and representation theory. The author would also like to thank the University of Limpopo for providing financial support.

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Table 4. The character table of $H_3 = 2^2{\times}A_5$

$[g]_{H_3}$	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$2g$	$3a$	$5a$	$5b$	$6a$	$6b$	$6c$	$10a$	$10b$	$10c$	$10d$	$10e$	$10f$
$\|C_{H_3}(g)\|$	240	240	240	240	16	16	16	16	12	20	20	12	12	12	20	20	20	20	10	10
$\hookrightarrow J_2$	$1A$	$2B$	$2B$	$2B$	$2B$	$2B$	$2B$	$2A$	$3B$	$5B$	$5A$	$6B$	$6B$	$6B$	$10B$	$10A$	$10A$	$10B$	$10B$	$10A$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	−1	−1	−1	−1	1	1	1	1	1	−1	1	−1	−1	1	−1	−1	1	−1
χ_3	1	−1	1	−1	1	−1	−1	1	1	1	1	−1	−1	1	1	−1	−1	−1	−1	1
χ_4	1	−1	−1	1	−1	1	−1	1	1	1	1	1	−1	−1	−1	−1	1	1	−1	−1
χ_5	3	3	3	3	−1	−1	−1	−1	0	A	A^*	0	0	0	A	A^*	A^*	A	A	A^*
χ_6	3	3	3	3	−1	−1	−1	−1	0	A^*	A	0	0	0	A^*	A	A	A^*	A^*	A
χ_7	3	3	−3	−3	1	1	−1	−1	0	A	A^*	0	0	0	$-A$	A^*	$-A^*$	$-A$	A	$-A^*$
χ_8	3	3	−3	−3	1	1	−1	−1	0	A^*	A	0	0	0	$-A^*$	A	$-A$	$-A^*$	A^*	$-A$
χ_9	3	−3	3	−3	−1	1	1	−1	0	A	A^*	0	0	0	A	$-A^*$	$-A^*$	$-A$	$-A$	A^*
χ_{10}	3	−3	3	−3	−1	1	1	−1	0	A^*	A	0	0	0	A^*	$-A$	$-A$	$-A^*$	$-A^*$	A
χ_{11}	3	−3	−3	3	1	−1	1	−1	0	A	A^*	0	0	0	$-A$	$-A^*$	A^*	A	$-A$	$-A^*$
χ_{12}	3	−3	−3	3	1	−1	1	−1	0	A^*	A	0	0	0	$-A^*$	$-A$	A	A^*	$-A^*$	$-A$
χ_{13}	4	4	4	4	0	0	0	0	1	−1	−1	1	1	1	−1	−1	−1	−1	−1	−1
χ_{14}	4	4	−4	−4	0	0	0	0	1	−1	−1	−1	1	−1	1	−1	1	1	−1	1
χ_{15}	4	−4	4	−4	0	0	0	0	1	−1	−1	−1	−1	1	−1	1	1	1	1	−1
χ_{16}	4	−4	−4	4	0	0	0	0	1	−1	−1	1	−1	−1	1	1	−1	−1	1	1
χ_{17}	5	5	5	5	1	1	1	1	−1	0	0	−1	−1	−1	0	0	0	0	0	0
χ_{18}	5	5	−5	−5	−1	−1	1	1	−1	0	0	1	−1	1	0	0	0	0	0	0
χ_{19}	5	−5	5	−5	1	−1	−1	1	−1	0	0	1	1	−1	0	0	0	0	0	0
χ_{20}	5	−5	−5	5	−1	1	−1	1	−1	0	0	−1	1	1	0	0	0	0	0	0

where in Table 4, $A = (1-\sqrt{5})/2$ and $A^* = (1+\sqrt{5})/2$.

Table 5. The character table of $\overline{G} = 4^6{:}J_2$.

$[g_i]_{J_2}$	1A			2A				2B								3A	3B		
$[g_{ij}]_{\overline{G}}$	$1a$	$2a$	$2b$	$2c$	$2d$	$4a$	$4b$	$2e$	$4c$	$4d$	$4e$	$4f$	$4g$	$4h$	$4i$	$3a$	$3b$	$6a$	$6b$
$\vert C_{\overline{G}}(g_{ij})\vert$	2477260800	1572864	983040	491520	32768	4096	4096	15360	15360	15360	15360	1024	1024	1024	1024	1080	576	192	48
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	14	14	14	−2	−2	−2	−2	2	2	2	2	2	2	2	2	5	−1	−1	−1
χ_3	14	14	14	−2	−2	−2	−2	2	2	2	2	2	2	2	2	5	−1	−1	−1
χ_4	21	21	21	5	5	5	5	−3	−3	−3	−3	−3	−3	−3	−3	3	0	0	0
χ_5	21	21	21	5	5	5	5	−3	−3	−3	−3	−3	−3	−3	−3	3	0	0	0
χ_6	36	36	36	4	4	4	4	0	0	0	0	0	0	0	0	9	0	0	0
χ_7	63	63	63	15	15	15	15	−1	−1	−1	−1	−1	−1	−1	−1	0	3	3	3
χ_8	70	70	70	−10	−10	−10	−10	−2	−2	−2	−2	−2	−2	−2	−2	7	1	1	1
χ_9	70	70	70	−10	−10	−10	−10	−2	−2	−2	−2	−2	−2	−2	−2	7	1	1	1
χ_{10}	90	90	90	10	10	10	10	6	6	6	6	6	6	6	6	9	0	0	0
χ_{11}	126	126	126	14	14	14	14	6	6	6	6	6	6	6	6	−9	0	0	0
χ_{12}	160	160	160	0	0	0	0	4	4	4	4	4	4	4	4	16	1	1	1
χ_{13}	175	175	175	15	15	15	15	−5	−5	−5	−5	−5	−5	−5	−5	−5	1	1	1
χ_{14}	189	189	189	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	0	0	0	0
χ_{15}	189	189	189	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	0	0	0	0
χ_{16}	224	224	224	0	0	0	0	−4	−4	−4	−4	−4	−4	−4	−4	8	−1	−1	−1
χ_{17}	224	224	224	0	0	0	0	−4	−4	−4	−4	−4	−4	−4	−4	8	−1	−1	−1
χ_{18}	225	225	225	−15	−15	−15	−15	5	5	5	5	5	5	5	5	0	3	3	3
χ_{19}	288	288	288	0	0	0	0	4	4	4	4	4	4	4	4	0	−3	−3	−3
χ_{20}	300	300	300	−20	−20	−20	−20	0	0	0	0	0	0	0	0	−15	0	0	0
χ_{21}	336	336	336	16	16	16	16	0	0	0	0	0	0	0	0	−6	0	0	0
χ_{22}	1 575	39	−25	135	7	−9	7	15	15	15	15	−1	−1	−1	−1	0	12	−4	0
χ_{23}	1 575	39	−25	−105	23	7	−9	15	15	15	15	−1	−1	−1	−1	0	12	−4	0
χ_{24}	3 150	78	−50	30	30	−2	−2	30	30	30	30	−2	−2	−2	−2	0	−12	4	0
χ_{25}	4 725	117	−75	−75	53	5	−11	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{26}	4 725	117	−75	165	37	−11	5	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{27}	4 725	117	−75	165	37	−11	5	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{28}	4 725	117	−75	165	37	−11	5	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{29}	4 725	117	−75	−75	53	5	−11	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{30}	4 725	117	−75	−75	53	5	−11	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{31}	9 450	234	−150	90	90	−6	−6	30	30	30	30	−2	−2	−2	−2	0	0	0	0
χ_{32}	18 900	468	−300	180	−76	−12	20	0	0	0	0	0	0	0	0	0	0	0	0
χ_{33}	18 900	468	−300	−300	−44	20	−12	0	0	0	0	0	0	0	0	0	0	0	0
χ_{34}	2 520	−40	24	120	−8	8	−8	48	−16	−16	−16	0	0	0	0	0	3	3	−1
χ_{35}	2 520	−40	24	120	−8	8	−8	−16	−16	−16	48	0	0	0	0	0	3	3	−1
χ_{36}	2 520	−40	24	120	−8	8	−8	−16	−16	48	−16	0	0	0	0	0	3	3	−1
χ_{37}	2 520	−40	24	120	−8	8	−8	−16	48	−16	−16	0	0	0	0	0	3	3	−1
χ_{38}	7 560	−120	72	−120	8	−8	8	−36	12	12	12	−4	12	−4	−4	0	0	0	0
χ_{39}	7 560	−120	72	−120	8	−8	8	−36	12	12	12	−4	12	−4	−4	0	0	0	0
χ_{40}	7 560	−120	72	−120	8	−8	8	12	12	12	−36	−4	−4	12	−4	0	0	0	0
χ_{41}	7 560	−120	72	−120	8	−8	8	12	12	12	−36	−4	−4	12	−4	0	0	0	0
χ_{42}	7 560	−120	72	−120	8	−8	8	12	12	−36	12	−4	−4	−4	12	0	0	0	0
χ_{43}	7 560	−120	72	−120	8	−8	8	12	12	−36	12	−4	−4	−4	12	0	0	0	0
χ_{44}	7 560	−120	72	−120	8	−8	8	12	−36	12	12	12	−4	−4	−4	0	0	0	0
χ_{45}	7 560	−120	72	−120	8	−8	8	12	−36	12	12	12	−4	−4	−4	0	0	0	0
χ_{46}	10 080	−160	96	0	0	0	0	12	−4	−4	−4	−4	12	−4	−4	0	3	3	−1
χ_{47}	10 080	−160	96	0	0	0	0	−4	−4	−4	12	−4	−4	12	−4	0	3	3	−1
χ_{48}	10 080	−160	96	0	0	0	0	−4	−4	12	−4	−4	−4	−4	12	0	3	3	−1
χ_{49}	10 080	−160	96	0	0	0	0	−4	12	−4	−4	12	−4	−4	−4	0	3	3	−1
χ_{50}	12 600	−200	120	120	−8	8	−8	60	−20	−20	−20	−4	12	−4	−4	0	−3	−3	1
χ_{51}	12 600	−200	120	120	−8	8	−8	−20	−20	−20	60	−4	−4	12	−4	0	−3	−3	1
χ_{52}	12 600	−200	120	120	−8	8	−8	−20	−20	60	−20	−4	−4	−4	12	0	−3	−3	1
χ_{53}	12 600	−200	120	120	−8	8	−8	−20	60	−20	−20	12	−4	−4	−4	0	−3	−3	1

Continued on next page

Table 5 (continued)

$[g_i]_{J_2}$	$4A$					$5A$		$5B$		$5C$	$5D$	$6A$	$6B$				$7A$	$8A$				$10A$				$10B$				$10C$	$10D$	$12A$	$15A$	$15B$
$[g_{ij}]_{\overline{G}}$	$4j$	$4k$	$4l$	$4m$	$4n$	$5a$	$10a$	$5b$	$10b$	$5c$	$5d$	$6c$	$6d$	$12a$	$12b$	$12c$	$7a$	$8a$	$8b$	$8c$	$8d$	$10c$	$20a$	$20b$	$20c$	$10d$	$20d$	$20e$	$20f$	$10e$	$10f$	$12d$	$15a$	$15b$
$\lvert C_{\overline{G}}(g_{ij})\rvert$	1536	512	512	512	256	4800	320	4800	320	50	50	24	48	48	48	48	7	32	32	32	32	80	80	80	80	80	80	80	80	10	10	12	15	15
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	2	2	2	2	2	A	A	A^*	A^*	G	G^*	1	−1	−1	−1	−1	0	0	0	0	0	$-F$	$-F$	$-F$	$-F$	$-F^*$	$-F^*$	$-F^*$	$-F^*$	F	F^*	−1	0	0
χ_3	2	2	2	2	2	A^*	A^*	A	A	G^*	G	1	−1	−1	−1	−1	0	0	0	0	0	$-F^*$	$-F^*$	$-F^*$	$-F^*$	$-F$	$-F$	$-F$	$-F$	F^*	F	−1	0	0
χ_4	1	1	1	1	1	B	B	B^*	B^*	H	H^*	−1	0	0	0	0	0	−1	−1	−1	−1	$-F^*$	$-F^*$	$-F^*$	$-F^*$	$-F$	$-F$	$-F$	$-F$	0	0	1	F	F^*
χ_5	1	1	1	1	1	B^*	B^*	B	B	H^*	H	−1	0	0	0	0	0	−1	−1	−1	−1	$-F$	$-F$	$-F$	$-F$	$-F^*$	$-F^*$	$-F^*$	$-F^*$	0	0	1	F^*	F
χ_6	4	4	4	4	4	−4	−4	−4	−4	1	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1	1	−1	−1
χ_7	3	3	3	3	3	3	3	3	3	−2	−2	0	−1	−1	−1	−1	0	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	0	0	0	0	0
χ_8	2	2	2	2	2	C	C	C^*	C^*	0	0	−1	1	1	1	1	0	0	0	0	0	F	F	F	F	F^*	F^*	F^*	F^*	0	0	−1	$-F$	$-F^*$
χ_9	2	2	2	2	2	C^*	C^*	C	C	0	0	−1	1	1	1	1	0	0	0	0	0	F^*	F^*	F^*	F^*	F	F	F	F	0	0	−1	$-F^*$	$-F$
χ_{10}	−2	−2	−2	−2	−2	5	5	5	5	0	0	1	0	0	0	0	−1	0	0	0	0	1	1	1	1	1	1	1	1	0	0	1	−1	−1
χ_{11}	2	2	2	2	2	1	1	1	1	1	1	−1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	−1	−1	−1	1	1
χ_{12}	0	0	0	0	0	−5	−5	−5	−5	0	0	0	1	1	1	1	−1	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	0	0	0	1	1
χ_{13}	−1	−1	−1	−1	−1	0	0	0	0	0	0	3	1	1	1	1	0	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0	−1	0	0
χ_{14}	−3	−3	−3	−3	−3	A	A	A^*	A^*	G	G^*	0	0	0	0	0	0	1	1	1	1	$-F$	$-F$	$-F$	$-F$	$-F^*$	$-F^*$	$-F^*$	$-F^*$	$-F$	$-F^*$	0	0	0
χ_{15}	−3	−3	−3	−3	−3	A^*	A^*	A	A	G^*	G	0	0	0	0	0	0	1	1	1	1	$-F^*$	$-F^*$	$-F^*$	$-F^*$	$-F$	$-F$	$-F$	$-F$	$-F^*$	$-F$	0	0	0
χ_{16}	0	0	0	0	0	D	D	D^*	D^*	$-H^*$	$-H$	0	−1	−1	−1	−1	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	F^*	F
χ_{17}	0	0	0	0	0	D^*	D^*	D	D	$-H$	$-H^*$	0	−1	−1	−1	−1	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	F	F^*
χ_{18}	−3	−3	−3	−3	−3	0	0	0	0	0	0	0	−1	−1	−1	−1	1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{19}	0	0	0	0	0	3	3	3	3	−2	−2	0	1	1	1	1	1	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	0	0	0	0	0
χ_{20}	4	4	4	4	4	0	0	0	0	0	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
χ_{21}	0	0	0	0	0	−4	−4	−4	−4	1	1	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	−1	−1
χ_{22}	15	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	−1	3	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{23}	3	3	3	3	−5	0	0	0	0	0	0	0	0	0	0	0	0	1	−3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{24}	18	2	2	2	−6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{25}	−3	−3	−3	13	−3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	−3	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{26}	9	−7	−7	9	1	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1	−1	3	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{27}	9	−7	9	−7	1	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1	3	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{28}	9	9	−7	−7	1	0	0	0	0	0	0	0	0	0	0	0	0	3	−1	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{29}	−3	−3	13	−3	−3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	−3	1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{30}	−3	13	−3	−3	−3	0	0	0	0	0	0	0	0	0	0	0	0	−3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{31}	−18	−2	−2	−2	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{32}	−12	4	4	4	−4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{33}	12	−4	−4	−4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{34}	0	0	0	0	0	15	−1	15	−1	0	0	0	3	−1	−1	−1	0	0	0	0	0	3	−1	−1	−1	3	−1	−1	−1	0	0	0	0	0
χ_{35}	0	0	0	0	0	15	−1	15	−1	0	0	0	−1	3	−1	−1	0	0	0	0	0	−1	3	−1	−1	−1	3	−1	−1	0	0	0	0	0
χ_{36}	0	0	0	0	0	15	−1	15	−1	0	0	0	−1	−1	3	−1	0	0	0	0	0	−1	−1	−1	3	−1	−1	3	−1	0	0	0	0	0
χ_{37}	0	0	0	0	0	15	−1	15	−1	0	0	0	−1	−1	−1	3	0	0	0	0	0	−1	−1	3	−1	−1	−1	−1	3	0	0	0	0	0
χ_{38}	0	0	0	0	0	E	$-F$	E^*	$-F^*$	0	0	0	0	0	0	0	0	0	0	0	0	A	$-F$	$-F$	$-F$	A^*	$-F^*$	$-F^*$	$-F^*$	0	0	0	0	0
χ_{39}	0	0	0	0	0	E^*	$-F^*$	E	$-F$	0	0	0	0	0	0	0	0	0	0	0	0	A^*	$-F^*$	$-F^*$	$-F^*$	A	$-F$	$-F$	$-F$	0	0	0	0	0
χ_{40}	0	0	0	0	0	E	$-F$	E^*	$-F^*$	0	0	0	0	0	0	0	0	0	0	0	0	$-F$	A	$-F$	$-F$	$-F^*$	A^*	$-F^*$	$-F^*$	0	0	0	0	0
χ_{41}	0	0	0	0	0	E^*	$-F^*$	E	$-F$	0	0	0	0	0	0	0	0	0	0	0	0	$-F^*$	A^*	$-F^*$	$-F^*$	$-F$	A	$-F$	$-F$	0	0	0	0	0
χ_{42}	0	0	0	0	0	E	$-F$	E^*	$-F^*$	0	0	0	0	0	0	0	0	0	0	0	0	$-F$	$-F$	$-F$	A	$-F^*$	$-F^*$	A^*	$-F^*$	0	0	0	0	0
χ_{43}	0	0	0	0	0	E^*	$-F^*$	E	$-F$	0	0	0	0	0	0	0	0	0	0	0	0	$-F^*$	$-F^*$	$-F^*$	A^*	$-F$	$-F$	A	$-F$	0	0	0	0	0
χ_{44}	0	0	0	0	0	E	$-F$	E^*	$-F^*$	0	0	0	0	0	0	0	0	0	0	0	0	$-F$	$-F$	A	$-F$	$-F^*$	$-F^*$	$-F^*$	A^*	0	0	0	0	0
χ_{45}	0	0	0	0	0	E^*	$-F^*$	E	$-F$	0	0	0	0	0	0	0	0	0	0	0	0	$-F^*$	$-F^*$	A^*	$-F^*$	$-F$	$-F$	$-F$	A	0	0	0	0	0
χ_{46}	0	0	0	0	0	−15	1	−15	1	0	0	0	3	−1	−1	−1	0	0	0	0	0	−3	1	1	1	−3	1	1	1	0	0	0	0	0
χ_{47}	0	0	0	0	0	−15	1	−15	1	0	0	0	−1	3	−1	−1	0	0	0	0	0	1	−3	1	1	1	−3	1	1	0	0	0	0	0
χ_{48}	0	0	0	0	0	−15	1	−15	1	0	0	0	−1	−1	3	−1	0	0	0	0	0	1	1	1	−3	1	1	−3	1	0	0	0	0	0
χ_{49}	0	0	0	0	0	−15	1	−15	1	0	0	0	−1	−1	−1	3	0	0	0	0	0	1	1	−3	1	1	1	1	−3	0	0	0	0	0
χ_{50}	0	0	0	0	0	0	0	0	0	0	0	0	−3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{51}	0	0	0	0	0	0	0	0	0	0	0	0	1	−3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{52}	0	0	0	0	0	0	0	0	0	0	0	0	1	1	−3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{53}	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	−3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

where in Table 5, $A = 3(1-\sqrt{5})/2$, $B = (7-\sqrt{5})/2$, $C = (1-\sqrt{5})/2$, $D = -1+2\sqrt{5}$, $E = 15(1-\sqrt{5})/2$, $F = 5(1-\sqrt{5})/2$, $G = (3-\sqrt{5})/2$, $H = 1-\sqrt{5}$ and if $K = a + b\sqrt{m}$ for some $a, b \in \mathbb{R}$ and $m \in \mathbb{N}$, then K^* denotes the number $a - b\sqrt{m}$.