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ON THE MODULAR SEQUENCE SPACES GENERATED BY THE CESÀRO MEAN

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Abstract: In this paper, the seminormed Cesàro difference sequence space $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ is defined by using the generalized Orlicz function. Some algebraic and topological properties of the space $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ are investigated. Various inclusion relations for this sequence space are also studied.

Keywords: Difference sequences, Orlicz function, Modular sequence, AK-space and BK-space.

1. Preliminaries and introduction

The notation $\omega(\mathcal{X})$ represents the spaces of all $\mathcal{X}$-valued sequence spaces, and $(\mathcal{X}, g)$ is a seminormed space. By ℓ_∞, c, and c_0, we indicate the spaces of all bounded, convergent, and null convergent sequences, respectively. Also, we denote the set of natural numbers including zero by $\mathbb{N}$ and the zero sequence by θ.

In [9], Kızmaz introduced the notion of difference sequence spaces $\lambda(\Delta)$, where λ denotes any one of the classical sequence spaces ℓ_∞, c, and c_0. Çolak and Et [5] further generalized the notion of difference sequence space $\lambda(\Delta^m)$ for $\lambda \in \{\ell_\infty, c, c_0\}$. Following [14], for $t, s \in \mathbb{N}$ and $\lambda = \ell_\infty, c, c_0$, we have

$$\lambda(\Delta^t_{(s)}) = \big\{x \in \omega : (\Delta^t_{(s)} x_i) \in \lambda\big\},$$

where

$$\Delta^t_{(s)} x_i = \Delta^{t-1}_{(s)} x_i - \Delta^{t-1}_{(s)} x_{i+s}, \quad \Delta^0_{(s)} x_i = x_i \quad \forall i \in \mathbb{N},$$

which has the following binomial expression:

$$\Delta^t_{(s)} x_i = \sum_{k=0}^{t} (-1)^k \binom{t}{k} x_{i+sk}.$$

For $s = t = 1$, we obtain the spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$.

A linear metric space $\mathcal{X}$ over $\mathbb{C}$ is said to be a *paranormed space* if there is a subadditive function $q : \mathcal{X} \to \mathbb{C}$ such that $q(0) = 0$, $q(x) = q(-x)$, and scalar multiplication is continuous; that is, $|\alpha_n - \alpha| \to 0$ and $q(x_i - x) \to 0$ imply $q(\alpha_i x_i - \alpha x) \to 0$ as $i \to \infty$ $\forall \alpha \in \mathbb{C}$ and $x \in \mathcal{X}$.

A paranorm q is called *total* if $q(x) = 0$ implies $x = 0$. The pair $(\mathcal{X}, q)$ is called a total paranormed space.

A convex function $M : \mathbb{R} \to \mathbb{R}$ such that $M(0) = 0$ and $M(x) > 0$ for all $x > 0$ is called an *Orlicz function*. Let X_M be the set of all sequences (x_n) such that $\sum_n M(|x_n|/p) < \infty$ for some $p > 0$; X_M is a Banach space with the norm

$$\|x_n\|_M = \inf\Big\{p > 0 : \sum_{n=1}^{\infty} M\Big(\frac{|x_n|}{p}\Big) \leq 1\Big\},$$

and $(X_M, \|\cdot\|)$ is called an *Orlicz sequence space*. An Orlicz function $\mathcal{F} : [0,\infty) \to [0,\infty)$ is called a *modulus function* if

$$\mathcal{F}(x+y) \leq \mathcal{F}(x) + \mathcal{F}(y) \quad \forall x, y \in [0,\infty).$$

An Orlicz function $\mathcal{F}$ is said to satisfy Δ_2-condition for all values of $u \geq 0$ if there exists $K > 0$ such that

$$\mathcal{F}(2u) \leq K\mathcal{F}(u).$$

This is equivalent to satisfying the inequality

$$\mathcal{F}(ru) \leq Kr\mathcal{F}(u)$$

for $r > 1$ and all values of $u \geq 0$. The Δ_2-condition implies

$$\mathcal{F}(ru) \leq Kr^{\log_2 K}\mathcal{F}(u)$$

for all values of $u \geq 0$ and for $r > 1$.

Two Orlicz functions M and N are said to be *equivalent* if there exist $\alpha, \beta > 0$, $0 < K \leq L$, and $a > 0$ such that $KM(\alpha x) \leq N(x) \leq LM(\beta x)$ for each $x \in [0, a]$. A BK-space is a Banach space of complex sequences with continuous coordinate maps. A sequence $x = (x_i) \in \nu$ is called sectionally convergent if

$$x^{[n]} = \sum_{i=1}^{n} x_i e_i \to x$$

as $n \to \infty$, where $e_i = (\delta_{ij})$ is the Kronecker symbol, that is, $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. A space ν is called an AK-space if and only if each element is sectionally convergent.

Orlicz [13] studied the Orlicz functions and introduced the sequence space $\ell_{\mathcal{F}}$. Orlicz spaces have many applications in various fields including the theory of nonlinear integral equations. Also, Orlicz sequence spaces generalize ℓ_p-spaces, and ℓ_p-spaces are enveloped in Orlicz spaces. Many researchers have studied different sequence spaces using the Orlicz functions. For a more detailed study of the Orlicz functions, one can refer to [2–4, 6–8, 11, 15–17].

For a sequence $(\mathcal{F}_i)$ of Orlicz functions, the vector space $\ell(\mathcal{F}_i)$ defined by

$$\ell(\mathcal{F}_i) = \Big\{x = (x_i) \in w : \sum_{i=1}^{\infty} \mathcal{F}_i\Big(\frac{|x_i|}{\rho}\Big) < \infty \text{ for some } \rho > 0\Big\}$$

is a Banach space with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{i=1}^{\infty} \mathcal{F}_i\Big(\frac{|x_i|}{\rho}\Big) \leq 1\Big\}$$

and is called a *modular sequence space*. Furthermore, the space $\ell(\mathcal{F}_i)$ generalizes the notion of modular sequence space studied by Nakano [12] who introduced the space $\ell(\mathcal{F}_i)$ for $\mathcal{F}_i(x) = x^{\alpha_i}$, where $1 \leq \alpha_i < \infty$ for $i \in \{1, 2, \ldots\}$. In [10], Lindenstrauss and Tzafriri proved that every Orlicz sequence space contains a subspace isomorphic to ℓ_p for some $1 \leq p < \infty$. They also proved that every subspace of a separable Orlicz sequence space is isomorphic to ℓ_p for some $1 \le p < \infty$. Woo [18] extended these findings to the separable modular sequence spaces.

In this paper, we define and study the seminormed Cesàro difference sequence space $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ using the concept of the generalized Orlicz function. Throughout the paper, we use a well-known inequality which is explained as follows [1]:
let (q_j) be a sequence of positive real numbers with

$$0 \le q_j \le \sup_j q_j = H, \quad K = \max(1, 2^{H-1}),$$

then

$$|a_i + b_i|^{q_j} \le K\,|a_i|^{q_j} + K\,|b_i|^{q_j}$$

for any two complex numbers a_i and b_i, for each $i \in \mathbb{N}$.

2. Seminormed difference sequence space and Orlicz functions

Let $\mathcal{F} = (\mathcal{F}_j)$ be a sequence of Orlicz functions, let $(\mathcal{X}, g)$ be a seminormed space, and let (q_j) be a strictly bounded sequence of positive real numbers. Let $\mathcal{C}$ be the Cesàro matrix of order 1. Then, for a nonnegative real number r and a sequence of positive real numbers $\mu = (\mu_i)$, we define a difference sequence space $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ as follows:

$$\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}) = \left\{ x \in \omega(\mathcal{X}) : \sum_{j=0}^{\infty} j^{-r} \left[\mathcal{F}_j \left(g \left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)} \right) \right) \right]^{q_j} < \infty,\ \rho > 0 \right\}.$$

Theorem 1. *The sequence space $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ is a linear space over the field of complex numbers $\mathbb{C}$.*

P r o o f. Let $x = (x_i)$ and $y = (y_i)$ belong to $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$. Let a and b be two nonzero complex numbers. To establish the result, we need to find some $\rho_3 > 0$ such that

$$\sum_{j=0}^{\infty} j^{-r} \left[\mathcal{F}_j \left(g \left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} (a x_i + b y_i)}{\rho_3(j+1)} \right) \right) \right]^{q_j} < \infty.$$

For $x, y \in \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$, there exist $\rho_1, \rho_2 > 0$ such that

$$\sum_{j=0}^{\infty} j^{-r} \left[\mathcal{F}_j \left(g \left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho_1(j+1)} \right) \right) \right]^{q_j} < \infty \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{1}{j^r} \left[\mathcal{F}_j \left(g \left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} y_i}{\rho_2(j+1)} \right) \right) \right]^{q_j} < \infty.$$

Consider

$$\frac{1}{\rho_3} = \min \left\{ \frac{1}{|a|\,\rho_1},\ \frac{1}{|b|\,\rho_2} \right\}.$$

Since $\mathcal{F} = (\mathcal{F}_j)$ is nondecreasing, g is a seminorm, and $\Delta^t_{(s)}$ is linear, we obtain

$$\sum_{j=0}^{\infty} j^{-r} \left[\mathcal{F}_j \circ g \left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} (a x_i + b y_i)}{\rho_3(j+1)} \right) \right]^{q_j}$$
$$\le \sum_{j=0}^{\infty} j^{-r} \left[\mathcal{F}_j \left(g \left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} a x_i}{\rho_3(j+1)} \right) + g \left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} b y_i}{\rho_3(j+1)} \right) \right) \right]^{q_j}$$

$$\leq \sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho_1(j+1)}\right)+g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t y_i}{\rho_2(j+1)}\right)\right)\right]^{q_j}$$

$$\leq K\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho_1(j+1)}\right)\right)\right]^{q_j}+K\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t y_i}{\rho_2(j+1)}\right)\right)\right]^{q_j}<\infty.$$

Hence, $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta_{(s)}^t, \mathcal{C})$ is a linear space. □

Theorem 2. *The space $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta_{(s)}^t, \mathcal{C})$ is a paranormed space (not necessarily total paranormed) with the paranorm $\mathfrak{H}$ given by*

$$\mathfrak{H}_\Delta(x)=\inf\left\{\rho^{q_t/G}:\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho(j+1)}\right)\right)\right]\leq 1,\ \rho>0,\ t\in\mathbb{N}\right\},$$

where $G=\max\left\{1, H=\sup_{j\in\mathbb{N}} q_j\right\}$.

P r o o f. Trivially, $\mathfrak{H}_\Delta(x)=\mathfrak{H}_\Delta(-x)$. Since $\mathcal{F}_j(\theta)=0$ for all $j\in\mathbb{N}$, we obtain $\inf\{\rho^{q_n/G}\}=0$ for $x=\theta$.

Let $x=(x_i)$ and $y=(y_i)$ be two arbitrary sequences in $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta_{(s)}^t, \mathcal{C})$. Then, for some $\rho_1, \rho_2>0$, we have

$$\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho_1(j+1)}\right)\right)\right]\leq 1 \quad\text{and}\quad \sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t y_i}{\rho_2(j+1)}\right)\right)\right]\leq 1.$$

For $\rho=\rho_1+\rho_2$, we obtain

$$\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t (x_i+y_i)}{\rho(j+1)}\right)\right)\right]$$

$$\leq\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho_1(j+1)}\right)\right)\right]+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum\limits_{i=0}^{j}\mu_i\Delta_{(s)}^t y_i}{\rho_2(j+1)}\right)\right)\right]<1.$$

Thus,

$$\mathfrak{H}_\Delta(x+y)=\inf\left\{(\rho_1+\rho_2)^{q_t/G}:\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t (x_i+y_i)}{(\rho_1+\rho_2)(j+1)}\right)\right)\right]\leq 1,\ \rho_1>0,\ \rho_2>0\right\}$$

$$\leq\inf\left\{(\rho_1)^{q_t/G}:\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho_1(j+1)}\right)\right)\right]\leq 1, \rho_1>0, t\in\mathbb{N}\right\}$$

$$+\inf\left\{(\rho_2)^{q_t/G}:\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t y_i}{\rho_2(j+1)}\right)\right)\right]\leq 1, \rho_2>0, t\in\mathbb{N}\right\}\leq\mathfrak{H}_\Delta(x)+\mathfrak{H}_\Delta(y).$$

Finally, for any scalar $\gamma\neq 0$ and $r=\rho/|\gamma|$, we have

$$\mathfrak{H}_\Delta(\gamma x)=\inf\left\{\rho^{q_t/G}:\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t (\gamma x_i)}{\rho(j+1)}\right)\right)\right]\leq 1,\ \rho>0,\ t\in\mathbb{N}\right\}$$

$$=\inf\left\{(|\gamma|r)^{q_t/G}:\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{r(j+1)}\right)\right)\right]\leq 1,\ r>0,\ t\in\mathbb{N}\right\}.$$

Hence, $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ is a paranormed sequence space. □

Theorem 3. *Let $\mathcal{F} = (\mathcal{F}_j)$ and $\mathcal{T} = (\mathcal{T}_j)$ be two sequences of Orlicz functions. Then*

$$\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}) \cap \ell(\mathcal{T}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}) \subset \ell(\mathcal{F}_j + \mathcal{T}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}).$$

P r o o f. Let

$$x \in \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}) \cap \ell(\mathcal{T}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}).$$

Then there exist $\rho_1, \rho_2 > 0$ such that

$$\sum_{j=0}^{\infty} j^{-r} \Big[\mathcal{F}_j\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho_1(j+1)}\Big)\Big)\Big]^{q_j} < \infty \quad \text{and} \quad \sum_{j=0}^{\infty} j^{-r} \Big[\mathcal{T}_j\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho_2(j+1)}\Big)\Big)\Big]^{q_j} < \infty.$$

Taking $1/\rho = \min\{1/\rho_1, 1/\rho_2\}$, we obtain

$$\sum_{j=0}^{\infty} j^{-r} \Big[(\mathcal{F}_j + \mathcal{T}_j)\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\Big)\Big)\Big]^{q_j}$$

$$\leq K\Big[\sum_{j=0}^{\infty} j^{-r} \Big[\mathcal{F}_j\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho_1(j+1)}\Big)\Big)\Big]^{q_j}\Big] + K\Big[\sum_{j=0}^{\infty} j^{-r} \Big[\mathcal{T}_j\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho_2(j+1)}\Big)\Big)\Big]^{q_j}\Big] < \infty.$$

Therefore,

$$\sum_{j=0}^{\infty} j^{-r} \Big[(\mathcal{F}_j + \mathcal{T}_j)\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\Big)\Big)\Big]^{q_j} < \infty.$$

Hence, $x \in \ell(\mathcal{F}_j + \mathcal{T}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$. □

Theorem 4. *For $t \geq 1$, the inclusion $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^{t-1}_{(s)}, \mathcal{C}) \subset \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ is strict.*

P r o o f. Let $x \in \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^{t-1}_{(s)}, \mathcal{C})$. Then, we have

$$\sum_{j=0}^{\infty} j^{-r} \Big[\mathcal{F}_j\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i \Delta^{t-1}_{(s)} x_i}{\rho(j+1)}\Big)\Big)\Big]^{q_j} < \infty \quad \text{for some} \quad \rho > 0.$$

Since $\mathcal{F} = (\mathcal{F}_j)$ is nondecreasing and g is a seminorm, we obtain

$$\sum_{j=0}^{\infty} j^{-r} \Big[\mathcal{F}_j\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\Big)\Big)\Big]^{q_j} \leq \sum_{j=0}^{\infty} j^{-r} \Big[\mathcal{F}_j\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i (\Delta^{t-1}_{(s)} x_i - \Delta^{t-1}_{(s)} x_{i+1})}{\rho(j+1)}\Big)\Big)\Big]^{q_j}$$

$$\leq K\Big[\sum_{j=0}^{\infty} j^{-r} \Big[\mathcal{F}_j\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i \Delta^{t-1}_{(s)} x_i}{\rho(j+1)}\Big)\Big)\Big]^{q_j}\Big] + K\Big[\sum_{j=0}^{\infty} j^{-r} \Big[\mathcal{F}_j\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i \Delta^{t-1}_{(s)} x_{i+1}}{\rho(j+1)}\Big)\Big)\Big]^{q_j}\Big] < \infty.$$

Therefore,

$$\sum_{j=0}^{\infty} j^{-r} \Big[\mathcal{F}_j\Big(g\Big(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\Big)\Big)\Big]^{q_j} < \infty.$$

Hence, $x \in \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$. $\square$

In general, $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^i_{(s)}, \mathcal{C}) \subset \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ for $i = 1, 2, \ldots, t-1$, and the inclusion is strict.

Theorem 5. *Let (q_j) be a sequence of positive real numbers. Then*

(a) $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}) \subset \ell(\mathcal{F}_j, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ *for* $0 < \inf_j q_j \le q_j \le 1$.
(b) $\ell(\mathcal{F}_j, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}) \subset \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ *for* $1 \le q_j \le \sup_j q_j < \infty$.

P r o o f. (a) Let $x \in \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$. Then

$$\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\right)\right)\right]^{q_j} < \infty \quad \text{for some} \quad \rho > 0.$$

Since $0 < \inf_j q_j \le q_j \le 1$, we have

$$\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\right)\right)\right] \le \sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\right)\right)\right]^{q_j} < \infty.$$

This implies that $x \in \ell(\mathcal{F}_j, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$. Hence, $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}) \subset \ell(\mathcal{F}_j, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$.

(b) Let $q_j \ge 1$ for all j, $\sup_j q_j < \infty$, and $x \in \ell(\mathcal{F}_j, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$.
Then

$$\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\right)\right)\right] < \infty \quad \text{for some} \quad \rho > 0. \tag{2.1}$$

Since $1 \le q_j \le \sup_j q_j < \infty$, we have

$$\sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\right)\right)\right]^{q_j} \le \sum_{j=0}^{\infty} j^{-r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\right)\right)\right] < \infty.$$

Thus, $x \in \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$. Hence, $\ell(\mathcal{F}_j, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}) \subset \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$. $\square$

Theorem 6. *Let $(\mathcal{F}_j)$ and $(\mathcal{T}_j)$ be two sequences of Orlicz functions satisfying the Δ_2-condition, and let $r > 1$. Then $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}) \subset \ell(\mathcal{T}_j \circ \mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$.*

P r o o f. Let $x \in \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ and $\varepsilon > 0$. Choose $0 < \delta < 1$ such that $\mathcal{F}_j(v) < \varepsilon$ for $0 \le v \le \delta$. Write

$$y_j = \mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\right)\right) \quad \text{for each} \quad j \in \mathbb{N}.$$

Consider the equality

$$\sup_j \sum_{j=0}^{\infty} j^{-r}\left[\mathcal{T}_j(y_j)\right]^{q_j} = \sup_j \sum_{1} j^{-r}\left[\mathcal{T}_j(y_j)\right]^{q_j} + \sup_j \sum_{2} j^{-r}\left[\mathcal{T}_j(y_j)\right]^{q_j},$$

where $y_j \leq \delta$ for the first summation and $y_j > \delta$ for the second summation. Thus, for $r > 1$, we have

$$\sup_j \sum_1 j^{-r} \left[\mathcal{T}_j(y_j)\right]^{q_j} < \max(1, \varepsilon^H) \sum j^{-r} < \infty.$$

For $y_j > \delta$, we get $y_j < y_j/\delta \leq 1 + y_j/\delta$.

Since each $\mathcal{T}_j$ is nondecreasing, convex, and satisfies the Δ_2-condition, it follows that

$$\mathcal{T}_j(y_j) < \mathcal{T}\left(1 + \frac{y_j}{\delta}\right) < \frac{1}{2}\mathcal{T}_j(2) + \frac{1}{2}\mathcal{T}_j\left(\frac{2y_j}{\delta}\right)$$
$$< \frac{1}{2}K\frac{y_j}{\delta}\mathcal{T}_j(2) + \frac{1}{2}K\frac{y_j}{\delta}\mathcal{T}_j(2) < Ky_j\delta^{-1}\mathcal{T}_j(2) \quad \text{for each} \quad j \in \mathbb{N}.$$

Therefore,

$$\sup_j \sum_2 j^{-r} \left[\mathcal{T}_j(y_j)\right]^{q_j} < \max(1, (K\delta^{-1}\mathcal{F}(2))^H) \sum_2 j^{-r} (y_j)^{q_j} < \infty.$$

Thus, (2.1) yields

$$\sup_j \sum_{j=0}^{\infty} j^{-r} \left[\mathcal{T}_j(y_j)\right]^{q_j} \leq \max(1, \varepsilon^j) \sum_{j=1}^{\infty} j^{-r} + \max(1, (K\delta^{-1}\mathcal{F}(2))^H) \sum_{j=2}^{\infty} j^{-r} (y_j)^{q_j} < \infty.$$

Hence, $x \in \ell(\mathcal{T}_j \circ \mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$. □

Corollary 1. *Let $(\mathcal{F}_j)$ be any sequence of Orlicz functions satisfying the Δ_2-condition, and let $r > 1$. If $\mathcal{F}_j(x) = x$ for all $x \in [0, \infty)$ and for all $\in \mathbb{N}$, then $\ell(q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}) \subseteq \ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$.*

Corollary 2. *If $\mathcal{F}_j$ and $\mathcal{T}_j$ are Orlicz functions that are equivalent for each $j \in \mathbb{N}$, then $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C}) = \ell(\mathcal{T}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$.*

For $r = 0$, the space $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ reduces to a sequence space as follows:

$$\ell(\mathcal{F}_j, q, g, \mu, \Delta^t_{(s)}, \mathcal{C}) = \left\{x \in \omega(\mathcal{X}) : \sum_{j=0}^{\infty} \left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\right)\right)\right]^{q_j} < \infty \text{ for some } \rho > 0\right\}.$$

Theorem 7. *Let $(\mathcal{F}_j)$ be a sequence of Orlicz functions, let $q_j \in \ell_\infty$, and let $(\mathcal{X}, g)$ be a complete seminormed space. Then $\ell(\mathcal{F}_j, q, g, \mu, \Delta^t_{(s)}, \mathcal{C})$ is a complete paranormed endowed with the paranorm $\mathfrak{H}_\Delta$ defined by*

$$\mathfrak{H}_\Delta(x) = \inf\left\{\rho^{q_t/K} : \sum_{j=0}^{\infty} \left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j} \mu_i \Delta^t_{(s)} x_i}{\rho(j+1)}\right)\right)\right] \leq 1,\ \rho > 0,\ t \in \mathbb{N}\right\},$$

where $K = \max\{1, H = \sup_{j \in \mathbb{N}} q_j\}$.

P r o o f. Let (x_i) be a Cauchy sequence in $\ell(\mathcal{F}_j, q, g, \mu, \Delta^t_{(s)}, \mathcal{C})$. Let $\delta > 0$ be fixed, and let $s > 0$ be such that, for given $0 < \varepsilon < 1$, $\varepsilon/s\delta > 0$ and $s\delta \geq 1$. Then, there exists a positive integer n_0 such that

$$h(x^m - x^n) < \frac{\varepsilon}{s\delta} \quad \forall\, m, n \geq n_0.$$

Thus,

$$\inf\left\{\rho^{q_t/K} : \sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i(\Delta^t_{(s)}x_i^m - \Delta^t_{(s)}x_i^n)}{\rho(j+1)}\right)\right)\right] \leq 1, t \in \mathbb{N}\right\} < \frac{\varepsilon}{s\delta} \quad \forall\, m, n \geq n_0.$$

It implies that

$$\sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i(\Delta^t_{(s)}x_i^m - \Delta^t_{(s)}x_i^n)}{h(x^m - x^n)(j+1)}\right)\right)\right] \leq 1 \quad \forall\, m, n \geq n_0.$$

Therefore,

$$\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i(\Delta^t_{(s)}x_i^m - \Delta^t_{(s)}x_i^t)}{h(x^m - x^n)(j+1)}\right)\right) \leq 1 \quad \forall\, m, n \geq n_0 \quad \text{and} \quad j \in \mathbb{N}.$$

For $s > 0$ with $\mathcal{F}_j(s\delta/2) \geq 1$, we obtain

$$\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i(\Delta^t_{(s)}x_i^m - \Delta^t_{(s)}x_i^n)}{h(x^m - x^n)(j+1)}\right)\right) \leq \mathcal{F}_j\left(\frac{s\delta}{2}\right) \quad \forall\, m, n \geq n_0 \quad \text{and} \quad j \in \mathbb{N}.$$

Since $\mathcal{F}_j$ is nondecreasing for each $j \in \mathbb{N}$, we have

$$g\left(\frac{\sum_{i=0}^{j}\mu_i(\Delta^t_{(s)}x_i^m - \Delta^t_{(s)}x_i^n)}{j+1}\right) \leq \frac{s\delta}{2} \times \frac{\varepsilon}{s\delta} = \frac{\varepsilon}{2}.$$

Hence, $(\Delta^t_{(s)}x_i^m)$ is a Cauchy sequence in $(\mathcal{X}, g)$ for each $i \in \mathbb{N}$. However, $(\mathcal{X}, g)$ is complete and so $(\Delta^t_{(s)}x_i^m)$ is convergent in $(\mathcal{X}, g)$ for all $i \in \mathbb{N}$.

Let $\lim\limits_{m\to\infty}\mu_i\Delta^t_{(s)}x_i^m = x_i$ exists for each $i \geq 1$. For $i = 1$, we obtain

$$\lim_{m\to\infty}\mu_1\Delta^t_{(s)}x_1^m = \lim_{m\to\infty}\mu_1\sum_{k=0}^{t}(-1)^k\binom{t}{k}x_{1+sk} = \lim_{m\to\infty}\mu_1 x_1^m = x_1. \tag{2.2}$$

Similarly,

$$\lim_{m\to\infty}\mu_i\Delta^t_{(s)}x_i^m = \lim_{m\to\infty}\mu_i x_i^m = x_i \quad \text{for} \quad i = 1, \ldots, ts. \tag{2.3}$$

From (2.2) and (2.3), it follows that $\lim\limits_{m\to\infty}\mu_i x^m_{1+ts}$ exists.

Let $\lim\limits_{m\to\infty}\mu_i x^m_{1+ts} = \mu_i\, x_{1+ts}$. Then, by induction, $\lim\limits_{m\to\infty}\mu_i x_i^m = x_i$ for all $i \in \mathbb{N}$.

Now, for each $m, n \geq n_0$, we have

$$\inf\left\{\rho^{q_t/K} : \sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i(\Delta^t_{(s)}x_i^m - \Delta^t_{(s)}x_i^n)}{\rho(j+1)}\right)\right)\right] \leq 1,\ t \in \mathbb{N}\right\} < \varepsilon.$$

Thus,

$$\lim_{n\to\infty}\left\{\inf\left\{\rho^{q_t/K}:\sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i(\Delta_{(s)}^t x_i^m-\Delta_{(s)}^t x_i^n)}{\rho(j+1)}\right)\right)\right]\le 1,\ t\in\mathbb{N}\right\}\right\}<\varepsilon \quad \forall m,n\ge n_0.$$

Using the continuity of Orlicz functions, we obtain

$$\inf\left\{\rho^{q_t/K}:\sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i(\Delta_{(s)}^t x_i^m-\Delta_{(s)}^t \lim_{n\to\infty} x_i^n)}{\rho(j+1)}\right)\right)\right]\le 1,\ t\in\mathbb{N}\right\}<\varepsilon \quad \forall m\ge n_0.$$

This implies that

$$\inf\left\{\rho^{q_t/K}:\sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i(\Delta_{(s)}^t x_i^m-\Delta_{(s)}^t x_i)}{\rho(j+1)}\right)\right)\right]\le 1,\ t\in\mathbb{N}\right\}<\varepsilon \quad \forall n\ge n_0.$$

Hence, $(x^m-x)\in\ell(\mathcal{F}_j,q,g,\mu,\Delta_{(s)}^t,\mathcal{C})$, and then $x=x^m-(x^m-x)\in\ell(\mathcal{F}_j,q,g,\mu,\Delta_{(s)}^t,\mathcal{C})$. □

For $r=0,\ q_j=q$, a constant, the space $\ell(\mathcal{F}_j,q,g,r,\mu,\Delta_{(s)}^t,\mathcal{C})$ reduces to a sequence space as follows:

$$\ell(\mathcal{F}_j,g,\mu,\Delta_{(s)}^t,\mathcal{C})=\left\{x\in\omega(\mathcal{X}):\sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho(j+1)}\right)\right)\right]<\infty \text{ for some } \rho>0\right\}.$$

Theorem 8. *Let $(\mathcal{X},g)$ be a complete normed space. Then, $\ell(\mathcal{F}_j,g,\mu,\Delta_{(s)}^t,\mathcal{C})$ is a Banach space with a norm $\|\cdot\|$ defined by*

$$\|x\|=\inf\left\{\rho:\sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho(j+1)}\right)\right)\right]\le 1\right\}. \tag{2.4}$$

P r o o f. To prove that $\|\cdot\|$ is a norm in $\ell(\mathcal{F}_j,g,\mu,\Delta_{(s)}^t,\mathcal{C})$, we can verify the completeness of $\ell(\mathcal{F}_j,g,\mu,\Delta_{(s)}^t,\mathcal{C})$ as in the proof of Theorem 7.

If $x=\theta$, then clearly $\|x\|=0$.

Conversely, suppose that $\|x\|=0$. Then,

$$\inf\left\{\rho:\sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho(j+1)}\right)\right)\right]\le 1\right\}=0.$$

Thus, for given $\varepsilon>0$, there exists ρ_ε $(0<\rho_\varepsilon<\varepsilon)$ such that

$$\sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho_\varepsilon(j+1)}\right)\right)\right]\le 1.$$

This implies that

$$\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho_\varepsilon(j+1)}\right)\right)\le 1 \quad \forall j\in\mathbb{N}.$$

Therefore, we have

$$\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\varepsilon(j+1)}\right)\right) \le \mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho_\varepsilon(j+1)}\right)\right) \le 1 \quad \forall j \in \mathbb{N}.$$

Suppose that

$$\frac{\sum_{i=0}^{n_j}\mu_i\Delta_{(s)}^t x_i}{(n_j+1)} \ne 0$$

for some n_j. Then,

$$\frac{\sum_{i=0}^{n_j}\mu_i\Delta_{(s)}^t x_i}{\varepsilon(n_j+1)} \to \infty$$

as $\varepsilon \to 0$. This implies that

$$\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\epsilon(j+1)}\right)\right) \to \infty \quad \text{as} \quad \varepsilon \to 0 \quad \text{for some} \quad n_j \in \mathbb{N},$$

which leads to a contradiction. Therefore,

$$\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{(j+1)} = 0 \quad \forall j \in \mathbb{N}.$$

If $j = 0$, then $\mu_0\Delta_{(s)}^t x_0 = 0$ and $\mu_0 x_0 = 0$; $\mu_1 x_1 = 0$ for $j = 1$.

Similarly, $x_j = 0$ for all $j \ge 1$. Hence, $x = \theta$.

Further, the properties $\|x+y\| \le \|x\| + \|y\|$ and $\|\alpha x\| = |\alpha|\,\|x\|$ for any scalar α can be proved as in the proof of Theorem 2. □

The above proof makes it easy to prove that $\|x^n\| \to 0$ implies that $x_i^n \to 0$ for each $n \ge 1$. Now, we state the following result.

Proposition 1. *The space $\ell(\mathcal{F}_j, g, \mu, \Delta_{(s)}^t, \mathcal{C})$ is a BK-space.*

To prove the AK-property of the space $\ell(\mathcal{F}_j, g, \mu, \Delta_{(s)}^t, \mathcal{C})$, we give the following definition and prove some related results.

Definition 1. *Let $\mathcal{F} = (\mathcal{F}_j)$ be any sequence of Orlicz functions. Define*

$$\ell'(\mathcal{F}_j, g, \mu, \Delta_{(s)}^t, \mathcal{C}) = \left\{x \in \omega(\mathcal{X}) : \sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta_{(s)}^t x_i}{\rho(j+1)}\right)\right)\right] < \infty \quad \text{for every} \quad \rho > 0\right\}.$$

Evidently, $\ell'(\mathcal{F}_j, g, \mu, \Delta_{(s)}^t, \mathcal{C})$ is a subspace of $\ell(\mathcal{F}_j, g, \mu, \Delta_{(s)}^t, \mathcal{C})$, and its topology is inherited from $\|\cdot\|$.

Theorem 9. *Let $(\mathcal{F}_j)$ be a sequence of Orlicz functions satisfying the Δ_2-condition. Then $\ell(\mathcal{F}_j, g, \mu, \Delta_{(s)}^t, \mathcal{C}) = \ell'(\mathcal{F}_j, g, \mu, \Delta_{(s)}^t, \mathcal{C})$.*

P r o o f. Let $x \in \ell(\mathcal{F}_j, g, \mu, \Delta^t_{(s)}, \mathcal{C})$. Then, for some $\rho > 0$, we have

$$\sum_{j=0}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\rho(j+1)}\right)\right)\right] < \infty.$$

Consider any arbitrary $\eta > 0$. If $\rho \leq \eta$, then

$$\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\eta(j+1)}\right)\right) < \mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\rho(j+1)}\right)\right) < \infty \quad \text{for each} \quad j \in \mathbb{N}.$$

Let $\eta < \rho$. Since $\mathcal{F}_j$ satisfies the Δ_2-condition, there exists a constant $K_j > 0$ such that

$$\mathcal{F}_j\left(g\left(\frac{\sum\limits_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\eta(j+1)}\right)\right) \leq K_j\left(\frac{\rho}{\eta}\right)^{\log_2 K_j}\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\rho(j+1)}\right)\right) \quad \text{for each} \quad j \in \mathbb{N}.$$

Now, we can find $R_j > 0$ such that

$$R_j = \sup_j K_j\left(\frac{\rho}{\eta}\right)^{\log_2 K_j}.$$

Then, for fixed $\eta > 0$ and for each $j \in \mathbb{N}$, we have

$$\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\eta(j+1)}\right)\right) \leq R_j\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\rho(j+1)}\right)\right) < \infty.$$

It follows the result. □

Theorem 10. *Let $(\mathcal{X}, g)$ be a complete normed space. Then $\ell'(\mathcal{F}_j, g, \mu, \Delta^t_{(s)}, \mathcal{C})$ is an AK-space.*

P r o o f. Let $x \in \ell'(\mathcal{F}_j, g, \mu, \Delta^t_{(s)}, \mathcal{C})$. Then, for each ε $(0 < \varepsilon < 1)$, we can find r_0 such that

$$\sum_{j\geq r_0}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\epsilon(j+1)}\right)\right)\right] \leq 1.$$

Therefore, for $r \geq r_0$, we have

$$\|x - x^{[r]}\| = \inf\left\{\rho : \sum_{j\geq r+1}^{\infty}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\rho(j+1)}\right)\right)\right] \leq 1\right\}$$
$$\leq \inf\left\{\rho : \sum_{j\geq r}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\rho(j+1)}\right)\right)\right] \leq 1\right\} < \varepsilon.$$

Hence, $\ell'(\mathcal{F}_j, g, \mu, \Delta^t_{(\mu s)}, \mathcal{C})$ is an AK-space. □

Now, using Proposition 1 and Theorem 9, we establish the following result.

Corollary 3. *Let* $(\mathcal{F}_j)$ *be a sequence of Orlicz functions satisfying the* Δ_2*-condition. Then* $\ell(\mathcal{F}_j, g, \mu, \Delta^t_{(s)}, \mathcal{C})$ *is an AK-space.*

Theorem 11. *The space* $\ell'(\mathcal{F}_j, g, \mu, \Delta^t_{(s)}, \mathcal{C})$ *is a closed subspace of* $\ell(\mathcal{F}_j, g, \mu, \Delta^t_{(s)}, \mathcal{C})$*.*

P r o o f. Let (x^r) be a sequence in $\ell'(\mathcal{F}_j, g, \mu, \Delta^t_{(s)}, \mathcal{C})$ such that $\|x^r - x\| \to 0$.
It suffices to show that $x \in \ell'(\mathcal{F}_j, g, \mu, \Delta^t_{(s)}, \mathcal{C})$, i.e.,

$$\sum_{j\geq 0}\left[\mathcal{F}_j\left(g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\rho(j+1)}\right)\right)\right] < \infty \quad \text{for every} \quad \rho > 0.$$

For $\rho > 0$, there exists m such that $\|x^m - x\| \leq \rho/2$. Since $\mathcal{F}_j$ is a convex function for each $j \in \mathbb{N}$, we have

$$\sum_{j\geq 0}\mathcal{F}_j\left[g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\rho(j+1)}\right)\right]$$

$$= \sum_{j\geq 0}\mathcal{F}_j\left[g\left(\frac{2\left(\left|\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i^m\right| - \left|\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i^m\right| + \left|\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i\right|\right)}{\rho(j+1)}\right)\right]$$
$$\leq \frac{1}{2}\sum_{j\geq 0}\mathcal{F}_j\left[g\left(\frac{2\left|\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i^m\right|}{\rho(j+1)}\right)\right] + \frac{1}{2}\sum_{j\geq 0}\mathcal{F}_j\left[g\left(\frac{2\left|\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}(x_i^m - x_i)\right|}{\rho(j+1)}\right)\right]$$
$$\leq \frac{1}{2}\sum_{j\geq 0}\mathcal{F}_j\left[g\left(\frac{2\left|\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i^m\right|}{\rho(j+1)}\right)\right] + \frac{1}{2}\sum_{j\geq 0}\mathcal{F}_j\left[g\left(\frac{2\left|\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}(x_i^m - x_i)\right|}{\|x^m - x\|(j+1)}\right)\right].$$

From (2.4), we get

$$\sum_{j\geq 0}\mathcal{F}_j\left[g\left(\frac{2\left|\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}(x_i^m - x_i)\right|}{\|x^m - x\|(j+1)}\right)\right] \leq 1.$$

Thus,

$$\sum_{j\geq 0}\mathcal{F}_j\left[g\left(\frac{\sum_{i=0}^{j}\mu_i\Delta^t_{(s)}x_i}{\rho(j+1)}\right)\right] < \infty \quad \text{for every} \quad \rho > 0.$$

Hence, $x \in \ell'(\mathcal{F}_j, g, \mu, \Delta^t_{(s)}, \mathcal{C})$. □

Corollary 4. *The space* $\ell'(\mathcal{F}_j, g, \mu, \Delta^t_{(s)}, \mathcal{C})$ *is a BK-space.*

3. Conclusion

We have investigated the convergence of the difference sequence for the Cesàro mean of order 1, along with the generalized Orlicz function, using the technique of seminorm. In our study, we established that the newly defined sequence space $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ is a paranormed sequence space. We examined both the algebraic and topological properties of this sequence space. Additionally, we verified that $\ell(\mathcal{F}_j, q, g, r, \mu, \Delta^t_{(s)}, \mathcal{C})$ is indeed a separable sequence space. In our upcoming research, we aim to extend this concept to the case of statistical convergence and the Cesàro mean of higher order.

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