URAL MATHEMATICAL JOURNAL, Vol. 9, No. 2, 2023, pp. 165–174

DOI: 10.15826/umj.2023.2.0141

SOME INEQUALITIES BETWEEN THE BEST SIMULTANEOUS APPROXIMATION AND THE MODULUS OF CONTINUITY IN A WEIGHTED BERGMAN SPACE

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Abstract: Some inequalities between the best simultaneous approximation of functions and their intermediate derivatives, and the modulus of continuity in a weighted Bergman space are obtained. When the weight function is $\gamma(\rho)=\rho^{\alpha},\ \alpha>0$, some sharp inequalities between the best simultaneous approximation and an mth order modulus of continuity averaged with the given weight are proved. For a specific class of functions, the upper bound of the best simultaneous approximation in the space $B_{2,\gamma_1},\ \gamma_1(\rho)=\rho^{\alpha},\ \alpha>0$, is found. Exact values of several n-widths are calculated for the classes of functions $W_p^{(r)}(\omega_m,q)$.

Keywords: The best simultaneous approximation, Modulus of continuity, Upper bound, n-widths.

1. Introduction

Extremal problems of polynomial approximation of functions in a Bergman space were studied, for example, in [8, 13–15]. Here, we will continue our research in this direction and study the simultaneous approximation of functions and their intermediate derivatives in a weighted Bergman space based on the works [4–6, 10]. Note that the problem of simultaneous approximation of periodic functions and their intermediate derivatives by trigonometric polynomials in the uniform metric was studied by Garkavi [1]. In the case of entire functions, this problem was studied by Timan [12].

To solve the problem, we first will prove an analog of Ligun's inequality [2].

Let us introduce the necessary definitions and notation to formulate our results. Let

$$U:=\{z\in\mathbb{C}:|z|<1\}$$

be the unit disk in $\mathbb{C}$, and let $\mathcal{A}(U)$ be the set of functions analytic in the disk U. Denote by $B_{2,\gamma}$ the weighted Bergman space of analytic functions $f\in\mathcal{A}(U)$ such that [8]

$$\|f\|_{2,\gamma}:=\left(\frac{1}{2\pi}\iint_{(U)}|f(z)|^2\gamma(|z|)d\sigma\right)^{1/2}<\infty, \tag{1.1}$$

$d\sigma$ is an area element, $\gamma:=\gamma(|z|)$ is a nonnegative measurable function that is not identically zero, and the integral is understood in the Lebesgue sense. It is obvious, that the norm (1.1) can be written in the form

$$\|f\|_{2,\gamma}=\left(\frac{1}{2\pi}\int_0^1\int_0^{2\pi}\rho\gamma(\rho)|f(\rho e^{it})|^2d\rho dt\right)^{1/2}.$$

In the particular case of $\gamma \equiv 1$, $B_q := B_{q,1}$ is the usual Bergman space. The mth order modulus of continuity in $B_{2,\gamma}$ is defined as

$$\omega_m(f,t)_{2,\gamma} = \sup\left\{\|\Delta_m(f,\cdot,\cdot,h)\|_{2,\gamma} \colon |h| \le t\right\} =$$
$$= \sup\left\{\left(\frac{1}{2\pi}\int_0^1\int_0^{2\pi}\rho\gamma(\rho)|\Delta_m(f;\rho,u,h)|^2 d\rho du\right)^{1/2} \colon |h| \le t\right\},$$

where

$$\Delta_m(f;\rho,u,h) = \sum_{k=0}^{m}(-1)^k C_m^k f\left(\rho e^{i(u+kh)}\right).$$

Let $\mathcal{P}_n$ be the set of complex polynomials of order at most n. Consider the best approximation of functions $f \in B_{2,\gamma}$:

$$E_{n-1}(f)_{2,\gamma} = \inf\left\{\|f - p_{n-1}\|_{2,\gamma} \colon p_{n-1} \in \mathcal{P}_{n-1}\right\}$$

Denote by $\mathscr{B}_{2,\gamma}^{(r)}$ and $\mathscr{B}_2^{(r)}$, $r \in \mathbb{N}$ the class of functions $f \in \mathcal{A}(U)$ whose rth order derivatives

$$f^{(r)}(z) = d^r f/dz^r$$

belong to the spaces $B_{2,\gamma}$ and B_2, respectively. Define

$$\alpha_{n,r} = n(n-1)\cdots(n-r+1), \quad n > r.$$

It is well known [7, 8] that the best approximation of functions

$$f = \sum_{k=0}^{\infty} c_k(f) z^k \in B_{2,\gamma}$$

is equal to

$$\begin{gathered} E_{n-1}(f)_{2,\gamma} = \left(\sum_{k=n}^{\infty}|c_k(f)|^2\int_0^1\rho^{2k+1}\gamma(\rho)d\rho\right)^{1/2}, \\ E_{n-s-1}\left(f^{(s)}\right)_{2,\gamma} = \left(\sum_{k=n}^{\infty}|c_k(f)|^2\,\alpha_{k,s}^2\int_0^1\rho^{2(k-s)+1}\gamma(\rho)d\rho\right)^{1/2}, \end{gathered} \tag{1.2}$$

and the modulus of continuity of $f \in B_{2,\gamma}$ is

$$\omega_m\left(f^{(r)},t\right)_{2,\gamma} = 2^{m/2}\sup_{|h|\le t}\left\{\sum_{k=r}^{\infty}\alpha_{k,r}^2|c_k(f)|^2(1-\cos(k-r)h)^m\int_0^1\rho^{2(k-r)+1}\gamma(\rho)d\rho\right\}^{1/2}. \tag{1.3}$$

Denote by

$$\mu_s(\gamma) = \int_0^1\gamma(\rho)\rho^s d\rho, \quad s = 0,1,2,\ldots \tag{1.4}$$

the moments of order s of the weight function $\gamma(\rho)$ on $[0,1]$. According to notation (1.4), we write equalities (1.2) and (1.3) in compact form:

$$\begin{gathered} E_{n-1}(f)_{2,\gamma} = \left(\sum_{k=n}^{\infty}|c_k(f)|^2\mu_{2k+1}(\gamma)\right)^{1/2}, \\ E_{n-s-1}\left(f^{(s)}\right)_{2,\gamma} = \left(\sum_{k=n}^{\infty}|c_k(f)|^2\,\alpha_{k,s}^2\,\mu_{2(k-s)+1}(\gamma)\right)^{1/2}, \\ \omega_m\left(f^{(r)},t\right)_{2,\gamma} = 2^{m/2}\sup_{|h|\le t}\left\{\sum_{k=r}^{\infty}\alpha_{k,r}^2|c_k(f)|^2(1-\cos(k-r)h)^m\,\mu_{2(k-r)+1}(\gamma)\right\}^{1/2}. \end{gathered} \tag{1.5}$$

2. Analog of Ligun's inequality

For compact statement of the results, we introduce the following extremal characteristic:

$$\mathscr{K}_{m,n,r,s,p}(q,\gamma,h)=\sup_{f\in\mathscr{B}_{2,\gamma}^{(r)}}\frac{2^{m/2}E_{n-s-1}\left(f^{(s)}\right)_{2,\gamma}}{\left(\int_0^h\omega_m^p(f^{(r)},t)_{2,\gamma}q(t)dt\right)^{1/p}},$$

where $m,n\in\mathbb{N}$, $r\in\mathbb{Z}_+$, $n>r\geq s$, $0<p<2$, $0<h\leq\pi/(n-r)$, and $q(t)$ is a real, nonnegative, measurable weight function that is not identically zero on $[0,h]$.

Theorem 1. *Let $k,m,n\in\mathbb{N}$, $r,s\in\mathbb{Z}_+$, $k>n>r\geq s$, $0<p<2$, $0<h\leq\pi/(n-r)$, and let $q(t)$ be a nonnegative, measurable function that is not identically zero on $[0,h]$. Then*

$$\frac{1}{\mathscr{L}_{n,r,s,p}(q,\gamma,h)}\leq\mathscr{K}_{m,n,r,s,p}(q,\gamma,h)\leq\frac{1}{\inf\limits_{n\leq k<\infty}\mathscr{L}_{k,r,s,p}(q,\gamma,h)},\tag{2.1}$$

where

$$\mathscr{L}_{k,r,s,p}(q,\gamma,h)=\frac{\alpha_{k,r}}{\alpha_{k,s}}\left(\frac{\mu_{2(k-r)+1}(\gamma)}{\mu_{2(k-s)+1}(\gamma)}\right)^{1/2}\left(\int_0^h(1-\cos(k-r)t)^{mp/2}q(t)dt\right)^{1/p}.$$

P r o o f. Consider the simplified variant of Minkowski's inequality [3, p. 104]:

$$\left(\int_0^h\left(\sum_{k=n}^{\infty}|g_k(t)|^2\right)^{p/2}dt\right)^{1/p}\geq\left(\sum_{k=n}^{\infty}\left(\int_0^h|g_k(t)|^pdt\right)^{2/p}\right)^{1/2},\tag{2.2}$$

which is hold for all $0<p\leq 2$ and $h\in\mathbb{R}_+$. Setting

$$g_k=f_kq^{1/p}\quad(0<p\leq 2)$$

in (2.2), we get

$$\left(\int_0^h\left(\sum_{k=n}^{\infty}|f_k(t)|^2\right)^{p/2}q(t)dt\right)^{1/p}\geq\left(\sum_{k=n}^{\infty}\left(\int_0^h|f_k(t)|^pq(t)dt\right)^{2/p}\right)^{1/2}.\tag{2.3}$$

From (1.3) with respect to (2.3), we get

$$\begin{aligned}
\left\{\int_0^h\omega_m^p\left(f^{(r)},t\right)_{2,\gamma}q(t)dt\right\}^{1/p}&=\left\{\int_0^h\left(\omega_m^2(f^{(r)},t)_{2,\gamma}\right)^{p/2}q(t)dt\right\}^{1/p}\\
&\geq\left\{\int_0^h\left(2^m\sum_{k=n}^{\infty}\alpha_{k,r}^2|c_k(f)|^2(1-\cos(k-r)t)^m\mu_{2(k-r)+1}(\gamma)\right)^{p/2}q(t)dt\right\}^{1/p}\\
&\geq\left\{\sum_{k=n}^{\infty}\left[2^{mp/2}\alpha_{k,r}^p|c_k(f)|^p\int_0^h(1-\cos(k-r)t)^{mp/2}\left(\mu_{2(k-r)+1}(\gamma)\right)^{p/2}q(t)dt\right]^{2/p}\right\}^{1/2}\\
&=2^{m/2}\left\{\sum_{k=n}^{\infty}|c_k(f)|^2\mu_{2(k-r)+1}(\gamma)\left[\alpha_{k,r}^p\int_0^h(1-\cos(k-r)t)^{mp/2}q(t)dt\right]^{2/p}\right\}^{1/2}\\
&=2^{m/2}\left\{\sum_{k=n}^{\infty}|c_k(f)|^2\alpha_{k,s}^2\,\mu_{2(k-s)+1}(\gamma)\mu_{2(k-r)+1}(\gamma)\left(\mu_{2(k-s)+1}(\gamma)\right)^{-1}\right.
\end{aligned}$$

$$\left[\left(\frac{\alpha_{k,r}}{\alpha_{k,s}}\right)^p\int_0^h(1-\cos(k-r)t)^{mp/2}q(t)dt\right]^{2/p}\Bigg\}^{1/2}$$

$$\geq 2^{m/2}\inf_{n\leq k<\infty}\left\{\frac{\alpha_{k,r}}{\alpha_{k,s}}\left(\frac{\mu_{2(k-r)+1}(\gamma)}{\mu_{2(k-s)+1}(\gamma)}\right)^{1/2}\left(\int_0^h(1-\cos(k-r)t)^{mp/2}q(t)dt\right)^{1/p}\right\}$$

$$\times\left(\sum_{k=n}^{\infty}|c_k(f)|^2\alpha_{k,s}^2\,\mu_{2(k-s)+1}(\gamma)\right)^{1/2}=2^{m/2}E_{n-s-1}(f^{(s)})_{2,\gamma}\inf_{n\leq k<\infty}\mathscr{L}_{k,r,s,p}(q,\gamma,h),$$

and this yields the inequality

$$\frac{2^{m/2}E_{n-s-1}\left(f^{(s)}\right)_{2,\gamma}}{\left(\int_0^h\omega_m^p(f^{(r)},t)_{2,\gamma}q(t)dt\right)^{1/p}}\leq\frac{1}{\inf\limits_{n\leq k<\infty}\mathscr{L}_{k,r,s,p}(q,\gamma,h)}\tag{2.4}$$

or

$$\mathscr{K}_{m,n,r,s,p}(q,\gamma,h)\leq\frac{1}{\inf\limits_{n\leq k<\infty}\mathscr{L}_{k,r,s,p}(q,\gamma,h)}.\tag{2.5}$$

To estimate the value in (2.1) from below, consider the function

$$f_0(z)=z^n\in\mathscr{B}_{2,\gamma}^{(r)}.$$

Simple calculation leads to the following relations:

$$E_{n-s-1}(f_0^{(s)})_{2,\gamma}=\alpha_{n,s}\left(\int_0^1\rho^{2(n-s)+1}\gamma(\rho)d\rho\right)^{1/2}=\alpha_{n,s}\left(\mu_{2(n-s)+1}(\gamma)\right)^{1/2},$$

$$\omega_m^2\left(f_0^{(r)},t\right)_{2,\gamma}=2^m\alpha_{n,r}^2\left(1-\cos(n-r)t\right)^m\int_0^1\rho^{2(n-r)+1}\gamma(\rho)d\rho$$
$$=2^m\alpha_{n,r}^2\left(1-\cos(n-r)t\right)^m\mu_{2(n-r)+1}(\gamma),$$

using which, we get the lower estimate

$$\mathscr{K}_{m,n,r,p}(q,\gamma,h)\geq\frac{2^{m/2}E_{n-s-1}\left(f_0^{(s)}\right)_{2,\gamma}}{\left(\int_0^h\omega_m^p(f_0^{(r)},t)_{2,\gamma}q(t)dt\right)^{1/p}}$$
$$=\frac{2^{m/2}\alpha_{n,s}\left(\mu_{2(n-s)+1}(\gamma)\right)^{1/2}}{\left(2^{mp/2}\alpha_{n,r}^p\left(\mu_{2(n-r)+1}(\gamma)\right)^{p/2}\int_0^h(1-\cos(n-r)t)^{mp/2}q(t)dt\right)^{1/p}}=\frac{1}{\mathscr{L}_{n,r,s,p}(q,\gamma,h)}.\tag{2.6}$$

Comparing the upper estimate (2.5) and the lower estimate (2.6), we obtain the required two-sided inequality (2.1). This completes the proof of Theorem 1. □

Corollary 1. *The following two-sided inequality holds for $\gamma_1(\rho)=\rho^\alpha$, $\alpha\geq 0$, in Theorem 1:*

$$\frac{1}{\mathscr{G}_{n,r,s,p,\alpha}(q,h)}\leq\mathscr{K}_{m,n,r,s,p}(q,\gamma_1,h)\leq\frac{1}{\inf\limits_{n\leq k<\infty}\mathscr{G}_{k,r,s,p,\alpha}(q,h)},\tag{2.7}$$

where

$$\mathscr{G}_{k,r,s,p,\alpha}(q,h)=\frac{\alpha_{k,r}}{\alpha_{k,s}}\left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha}\right)^{1/2}\left(\int_0^h(1-\cos(k-r)t)^{mp/2}q(t)dt\right)^{1/p}.\tag{2.8}$$

The following problem naturally arises from (2.7): to find an exact upper bound for the extremal characteristic

$$\mathscr{K}_{m,n,r,s,p}(q,\gamma_1,h)=\sup_{f\in\mathscr{B}^{(r)}_{2,\gamma_1}}\frac{2^{m/2}E_{n-s-1}(f^{(s)})_{2,\gamma_1}}{\left(\displaystyle\int_0^h\omega_m^p(f^{(r)},t)_{2,\gamma_1}q(t)dt\right)^{1/p}},$$

where $m,n\in\mathbb{N}$, $r,s\in\mathbb{Z}_+$, $n>r\geq s$, $0<p<2$, $0<h\leq\pi/(n-r)$, $\gamma_1(\rho)=\rho^\alpha$, and $\alpha\geq 0$.

Theorem 2. *Let a weight function $q(t)$, $t\in[0,h]$, be continuous and differentiable on the interval. If the differential inequality*

$$\left(\sum_{l=s}^{r-1}\frac{p}{k-l}-\frac{2p(r-s)}{[2(k-r+1)+\alpha](2(k-s+1)+\alpha)}-\frac{1}{k-r}\right)q(t)-\frac{1}{k-r}tq'(t)\geq 0\tag{2.9}$$

holds for all $k\in\mathbb{N}$, $r,s\in\mathbb{Z}_+$, $k>n>r\geq s$, $0<p\leq 2$, and $\alpha\geq 0$, then the following equality holds for all $m,n\in\mathbb{N}$ and $0<h\leq\pi/(n-r)$:

$$\mathscr{K}_{m,n,r,s,p}(q,\gamma_1,h)=\frac{\alpha_{n,s}}{\alpha_{n,r}}\left(\frac{2(n-r+1)+\alpha}{2(n-s+1)+\alpha}\right)^{1/2}\left(\int_0^h(1-\cos(n-r)t)^{mp/2}q(t)dt\right)^{1/p}.\tag{2.10}$$

P r o o f. To prove equality (2.10), it suffices to show that the following equality holds in (2.7):

$$\inf_{n\leq k<\infty}\mathscr{G}_{k,r,s,p,\alpha}(q,h)=\mathscr{G}_{n,r,s,p,\alpha}(q,h).\tag{2.11}$$

We should note that a similar problem of finding a lower bound in (2.11) for some specific weights for $p=2$ was considered in [2]. In the general case, this problem was studied in [9], where it was proved that, if the weight function $q\in C^{(1)}[0,h]$ for $1/r<p\leq 2$, $r\geq 1$, and $0<t\leq h$ satisfies the differential equation

$$(rp-1)q(t)-tq'(t)\geq 0,$$

then (2.11) holds.

Let us now show that, under all constrains on the parameters k, r, s, m, p, α, and h in Theorem 2, the function

$$\psi(k)=\left(\frac{\alpha_{k,r}}{\alpha_{k,s}}\right)^p\left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha}\right)^{p/2}\int_0^h\left(1-\cos(k-r)t\right)^{mp/2}q(t)dt\tag{2.12}$$

increases for $n\leq k<\infty$. Indeed, differentiating (2.12) and using the identity

$$\frac{d}{dk}(1-\cos(k-r)t)^{mp/2}=\frac{t}{k-r}\frac{d}{dt}(1-\cos(k-r)t)^{mp/2},$$

we obtain

$$\psi'(k)=\left(\frac{\alpha_{k,r}}{\alpha_{k,s}}\right)^p\sum_{l=s}^{r-1}\frac{p}{k-l}\left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha}\right)^{p/2}\int_0^h(1-\cos(k-r)t)^{mp/2}q(t)dt$$
$$+\left(\frac{\alpha_{k,r}}{\alpha_{k,s}}\right)^p\frac{p}{2}\left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha}\right)^{p/2-1}\frac{4s-4r}{[2(k-r+1)+\alpha]^2}\int_0^h(1-\cos(k-r)t)^{mp/2}q(t)dt$$
$$+\left(\frac{\alpha_{k,r}}{\alpha_{k,s}}\right)^p\left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha}\right)^{p/2}\int_0^h\frac{d}{dk}(1-\cos(k-r)t)^{mp/2}q(t)dt$$

$$= \int_0^h (1-\cos(k-r)t)^{mp/2} q(t)dt \Bigg\{ \left(\frac{\alpha_{k,r}}{\alpha_{k,s}}\right)^p \sum_{l=s}^{r-1} \frac{p}{k-l} \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha}\right)^{p/2}$$
$$- \left(\frac{\alpha_{k,r}}{\alpha_{k,s}}\right)^p \frac{2p(r-s)}{[2(k-r+1)+\alpha](2(k-s+1)+\alpha)} \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha}\right)^{p/2} \Bigg\}$$
$$+ \left(\frac{\alpha_{k,r}}{\alpha_{k,s}}\right)^p \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha}\right)^{p/2} \int_0^h \frac{t}{k-r} \frac{d}{dt}(1-\cos(k-r)t)^{mp/2} q(t)dt$$
$$= \left(\frac{\alpha_{k,r}}{\alpha_{k,s}}\right)^p \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha}\right)^{p/2} \Bigg\{ \frac{h}{k-r}(1-\cos(k-r)h)^{mp/2} q(h) + \int_0^h (1-\cos(k-r)t)^{mp/2}$$
$$\times \left[\left(\sum_{l=s}^{r-1} \frac{p}{k-l} - \frac{2p(r-s)}{[2(k-r+1)+\alpha](2(k-s+1)+\alpha)} - \frac{1}{k-r} \right) q(t) - \frac{1}{k-r} t q'(t) \right] dt \Bigg\}.$$

This relation and condition (2.9) imply that $\psi(k) > 0$, $k \geq n > r \geq s$, and we obtain equality (2.10). Theorem 2 is proved. □

Denote by $W_p^{(r)}(\omega_m, q)$ $(r \in \mathbb{Z}_+,\ 0 < p \leq 2)$ the set of functions $f \in \mathscr{B}_{2,\gamma_1}^{(r)}$ whose rth derivatives $f^{(r)}$ satisfy the following condition for all $0 < h \leq \pi/(n-r)$ and $n > r$:

$$\int_0^h \omega_m^p \big(f^{(r)}, t\big)_{2,\gamma_1} q(t)dt \leq 1.$$

Since, for $f \in \mathscr{B}_{2,\gamma_1}^{(r)}$, its intermediate derivatives $f^{(s)}$ $(1 \leq s \leq r-1)$ also belong to L_2, the behavior of the value $E_{n-s-1}\big(f^{(s)}\big)_2$ for some classes $\mathfrak{M}^{(r)} \subset \mathscr{B}_{2,\gamma_1}^{(r)}$, $n > r \geq s$, $n \in \mathbb{N}$, and $r, s \in \mathbb{Z}_+$, is of interest. More precisely, it is required to find the value

$$\mathscr{A}_{n,s}\big(\mathfrak{M}^{(r)}\big) := \sup\big\{E_{n-s-1}(f^{(s)})_{2,\gamma_1} :\ f \in \mathfrak{M}^{(r)}\big\}.$$

Corollary 2. *The following equality holds for all* $n \in \mathbb{N}$, $n > r \geq s$, $0 < p \leq 2$, *and* $0 < h \leq \pi/(n-r)$:

$$\mathscr{A}_{n,s}\big(W_p^{(r)}(\omega_m, q)\big) := \sup\Big\{E_{n-s-1}(f^{(s)})_{2,\gamma_1} : f \in W_p^{(r)}(\omega_m, q)\Big\} = \frac{1}{2^{m/2}\mathscr{G}_{n,r,s,p,\alpha}(q,h)}. \tag{2.13}$$

Moreover, there is a function $g_0 \in W_p^{(r)}(\omega_m, q)$ *on which the upper bound in* (2.13) *is attained.*

P r o o f. Assuming that $\gamma = \gamma_1(\rho) = \rho^\alpha$ in (2.4), with respect to (2.8), we can write

$$E_{n-s-1}(f^{(s)})_{2,\gamma_1} \leq \frac{\left(\int_0^h \omega_m^p(f^{(r)}, t)_{2,\gamma_1} q(t)dt\right)^{1/p}}{2^{m/2} \inf\limits_{n\leq k<\infty} \mathscr{L}_{k,r,s,p}(q,\gamma_1,h)} = \frac{\left(\int_0^h \omega_m^p(f^{(r)}, t)_{2,\gamma_1} q(t)dt\right)^{1/p}}{2^{m/2} \inf\limits_{n\leq k<\infty} \mathscr{G}_{k,r,s,p,\alpha}(q,h)}.$$

Using equality (2.11) and the definition of the class $W_p^{(r)}(\omega_m, q)$, we get

$$E_{n-s-1}(f^{(s)})_{2,\gamma_1} \leq \frac{1}{2^{m/2}\mathscr{G}_{n,r,s,p,\alpha}(q,h)}. \tag{2.14}$$

From (2.14), it follows the upper estimate of the value on the left-hand side of (2.13):

$$\mathscr{A}_{n,s}\left(W_p^{(r)}(\omega_m; q, \Phi)\right) \leq \frac{1}{2^{m/2}\mathscr{G}_{n,r,s,p,\alpha}(q,h)}. \tag{2.15}$$

To obtain the lower estimate for this value, consider the function

$$g_0(z) = \frac{\sqrt{2(n-r+1)+\alpha}}{2^{m/2}\alpha_{n,r}} \left(\int_0^h (1-\cos(n-r)t)^{mp/2}\, q(t)dt\right)^{-1/p} z^n$$

and show that g_0 belongs to $W_p^{(r)}(\omega_m, q)$. Differentiating this function r times, we obtain

$$g_0^{(r)}(z) = \sqrt{\frac{2(n-r+1)+\alpha}{2^m}} \left(\int_0^h (1-\cos(n-r)t)^{mp/2}\, q(t)dt\right)^{-1/p} z^{n-r}.$$

Using this equality and formulas (1.3), we get

$$\omega_m\left(g_0^{(r)}, t\right)_{2,\gamma_1} = \frac{[1-\cos(n-r)t]^{m/2}}{\left(\int_0^h (1-\cos(n-r)t)^{mp/2}\, q(t)dt\right)^{1/p}}.$$

Raising both sides of this inequality to a power p $(0 < p \leq 2)$, multiplying them by the weight function $q(t)$, and integrating with respect to t from 0 to h, we obtain

$$\int_0^h \omega_m^p(g_0^{(r)}, t)_{2,\gamma_1} q(t)dt = 1$$

or, equivalently,

$$\left(\int_0^h \omega_m^p(g_0^{(r)}, t)_{2,\gamma_1} q(t)dt\right)^{1/p} = 1.$$

Thus, the inclusion $g_0 \in W_p^{(r)}(\omega_m, q)$ is proved.

Since the relation

$$g_0^{(s)}(z) = \sqrt{\frac{2(n-r+1)+\alpha}{2^m}}\, \frac{\alpha_{n,s}}{\alpha_{n,r}} \left(\int_0^h (1-\cos(n-r)t)^{mp/2}\, q(t)dt\right)^{-1/p} z^{n-s}$$

holds for all $0 \leq s \leq r < n$, $n \in \mathbb{N}$, and $r, s \in \mathbb{Z}_+$, according to (1.5), we have

$$\begin{aligned} E_{n-s-1}\left(g_0^{(s)}\right)_{2,\gamma_1} &= \frac{1}{2^{m/2}}\frac{\alpha_{n,s}}{\alpha_{n,r}} \sqrt{\frac{2(n-r+1)+\alpha}{2(n-s+1)+\alpha}} \left(\int_0^h [1-\cos(n-r)t]^{mp}\, q(t)dt\right)^{-1/p} \\ &= \frac{1}{2^{m/2}\,\mathscr{G}_{n,r,s,p,\alpha}(q,h)}. \end{aligned}$$

Using this equality, we obtain the lower estimate

$$\sup\left\{E_{n-s-1}(f^{(s)})_{2,\gamma_1} :\ f \in W_p^{(r)}(\Omega_m, q)\right\} \geq E_{n-s-1}(g_0^{(s)})_{2,\gamma_1} = \frac{1}{2^{m/2}\,\mathscr{G}_{n,r,s,p,\alpha}(q,h)}. \tag{2.16}$$

Comparing the upper estimate (2.15) and the lower estimate (2.16), we obtain the required equality (2.13). $\square$

3. Exact values of n-widths for the classes $W_p^{(r)}(\omega_m,q)$ $(r\in\mathbb{Z}_+,\ 0<p\le 2)$

Recall definitions and notation needed in what follows. Let X be a Banach space, let S be the unit ball in X, let $\Lambda_n\subset X$ be an n-dimensional subspace, let $\Lambda^n\subset X$ be a subspace of codimension n, let $\mathscr{L}:X\to\Lambda_n$ be a continuous linear operator, let $\mathscr{L}^{\perp}\colon X\to\Lambda_n$ be a continuous linear projection operator, and let $\mathfrak{M}$ be a convex centrally symmetric subset of X. The quantities

$$\begin{gathered}
b_n(\mathfrak{M},X)=\sup\big\{\sup\{\varepsilon>0;\ \varepsilon S\cap\Lambda_{n+1}\subset\mathfrak{M}\}:\Lambda_{n+1}\subset X\big\},\\
d_n(\mathfrak{M},X)=\inf\big\{\sup\{\inf\{\|f-g\|_X:g\in\Lambda_n\}:f\in\mathfrak{M}\}:\Lambda_n\subset X\big\},\\
\delta_n(\mathfrak{M},X)=\inf\big\{\inf\{\sup\{\|f-\mathscr{L}f\|_X:f\in\mathfrak{M}\}:\mathscr{L}X\subset\Lambda_n\}:\Lambda_n\subset X\big\},\\
d^n(\mathfrak{M},X)=\inf\big\{\sup\{\|f\|_X:f\in\mathfrak{M}\cap\Lambda^n\}:\Lambda^n\subset X\big\},\\
\Pi_n(\mathfrak{M},X)=\inf\big\{\inf\{\sup\{\|f-\mathscr{L}^{\perp}f\|_X:f\in\mathfrak{M}\}:\mathscr{L}^{\perp}X\subset\Lambda_n\}:\Lambda_n\subset X\big\}
\end{gathered}$$

are called the *Bernstein, Kolmogorov, linear, Gelfand, and projection n-widths* of a subset $\mathfrak{M}$ in the space X, respectively. These n-widths are monotone in n and related as follows in a Hilbert space X (see, e.g., [3, 11]):

$$b_n(\mathfrak{M},X)\le d^n(\mathfrak{M},X)\le d_n(\mathfrak{M},X)=\delta_n(\mathfrak{M},X)=\Pi_n(\mathfrak{M},X). \tag{3.1}$$

For an arbitrary subset $\mathfrak{M}\subset X$, we set

$$E_{n-1}(\mathfrak{M})_X:=\sup\big\{E_{n-1}(f)_2\colon f\in\mathfrak{M}\big\}.$$

Theorem 3. *The following equalities hold for all $m,n\in\mathbb{N}$, $r\in\mathbb{Z}_+$, $n>r$, and $0\le h\le\pi/(n-r)$:*

$$\begin{gathered}
\lambda_n(W_p^{(r)}(\omega_m,q),B_{2,\gamma_1})=E_{n-1}(W_p^{(r)}(\omega_m,q),B_{2,\gamma_1})\\
=\frac{1}{2^{m/2}\alpha_{n,r}}\sqrt{\frac{2(n-r+1)+\alpha}{2(n+1)+\alpha}}\left(\int_0^h[1-\cos(n-r)t]^{mp}\,q(t)dt\right)^{-1/p},
\end{gathered}\tag{3.2}$$

where $\lambda_n(\cdot)$ is any of the n-widths $b_n(\cdot)$, $d_n(\cdot)$, $d^n(\cdot)$, $\delta_n(\cdot)$, and $\Pi_n(\cdot)$.

P r o o f. We obtain the upper estimates of all n-widths for the class $W_p^{(r)}(\omega_m,q)$ with $s=0$ from (2.14) since

$$\begin{gathered}
E_{n-1}\big(W_p^{(r)}(\omega_m,q)\big)_{2,\gamma_1}=\sup\big\{E_{n-1}(f)_{2,\gamma_1}:f\in W_p^{(r)}(\omega_m,q)\big\}\\
\le\frac{1}{2^{m/2}\alpha_{n,r}}\sqrt{\frac{2(n-r+1)+\alpha}{2(n+1)+\alpha}}\left(\int_0^h[1-\cos(n-r)t]^{mp}\,q(t)dt\right)^{-1/p}.
\end{gathered}$$

Using relations (3.1) between the n-widths, we obtain the upper estimate in (3.2):

$$\begin{gathered}
\lambda_n\big(W_p^{(r)}(\omega_m,q)\big)\le E_{n-1}\big(W_p^{(r)}(\omega_m,q)\big)_{2,\gamma_1}\\
\le\frac{1}{2^{m/2}\alpha_{n,r}}\sqrt{\frac{2(n-r+1)+\alpha}{2(n+1)+\alpha}}\left(\int_0^h[1-\cos(n-r)t]^{mp}\,q(t)dt\right)^{-1/p}.
\end{gathered}\tag{3.3}$$

To obtain the lower estimate on the right-hand side of (3.2) for all n-widths in the $(n+1)$-dimensional subspace of complex algebraic polynomials

$$\mathcal{P}_{n+1}=\Big\{p_n(z):p_n(z)=\sum_{k=0}^{n}a_kz^k,\ a_k\in\mathbb{C}\Big\},$$

we introduce the ball

$$\mathbb{B}_{n+1}:=\left\{p_n(z)\in\mathcal{P}_n:\|p_n\|\le\frac{1}{2^{m/2}\alpha_{n,r}}\sqrt{\frac{2(n-r+1)+\alpha}{2(n+1)+\alpha}}\left(\int_0^h[1-\cos(n-r)t]^{mp}\,q(t)dt\right)^{-1/p}\right\},$$

where $n>r$, $n\in\mathbb{N}$, $r\in\mathbb{Z}_+$, and show that $\mathbb{B}_{n+1}\subset W_p^{(r)}(\omega_m,q)$. Indeed, for all $p_n(z)\in\mathbb{B}_{n+1}$, from (1.3), we write

$$\begin{aligned}\omega_m^2\left(p_n^{(r)},t\right)_{2,\gamma_1}&=2^m\sum_{k=r}^{\infty}\frac{\alpha_{k,r}^2|a_k(f)|^2}{2(k-r+1)+\alpha}(1-\cos(k-r)h)^m\\&\le 2^m\max_{r\le k\le n}\left\{\alpha_{k,r}^2(1-\cos(k-r)h)^m\right\}\sum_{k=r}^{\infty}\frac{|a_k(f)|^2}{2(k-r+1)+\alpha}.\end{aligned}\tag{3.4}$$

We have to prove that

$$\max_{r\le k\le n}\left\{\alpha_{k,r}^2(1-\cos(k-r)h)^m\right\}=\alpha_{n,r}^2(1-\cos(n-r)h)^m,\quad 0\le h\le\pi/(n-r).$$

Consider the function

$$\varphi(k)=\alpha_{k,r}^2(1-\cos(k-r)h)^m,\quad r\le k\le n,\quad 0\le h\le\pi/(n-r).$$

We will show that the function $\varphi(k)$ is monotone increasing for all accepted values k and h. To this end, it suffices to show that $\varphi'(k)>0$. In fact

$$\varphi'(k)=2\alpha_{k,r}^2\sum_{l=0}^{r-1}\frac{1}{k-l}(1-\cos(k-r)h)^m+mh\alpha_{k,r}^2\sin(k-r)h(1-\cos(k-r)h)^{m-1}\ge 0.$$

Hence, we can write (3.4) in the form

$$\begin{aligned}\omega_m^2\left(p^{(r)},t\right)_{2,\gamma_1}&\le 2^m\alpha_{n,r}^2(1-\cos(n-r)h)^m\sum_{k=r}^{\infty}\frac{|a_k(f)|^2}{2(k-r+1)+\alpha}\\\le 2^m\alpha_{n,r}^2(1-\cos(n-r)h)^m&\sum_{k=0}^{\infty}\frac{|a_k(f)|^2}{2(k-r+1)+\alpha}=2^m\alpha_{n,r}^2(1-\cos(n-r)h)^m\|p_n\|_{2,\gamma_1}^2.\end{aligned}\tag{3.5}$$

From (3.5), we have

$$\omega_m\left(p^{(r)},t\right)_{2,\gamma_1}\le 2^{m/2}\alpha_{n,r}(1-\cos(n-r)h)^{m/2}\|p_n\|_{2,\gamma_1}.$$

Raising both sides of this inequality to a power p $(0<p\le 2)$, multiplying them by the weight function $q(t)$, and integrating with respect to t from 0 to h, we obtain

$$\int_0^h\omega_m^p\left(p^{(r)},t\right)_{2,\gamma_1}q(t)dt\le 2^{mp/2}\alpha_{n,r}^p\|p_n\|_{2,\gamma_1}^p\int_0^h(1-\cos(n-r)h)^{mp/2}q(t)dt\le 1$$

for all $p_n\in\mathbb{B}_{n+1}$. It follows that $\mathbb{B}_{n+1}\subset W_p^{(r)}(\omega_m,q)$. Then, according to the definition of the Bernstein n-width and (3.1), we can write the following lower estimate for all above listed n-widths:

$$\begin{aligned}&\lambda_n(W_p^{(r)}(\omega_m,q),B_{2,\gamma_1})\ge b_n(W_p^{(r)}(\omega_m,q),B_{2,\gamma_1})\ge b_n(\mathbb{B}_{n+1},B_{2,\gamma_1})\\&\ge\frac{1}{2^{m/2}\alpha_{n,r}}\sqrt{\frac{2(n-r+1)+\alpha}{2(n+1)+\alpha}}\left(\int_0^h[1-\cos(n-r)t]^{mp}\,q(t)dt\right)^{-1/p}.\end{aligned}\tag{3.6}$$

Comparing the upper estimate (3.3) and the lower estimate in (3.6), we obtain the required equality (3.2). Theorem 3 is proved. □

4. Conclusion

Upper and lower estimates have been proven for extremal characteristics in a weighted Bergman space. In the case of a power function considered instead of a general weight, the values of n-widths have been calculated for a specific class of functions.

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