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FIXED RATIO POLYNOMIAL TIME APPROXIMATION ALGORITHM FOR THE PRIZE-COLLECTING ASYMMETRIC TRAVELING SALESMAN PROBLEM[1]

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Abstract: We develop the first fixed-ratio approximation algorithm for the well-known Prize-Collecting Asymmetric Traveling Salesman Problem, which has numerous valuable applications in operations research. An instance of this problem is given by a complete node- and edge-weighted digraph G. Each node of the graph G can either be visited by the resulting route or skipped, for some penalty, while the arcs of G are weighted by non-negative transportation costs that fulfill the triangle inequality constraint. The goal is to find a closed walk that minimizes the total transportation costs augmented by the accumulated penalties. We show that an arbitrary α-approximation algorithm for the Asymmetric Traveling Salesman Problem induces an $(\alpha + 1)$-approximation for the problem in question. In particular, using the recent $(22 + \varepsilon)$-approximation algorithm of V. Traub and J. Vygen that improves the seminal result of O. Svensson, J. Tarnavski, and L. Végh, we obtain $(23 + \varepsilon)$-approximate solutions for the problem.

Keywords: Prize-Collecting Traveling Salesman Problem, Triangle inequality, Approximation algorithm, Fixed approximation ratio.

1. Introduction

The Prize-Collecting Traveling Salesman Problem (PCTSP) is of the most recognized problems of combinatorial optimization. Introduced by Egon Balas in [2], it has valuable applications in drone routing [10], ride-sharing [23], or metal production. Theoretically, the PCTSP is closely related to the well-known Traveling Salesman Problem (TSP) [17] and Orienteering Problem (OP) [30].

Following to [2], an informal statement of the PCTSP is to find a traveling plan across a given transportation network, which consists of *cities* and *roads*. This network is represented by some connected graph.

For each city, the salesperson gains a reward or pays a penalty depending on whether he or she visits this city or not. In addition, traveling on an arbitrary road is charged with an appropriate transportation cost.

The goal is to find a tour retaining at least the given amount of reward and having the smallest accumulated costs and penalties.

1.1. Related work

Since PCTSP embeds the classic Traveling Salesman Problem, its general setting is strongly NP-hard and hard to approximate [27]. Furthermore, the problem remains intractable even in very

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specific settings, e.g. on the Euclidean plane [25]. As for the majority of known combinatorial problems, algorithmic desingn of this problem develops in the following main directions.

The first direction is related to exact branch-and-bound-and-cut algorithms [5, 11, 12] and goes back to fundamental results by E. Balas and M. Fischetti and P. Toth that describe facet-inducing inequalities for the equivalent mixed-integer linear programs [2, 15], see also [20]. Despite the significant impact contributed by these polyhedral results to the theory of combinatorial optimization and notable recent success in hardware development, exact algorithms still remain applicable to rather small instances of the problem.

The second one deals with developing problem-specific versions of various heuristics and metaheuristics including variable neighborhood search [21], tabu search [26], simulated annealing [14], bio-inspired and genetic algorithms [13], and their combinations. Often heuristics demonstrate an amazing performance by finding optimal (or close-to-optimal) solutions in a few seconds for real-life instances that come from industrial applications. Unfortunately, an absence of theoretical guarantees entails experimental assessment of these algorithms and possible additional tuning of their external parameters in case of any novel modification of the problem or series of instances.

Finally, the third direction relates to approximation algorithms augmented by theoretical performance guarantees and polynomial (or quasipolynomial) time approximation schemes (PTAS and QPTAS, respectively). First, we should mention the famous 5/2-approximation algorithm by D. Bienstock, M. Goemans, D. Simchi-Levi, and D. Williamson [6] for for the metric PCTSP. These algorithm relies on the classic L.Lovász result for the Euclidean graphs, an original technique of LP-relaxation rounding, and exploit as a black box the classic Christofides 3/2-approximation algorithm [9] for the metric TSP. Further, incorporation of the classic primal-dual approach by M.Goemans and D.Williamson [16] leads to several improvements of this result including $(1 - 2/3e^{-1/3})^{-1}$-approximation algorithm and more recent $(2 - \varepsilon)$-approximation algorithm [1].

We should notice that fixed-ratio approximation appears to be the best approximation result that can be obtained for an arbitrary metric (unless P=NP) since the metric PCTSP is APX-hard. Nevertheless, for some metrics of a special kind, there exist much more promising results, including PTAS for the PCTSP formulated on planar graphs [4] and the PCTSP in doubling metrics [7]. The latter PTAS is based on the breakthrough result obtained by Y. Bartal, L.A. Gottlieb, and R. Krauthgamer [3] for the classic TSP and continued by the series of recent papers (see, e.g. [8, 19]), where the efficient approximation of the combinatorial optimization problems is managed to extend beyond the finite dimensional vector spaces.

All the aforementioned results were obtained for the symmetric version of the problem. Meantime, approximation of its asymmetric version known as the Prize-Collecting Asymmetric Traveling Salesman Problem (PCATSP) still remains weakly studied. To the best of our knowledge, $(1 + \lceil \log n \rceil)$-approximation algorithm proposed in [24] is the only result obtained in this field so far. In this paper, we try to bridge this gap.

1.2. Our contribution

For the PCATSP with the triangle inequality, we introduce the first fixed-ratio polynomial time algorithm, which finds $(23 + \varepsilon)$-approximate solution of the problem for and arbitrary $\varepsilon > 0$. Our approach appears to be a further extension of the classic splitting-of technique for the Eulerian graphs and rounding framework of [6], and exploit the seminal recent Svensson-Traub $(22 + \varepsilon)$-approximation algorithm for the Asymmetric Traveling Salesman Problem (ATSP) with the triangle inequality, as a main building block.

In Section 2, we give a formulation of the problem in question. Next, in Section 3, we recall the necessary definitions and notation. Section 4 opens the discussion of novel results. In this section, we introduce the proposed algorithm, whose approximation ratio and running time bounds are proved in Section 5. Finally, in Section 6, we summarize the obtained results and discuss some open questions.

2. Problem statement

In this paper, we study polynomial approximation of the Profitable Tour Problem (PTP), introduced by M. Dell'Amico, F. Maffioli, and P. Värbrand in [12], which a simplified version of the PCATSP introduced by E. Balas in [2]. An arbitrary PTP instance is given by a complete digraph $G = (V, A)$ augmented by an edge-weighting function $c\colon A \to \mathbb{R}_+$ that specifies transportation costs and fulfills the triangle inequality

$$c(u, v) + c(v, w) \geq c(u, w) \quad \text{for any} \quad \{u, v, w\} \subset V, \tag{2.1}$$

and node-weighting function $\pi\colon V \to \mathbb{R}_+$, defining the penalties for skipping nodes of the graph G. Unlike the classic Asymmetric Traveling Salesman Problem, feasible solution set of the PTP contains an arbitrary closed walk (including the *empty* tour that skips each node of the graph G). The problem is to find the minimum cost walk, where a cost of an arbitrary walk T is defined as follows:

$$\operatorname{cost}(T) = \sum_{a \in T} c_a + \sum_{v \notin T} \pi_v,$$

provided T visits some subset $W \subset V$.

Comparing the considered problem with the original one introduced in [2], we should mention that, similarly to [6]

(i) we assume that transportation costs satisfiy the triangle inequality (2.1),

(ii) visiting of an arbitrary node of the graph G has no additional profit,

(iii) as a consequence, we exclude the knapsack constraint that restricts minimum possible profit to collect.

3. Preliminaries

Results of this paper are mainly based on approximation algorithms proposed recently for the ATSP, where the edge-waiting function satisfies the triangle inequality, and on the well-known splitting-off property of the Eulerian graphs.

In the ATSP, we are given by a complete digraph $G = (V, A)$ and a weighting function $c\colon A \to \mathbb{R}_+$, which specifies transportation costs. Without loss of generality, we assume that c satisfies the triangle inequality. The goal is to construct a closed route that visits each node of the graph G (a tour) with the minimum transportation cost.

For our construction, another equivalent formulation of the ATSP appears to be more convenient. In this setting, we are required to find a minimum cost (multi-)subset $T \subset A$, such that (V, T) is Eulerian connected multigraph. Assuming that each arc a is contained in T with multiplicity x_a, the cost of T is defined by

$$c(T) = \sum_{a \in T} c_a x_a.$$

In this context, it is convenient to assume that each tour R is defined by its arc-multiplicity vector x.

In the following, we need some standard definitions and notation. As usual, by $\delta(U_1, U_2)$ we denote the set or arcs $\{(u, v) \in A\colon u \in U_1, v \in U_2\}$ for an arbitrary disjoint non-empty subsets

$U_1, U_2 \subset V$. In particular case, where $U_1 = W$ and $U_2 = V \setminus W$, $\delta(U_1, U_2)$ coincides with classic notation of outgoing and incoming cuts

$$\delta^+(W) = \delta(W, V \setminus W) = \{(u,v) \in A\colon u \in W,\ v \notin W\} \quad \text{and}$$
$$\delta^-(W) = \delta(V \setminus W, W) = \{(u,v) \in A\colon u \notin W,\ v \in W\},$$

respectively and the cut $\delta(X) = \delta^+(X) \cup \delta^-(X)$. Next, we use a short notation $\delta(v)$ for $X = \{v\}$.

Further, we use the classic Held-Karp Mixed Integer Linear (MILP)-model for the ATSP

$$\min \sum_{a \in A} c_a x_a \tag{3.1}$$
$$\text{s.t.}\quad x(\delta^+(v)) = x(\delta^-(v)) \quad (v \in V), \tag{3.2}$$
$$x(\delta(U)) \geq 2 \quad (\varnothing \neq U \subset V), \tag{3.3}$$
$$x_a \in \mathbb{Z}_+ \quad (a \in A). \tag{3.4}$$

Here, equation (3.2) ensures that an arbitrary feasible solution induces a Eulerian multi-subgraph and (3.3) is the classic subtour elimination constraint. We denote optimum values of problem (3.1)–(3.4) and its LP-relaxation as ATSP* and ATSP-LP*, respectively.

In their seminal paper [28], O. Svensson, J. Tarnavski, and L. Végh provided the first polynomial time approximation for the ATSP within a fixed ratio. A few years later, this breakthrough result was substantially improved by V. Traub and J. Vygen in [29]. We remind this result, since it is one of the main building blocks of our own contribution.

Theorem 1. *For an arbitrary positive ε, there exists a polynomial-time algorithm that finds a feasible tour T for the given ATSP instance, such that*

$$\text{ATSP}^* \leq c(T) \leq (22 + \varepsilon)\ \text{ATSP-LP}^*.$$

We employ this result to prove the similar approximation result for the special case of the PCATSP, where all feasible walks are restricted to visit a given pair of nodes $\{u,v\} \subset V$, we call this problem PCATSP$_{u,v}$. Following the known results (see, e.g. [6, 11]), we propose the MILP-model for this problem:

$$\min \sum_{a \in A} c_a x_a + \sum_{w \in V} \pi_w (1 - y_w) \tag{3.5}$$
$$\text{s.t.}\quad x(\delta^+(w)) = x(\delta^-(w)) \quad (w \in V), \tag{3.6}$$
$$x(\delta(S)) \geq 2 \quad (S \subset V : |S \cap \{u,v\}| = 1), \tag{3.7}$$
$$x(\delta(S)) \geq 2y_w \quad (\varnothing \neq S \subseteq V \backslash \{u,v\}, \quad w \in S), \tag{3.8}$$
$$x_a \in \mathbb{Z}_+, \quad y_w \in \{0,1\}, \tag{3.9}$$
$$y_u = y_v = 1, \tag{3.10}$$

where, for the tour W, the Boolean variable y_w indicates of whether W visits the node $w \in V$, total transportation costs and node-skipping penalties accumulated by W are represented by the objective function (3.5), equation (3.6) guarantees that the multigraph (V, W) is Eulerian and, together with (3.7)–(3.8), ensures its connectivity.

As for the Held-Karp model, we assign an LP-relaxation PCATSP-LP$_{u,v}$ to the problem PCATSP$_{u,v}$, where constraint (3.9) is relaxed by $x_a \geq 0$ and $y_w \in [0,1]$, and denote by PCATSP$^*_{u,v}$, PCATSP-LP$^*_{u,v}$, and $\mathcal{TC}$(PCATSP-LP$_{u,v}$) the optimum values of the problem PCATSP$_{u,v}$ and problem PCATSP-LP$_{u,v}$ and the time complexity of the PCATSP-LP$_{u,v}$, respectively.

4. Approximation algorithm

We start with an approximation algorithm (Algorithm $\mathcal{A}_{u,v}$) for the PCATSP$_{u,v}$ used as a subroutine in our main Algorithm $\mathcal{A}$ for the PCATSP.

Algorithm $\mathcal{A}_{u,v}$ employs two outer parameters. The former one is an arbitrary approximation Algorithm $\mathcal{A}_0$ for the asymmetric ATSP. For any ATSP instance I and some $\alpha \geq 0$, this algorithm finds an approximate solution $T = T(I)$, such that:

$$\text{ATSP}^* \leq c(T) \leq \alpha \cdot \text{ATSP-LP}^* . \tag{4.1}$$

The latter parameter gives a threshold value τ separating full-size auxiliary ATSP instances approximated using algorithm $\mathcal{A}_0$ from the smaller ones, which are solved to optimality.

Algorithm $\mathcal{A}_{u,v}$

Input: an instance of the PCATSP$_{u,v}$

Parameters: an approximation algorithm $\mathcal{A}_0$ for the ATSP with triangle inequality, a threshold $\tau \geq 3$

Output: an approximate solution $W_{u,v}$ of this instance

find an optimum solution $(\bar{x}, \bar{y})$ for the PCATSP-LP$_{u,v}$

define the subset $V_{u,v} \subset V$ as follows:

$$V_{u,v} = \left\{ w \in V : \bar{y}_w \geq \frac{\alpha}{\alpha + 1} \right\},$$

by construction, $\{u, v\} \subset V_{u,v}$

consider an ATSP instance $I_{u,v}$ on the subgraph $G\langle V_{u,v} \rangle$ induced by the subset $V_{u,v}$

if $|V_{u,v}| > \tau$ **then**

  set $W_{u,v}$ to an approximate solution of $I_{u,v}$ found by Algorithm $\mathcal{A}_0$

else

  set $W_{u,v}$ to an optimum solution of this instance

end if

output $W_{u,v}$

Algorithm $\mathcal{A}$ is based on the following simple decomposition idea:

(i) by construction, any feasible solution of the PCATSP is either a closed walk $W_{u,v}$ visiting at least two nodes u and v of the input graph G or an empty walk that does not visit any node at all;

(ii) in the former case, the walk $W_{u,v}$ is a feasible solution of the appropriate restricted problem PCATSP$_{u,v}$ and has the cost

$$\text{cost}(W_{u,v}) = \sum_{a \in W_{u,v}} c_a x_a + \sum_{w \notin V'} \pi_w,$$

where x_a denotes the inclusion multiplicity for the arc a in the walk $W_{u,v}$ and V' is the subset of nodes visited by this walk;

(iii) in the latter case, the cost of the empty walk is $\sum_{w \in V} \pi_w$.

Finally, the initial PCATSP is decomposed as follows:

$$\mathrm{PCATSP}^* = \min\Big\{\sum_{w\in V}\pi_w, \min\big\{\mathrm{PCATSP}^*_{u,v}\colon \{u,v\}\subset V\big\}\Big\}. \tag{4.2}$$

Algorithm $\mathcal{A}$

Input: an instance of the PCATSP

Output: an $(\alpha+1)$-approximate solution of this instance

initialize the set of candidate solutions $\mathcal{C} = \varnothing$
for all $\{u,v\}\subset V$ **do**
  construct the auxiliary instance $\mathrm{PCATSP}_{u,v}$
  employ Algorithm $\mathcal{A}_{u,v}$ to find its approximate solution $W_{u,v}$
  append the walk $W_{u,v}$ to $\mathcal{C}$
end for
let $\bar{W} = \arg\min\big\{\,\mathrm{cost}(W_{u,v})\colon W_{u,v}\in\mathcal{C}\big\}$
if $\mathrm{cost}(\bar{W}) \le \sum_{w\in V}\pi_w$ **then**
  output $\bar{W}$
else
  output the empty walk
end if

5. Theoretical guarantees

First of all, we show that Algorithm $\mathcal{A}_{u,v}$, as an algorithm for $\mathrm{PCATSP}_{u,v}$, inherits all the approximation features of algorithm $\mathcal{A}_0$ for the ATSP with the triangle inequality. In particular, if we take the Svensson-Traub algorithm, then Algorithm $\mathcal{A}_{u,v}$, in polynomial time, will find an approximate solution of the subproblem $\mathrm{PCATSP}_{u,v}$, whose cost does not exceed $(23+\varepsilon)$ $\mathrm{PCATSP\text{-}LP}^*_{u,v}$.

We start with the following technical lemma.

Lemma 1. *For each $V_{u,v}\subset V$, such that $|V_{u,v}|>3$, the optimum value of the problem*

$$\min \sum c_a x_a \tag{5.1}$$

$$s.t.\quad x(\delta^+(w)) = x(\delta^-(w)) \quad (w\in V), \tag{5.2}$$

$$x(\delta(U)) \ge 2\begin{pmatrix} U\subset V\colon V_{u,v}\cap U\neq\varnothing, \\ V_{u,v}\cap V\backslash U\neq\varnothing \end{pmatrix}, \tag{5.3}$$

$$x_a \ge 0 \quad (a\in A), \tag{5.4}$$

$$x(\delta(w))\begin{cases} \ge 2, & \text{if } w\in V_{u,v}, \\ = 0, & \text{otherwise} \end{cases} \tag{5.5}$$

is equal to the optimum value of problem (5.1)–(5.4).

Lemma 1 can be derived from a much more general result on connectivity of Eulerian graphs obtained by L. Lovász [22] and B. Jackson [18]. Nevertheless, we prefer to present its direct proof, since, in our case, it appears to be much simpler.

P r o o f. Indeed, denote by X_1^* and X_2^* optimal sets of the problem (5.1)–(5.4) and (5.1)–(5.5), respectively. We show that an arbitrary rational solution

$$x^* = \arg\min\Big\{\sum_{a\in A} x_a \colon x \in X_1^*\Big\} \tag{5.6}$$

belongs to X_2^*. To prove it, it is sufficient to show that $x^*(\delta(w)) = 0$ for an arbitrary $w \notin V_{u,v}$, since, for any $w \in V_{u,v}$, inequality $x^*(\delta(w)) \geq 2$ follows straightforwardly from equation (5.3). By construction, there exists a number $D \geq 1$, such that the vector $\xi^* = D \cdot x^*$ is an integer and fulfills the following constraints:

$$\xi(\delta^+(w)) = \xi(\delta^-(w)) \quad (w \in V), \tag{5.7}$$

$$\xi(\delta(S)) \geq 2D \quad (S \in \mathcal{S}_{u,v}), \tag{5.8}$$

$$\xi \geq 0, \tag{5.9}$$

where

$$\mathcal{S}_{u,v} = \{S \subset V \colon V_{u,v} \cap S \neq \varnothing,\ V_{u,v} \setminus S \neq \varnothing\}$$

and $\xi^*(\delta(S))$ is even for an arbitrary $S \subset V$.

Assume that there exists $w_0 \notin V_{u,v}$, for which $\xi^*(\delta(w_0)) > 0$. Obviously, inequality (5.8) holds tight for at least one subset $\bar{S} \in \mathcal{S}_{u,v}$. Indeed, otherwise, for some nodes w' and w'' neighboring to w_0, for which $\xi^*(a) > 0$, $a \in \{(w', w_0), (w_0, w'')\}$, there exists the vector $\tilde{\xi}$, built as follows:

$$\tilde{\xi}_a = \begin{cases} \xi_a^* - 1, & \text{for} \quad a = (w', w_0) \quad \text{or} \quad a = (w_0, w'') \\ \xi_a^* + 1, & \text{for} \quad a = (w', w'') \\ \xi_a^*, & \text{for any other arc.} \end{cases} \tag{5.10}$$

By construction, $\tilde{\xi}$ satisfies equations (5.7)–(5.9). Furthermore,

$$\sum_{a\in A} \tilde{\xi}_a = \sum_{a\in A} \xi_a^* - 1, \quad \text{and} \quad \sum_{a\in A} c_a \tilde{\xi}_a \leq \sum_{a\in A} c_a \xi_a^*,$$

by triangle inequality. Therefore, the vector $\tilde{x} = 1/D \cdot \tilde{\xi}$ is feasible in the problem (5.1)–(5.4), it belongs to X_1^*, and

$$\sum_{a\in A} \tilde{x}_a = \sum_{a\in A} x_a^* - \frac{1}{D}$$

that contradicts to (5.6).

Therefore, we proved that there exists a subset $\bar{S} \in \mathcal{S}_{u,v}$, $w_0 \notin \bar{S}$ and the neighbors $w', w'' \in \bar{S}$, such that $\xi^*(\delta(\bar{S})) = 2D$, and $\tilde{\xi}(\delta(\bar{S})) = 2D - 2$. Notice that, even in this case, any time, when $\{w', w''\} \cap V_{u,v} = \varnothing$, transform (5.10) still provides feasible solution $\tilde{x}$ that contradicts to minimality of x^*. Therefore, in the sequel, we assume without of loss of generality that $w' \in V_{u,v}$.

Denote by S' the maximal subset of $V \setminus \{w_0\}$, such that $S' \in \mathcal{S}_{u,v}$, $w' \in S'$, and $\xi^*(\delta(S')) = 2D$. Since

$$w_0 \notin V_{u,v}, \quad V_{u,v} \setminus (S' \cup \{w_0\}) \neq \varnothing \quad \text{and} \quad \xi^*(\delta(S' \cup \{w_0\})) \geq 2D.$$

Therefore, S' cannot contain all the neighbors of w_0, since, otherwise (see Fig. 1)

$$2D \leq \xi^*(\delta(S' \cup \{w_0\})) < \xi^*(\delta(S')) = 2D.$$

Further, consider the subsets S' and $\bar{S}$. Since

$$S' \cap \bar{S} \neq \varnothing, \quad V \setminus (S' \cup \bar{S}) \neq \varnothing, \quad \bar{S} \setminus S' \neq \varnothing, \quad \text{and} \quad S' \setminus \bar{S} \neq \varnothing,$$

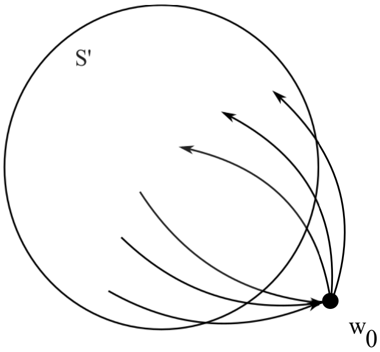

Figure 1. Example of subset S', node w_0, and its neighbors.

for an arbitrary vector ξ satisfying (5.7)–(5.9), the following inequalities

$$\xi(\delta(S' \setminus \bar{S})) + \xi(\delta(\bar{S} \setminus S')) \leq \xi(\delta(S')) + \xi(\delta(\bar{S})), \tag{5.11}$$

$$\xi(\delta(S' \cup \bar{S})) + \xi(\delta(S' \cap \bar{S})) \leq \xi(\delta(S')) + \xi(\delta(\bar{S})) \tag{5.12}$$

are valid. Indeed, as it follows from Fig. 2,

$$\begin{aligned} \xi(\delta(S' \setminus \bar{S})) &= a + b + d, \quad & \xi(\delta(\bar{S} \setminus S')) &= c + d + e, \\ \xi(\delta(S' \cup \bar{S})) &= a + e + f, \quad & \xi(\delta(S' \cap \bar{S})) &= b + c + f, \\ \xi(\delta(S')) &= a + c + d + f, \quad & \xi(\delta(\bar{S})) &= b + d + e + f. \end{aligned}$$

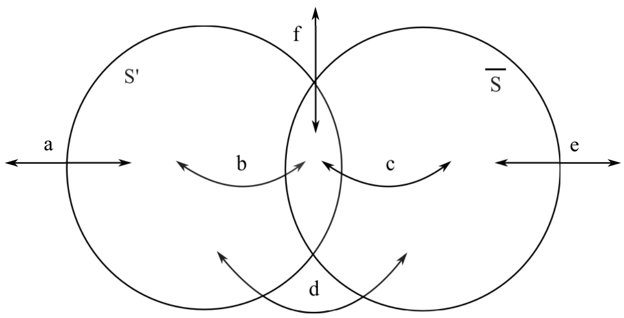

Figure 2. Cut sizes: $a = |\delta(S' \setminus \bar{S}, V \setminus (S' \cup \bar{S})|$; $b = |\delta(S' \setminus \bar{S}, S' \cap \bar{S})|$; $c = |\delta(\bar{S} \setminus S', S' \cap \bar{S})|$; $d = |\delta(S' \setminus \bar{S}, \bar{S} \setminus S')|$; $e = |\delta(\bar{S} \setminus S', V \setminus (S' \cup \bar{S}))|$; $f = |\delta(S' \cap \bar{S}, V \setminus (S' \cup \bar{S}))|$.

Therefore, inequalities (5.11) and(5.12) hold for any non-negative function c of transportation costs. Furthermore, for ξ^*,

$$\xi^*(\delta(S' \setminus \bar{S})) + \xi^*(\delta(\bar{S} \setminus S')) < \xi^*(\delta(S')) + \xi^*(\delta(\bar{S})), \tag{5.13}$$

by construction. Next, by assumption, $w' \in V_{u,v} \cap S' \cap \bar{S}$, $w'' \in \bar{S} \setminus S'$, and $w_0 \notin S'$, which imply $\xi^*(\delta(S' \cap \bar{S})) \geq 2D$. Thus, we obtain

$$\xi^*(\delta(S' \cup \bar{S})) \leq 2D,$$

due to (5.12) and the equality

$$\xi^*(\delta(S')) = \xi^*(\delta(\bar{S})) = 2D. \tag{5.14}$$

Consequently, $V_{u,v} \subset S' \cup \bar{S}$, since otherwise, taking into account the inequality $V_{u,v} \cap (\bar{S} \cap S') \neq \varnothing$, we come to contradiction with the maximality of the subset S'.

Coming back to the subsets $\bar{S} \setminus S'$ and $S' \setminus \bar{S}$, we can easily show that each belong to $\mathcal{S}_{u,v}$, which implies

$$\xi^*(\delta(S' \setminus \bar{S})) \geq 2D \quad \text{and} \quad \xi^*(\delta(\bar{S} \setminus S')) \geq 2D.$$

These equations together with (5.14) contradict (5.13). Lemma 1 is proved. □

Theorem 2. *If the Algorithm $\mathcal{A}_0$ approximates the ATSP in time $\mathcal{TC}(\mathcal{A}_0)$ within the accuracy bound (4.1), then the Algorithm $\mathcal{A}_{u,v}$, in time $\mathcal{TC}(\mathcal{A}_0)+\mathcal{TC}(\text{PCATSP-LP}_{u,v})$, finds an approximate solution $W_{u,v}$ of the* PCATSP$_{u,v}$, *for which*

$$\text{PCATSP}^*_{u,v} \leq \text{cost}(W_{u,v}) \leq (\alpha+1)\ \text{PCATSP-LP}^*_{u,v}\,. \tag{5.15}$$

P r o o f. Since the description of the Algorithm $\mathcal{A}_{u,v}$ leads to a straightforward upper bound of its time complexity, we proceed with the bound of its approximation ratio (5.15). Indeed, make a simple transformation of the fractional solution $(\bar{x}, \bar{y})$ obtained at Step 1 of the Algorithm $\mathcal{A}_{u,v}$ as follows:

$$\hat{x} = \frac{\alpha+1}{\alpha} \cdot \bar{x}, \quad \hat{y}_w = \begin{cases} 1, & \text{if} \quad \bar{y}_w \geq \alpha/(\alpha+1), \\ 0, & \text{otherwise.} \end{cases} \quad (w \in V). \tag{5.16}$$

Further, the set $V_{u,v}$ defined at Step 2 of the Algorithm $\mathcal{A}_{u,v}$ obeys the equation

$$V_{u,v} = \{w \in V : \hat{y}_w = 1\},$$

by construction. We consider the non-trivial case, where $|V_{u,v}| > \tau$. Then, $W_{u,v}$ coincides with an approximate solution of the auxiliary ATSP instance defined on the subgraph $G\langle V_{u,v}\rangle$ provided by Algorithm $\mathcal{A}_0$. Denote by x' the appropriate feasible solution of its MIP-model

$$\min \sum c_a x_a \tag{5.17}$$
$$s.t.\ \ x(\delta^+(w)) = x(\delta^-(w)) \quad (w \in V_{u,v}), \tag{5.18}$$
$$x(\delta(U)) \geq 2 \quad (\varnothing \neq U \subset V_{u,v}), \tag{5.19}$$
$$x_a \in \mathbb{Z}_+. \tag{5.20}$$

By condition,

$$\text{ATSP}^* \leq c(x') \leq \alpha \cdot \text{ATSP-LP}^*\,.$$

In turn, the LP-relaxation of problem (5.17)–(5.20), the problem ATSP LP, appears to be equivalent to the problem (5.1)–(5.5), whose optimum value is equal to the optimum value of the problem (5.1)–(5.4), by Lemma 1.

Further, we prove that the vector $\hat{x}$ is a feasible solution of problem (5.1)–(5.4). Indeed, equation (5.2) follows from (3.6), since $\hat{x} = (\alpha+1)/\alpha \cdot \bar{x}$. In order to prove that $\hat{x}$ satisfies equation (5.3), notice that, if $|\{u,v\} \cap S| = 1$, it easily follows from (3.7).

Next, suppose that $S \cap \{u,v\} = \varnothing$ (the case $\{u,v\} \subset S$ can be tackled by analogy). For an arbitrary $w \in V_{u,v} \cap S$, we have $\hat{y}_w = 1$, i.e. $\bar{y}_w \geq \alpha/(\alpha+1)$. Therefore,

$$\hat{x}(\delta(S)) = \sum_{e \in \delta(S)} \hat{x}_e = \frac{\alpha+1}{\alpha} \sum_{e \in \delta(S)} \bar{x}_e \geq \frac{\alpha+1}{\alpha} \cdot 2\bar{y}_w \geq 2,$$

since, $\bar{x}(\delta(w)) \geq 2\bar{y}_w$.

Thus, we showed that $\hat{x}$ is a feasible solution of the problem (5.1)–(5.4), whose optimum value equals to ATSP-LP*, i.e.

$$c(x') \le \alpha \cdot \text{ATSP-LP}^* \le \alpha \cdot c(\hat{x}).$$

Therefore, for the feasible solution $(x', \hat{y})$ (induced by the walk $W_{u,v}$) of the problem (3.5)–(3.10), we have

$$\text{PCATSP}^*_{u,v} \le \text{cost}(W_{u,v}) = c(x') + \sum_{w\in V} \pi_w(1-\hat{y}_w) \le \alpha \cdot \text{ATSP-LP}^*$$
$$+ \sum_{w\in V} \pi_w(1-\hat{y}_w) \le \alpha \cdot c(\hat{x}) + \sum_{w\in V} \pi_w(1-\hat{y}_w) = (\alpha+1)\cdot c(\bar{x}) + \sum_{w\in V} \pi_w(1-\hat{y}_w).$$

Taking into account the inequality

$$1 - \hat{y}_w \le (\alpha+1)(1-\bar{y}_w)$$

following straightforwardly from (5.16), we obtain

$$\frac{\text{cost}(W_{u,v})}{\text{PCATSP-LP}^*_{u,v}} \le \frac{(\alpha+1)\cdot c(\bar{x}) + (\alpha+1)\sum_{w\in V}\pi_w(1-\bar{y}_w)}{c(\bar{x}) + \sum_{w\in V}\pi_w(1-\bar{y}_w)} = \alpha + 1.$$

Theorem 2 is proved. □

Finally, we obtain our main result, which easily follows from Theorem 2.

Theorem 3. *From Theorem 2 it follows that Algorithm $\mathcal{A}$ finds $(\alpha+1)$-approximate feasible solution of the* PCATSP *in time*

$$O(n^2 \cdot (\mathcal{TC}(\mathcal{A}_0) + \mathcal{TC}(\text{PCATSP-LP}_{u,v}))). \tag{5.21}$$

P r o o f. First, we obtain an upper accuracy bound for Algorithm $\mathcal{A}$. Without loss of generality, we skip the trivial case, where, for the given PCATSP instance, an arbitrary non-empty walk is dominated by the empty walk and

$$\text{PCATSP}^* = \sum_{w\in V} \pi_w.$$

Then, in (4.2), the minimum is achieved at some pair $\{\tilde{u}, \tilde{v}\} \subset V$. Therefore, for the output walk $\bar{W}$, by Theorem 2 we have

$$\text{PCATSP}^* \le \text{cost}(\bar{W}) \le \text{cost}(W_{\tilde{u},\tilde{v}}) \le (\alpha+1)\cdot \text{PCATSP-LP}^*_{\tilde{u},\tilde{v}}$$
$$\le (\alpha+1)\cdot \text{PCATSP}^*_{\tilde{u},\tilde{v}} = (\alpha+1)\cdot \text{PCATSP}^*.$$

In turn, the complexity bound (5.21) easily follows from the construction of Algorithm $\mathcal{A}$ and Theorem 2. Indeed, the running time of Algorithm $\mathcal{A}$ is determined by the for-loop statement located between its Step 2 and Step 6. At each iteration of this loop, we employ Algorithm $\mathcal{A}_{u,v}$ to one of $O(n^2)$ auxiliary instances of the PCATSP$_{u,v}$. Theorem 3 is proved. □

Remark 1. Exploiting the recent $(22+\varepsilon)$-approximation algorithm for the ATSP, for an arbitrary $\varepsilon > 0$ we obtain the polynomial-time algorithm for the PCATSP within approximation ratio $(23+\varepsilon)$.

Remark 2. Since auxiliary instances PCATSP$_{u,v}$ are mutually independent, all of them can be approximated in parallel. In this case, the running-time bound of Algorithm $\mathcal{A}$ coincides asymptotically with the running-time bound of Algorithm $\mathcal{A}_{u,v}$.

6. Conclusion

In this paper, we proposed the first fixed-ratio $(23 + \varepsilon)$-approximation algorithm for the PCATSP. By its appearance, our algorithm owes to the recent breakthrough results of O. Svensson and V. Traub [28, 29] for the asymmetric TSP, who make possible polynomial time approximation for asymmetric versions of other routing problems within fixed ratios. To future work, we postpone reports on some algorithms of such kind including the algorithm for the general version of the PCATSP.

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