URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 99–111

DOI: 10.15826/umj.2024.1.009

STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES IN NEUTROSOPHIC 2-NORMED SPACES

Rahul Mondal

Department of Mathematics, Vivekananda Satavarshiki Mahavidyalaya,
Vidyasagar University,
Manikpara, Jhargram – 721513, West Bengal, India

imondalrahul@gmail.com

Nesar Hossain

Department of Mathematics, University of Burdwan,
Burdwan – 713104, West Bengal, India

nesarhossain24@gmail.com

Abstract: In this paper, we have studied the notion of statistical convergence for double sequences in neutrosophic 2-normed spaces. Also, we have defined statistically Cauchy double sequences and statistically completeness for double sequences and investigated some interesting results in connection with neutrosophic 2-normed space.

Keywords: Neutrosophic 2-normed space, Double natural density, Statistically double convergent sequence, Statistically double Cauchy sequence.

1. Introduction

In 1951, Fast [12] and Steinhaus [29] independently extended the concept of usual convergence of real sequences to statistical convergence of real sequences based on the natural density of a set. Later on, this idea has been studied in different directions and various spaces by many authors such as [8–10, 13, 14, 25, 26, 28, 31, 35], and many others.

After the introduction of the fuzzy set theory by Zadeh [37], there has been an extensive effort to find applications and fuzzy analogs of the classical theories and it is being applied in various branches of engineering and science [4, 15, 17, 19, 24]. Later on, the notion of the fuzzy set theory was developed effectively and generalized into new notions as its extensions like intuitionistic fuzzy set [1], interval-valued fuzzy set [36], interval-valued intuitionistic fuzzy set [2], and vague fuzzy set [3]. As a generalization of a crisp set, fuzzy set, intuitionistic fuzzy set, and Pythagorean fuzzy set, Smarandache [32] studied the concept of neutrosophic set. Later, Bera and Mahapatra introduced the notion of neutrosophic soft linear space [5] and neutrosophic soft normed linear space [6]. Recently, Kirişci and Şimşek [21] defined neutrosophic normed space and, in this space, many summability methods such as statistical convergence [21], statistical convergence of double sequences [18], ideal convergence [22], lacunary statistical convergence [23], deferred statistical convergence [11] etc.

Mursaleen and Edely [26] defined and studied statistical convergence and statistically Cauchy double sequences in $\mathbb{R}$. Sarabadan and Talebi [35] studied the notion of statistical convergence of double sequences in 2-normed spaces. Granados and Dhital [18] discussed statistical convergence and statistical Cauchy property for double sequences in neutrosophic normed spaces. Recently, Murtaza et al. [27] introduced neutrosophic 2-normed space and studied statistical convergence for single sequences. In the present paper, we study statistical convergence and statistically Cauchy double sequences in neutrosophic 2-normed spaces and prove some associated results in the line of investigations of them with respect to neutrosophic 2-norm.

2. Preliminaries

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ indicate the set of natural numbers and the set of reals, respectively; $|A|$ denotes the cardinality of the set A. First, we recall some basic definitions and notations.

Definition 1 [26]. *Let $\mathcal{K} \subseteq \mathbb{N}\times\mathbb{N}$ be a two-dimensional set of positive integers, and let $\mathcal{K}(m,n)$ be the number of (j,k) in $\mathcal{K}$ such that $j \leq m$ and $k \leq n$. Then, the two-dimensional analog of natural density can be defined as follows.*

The lower asymptotic density of the set $\mathcal{K} \subseteq \mathbb{N}\times\mathbb{N}$ is defined as

$$\underline{\delta_2}(\mathcal{K}) = \liminf_{m,n} \frac{\mathcal{K}(m,n)}{mn}.$$

In case the sequence $(\mathcal{K}(m,n)/(mn))$ has a limit in Pringsheim's sense, we say that $\mathcal{K}$ has double natural density defined as

$$\lim_{m,n} \frac{\mathcal{K}(m,n)}{mn} = \delta_2(\mathcal{K}).$$

Example 1. [26] Let

$$\mathcal{K} = \{(i^2, j^2) : i, j \in \mathbb{N}\}.$$

Then,

$$\delta_2(\mathcal{K}) = \lim_{m,n} \frac{\mathcal{K}(m,n)}{mn} \leq \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0;$$

i.e., the set $\mathcal{K}$ has double natural density zero, while the set $\{(i,2j) : i,j \in \mathbb{N}\}$ has double natural density 1/2.

Note that, setting $m = n$, we obtain the two-dimensional natural density due to Christopher [7].

Definition 2 [26]. *A real double sequence $\{l_{mn}\}$ is said to be statistically convergent to a number ξ if the set*

$$\{(m,n), m \leq i, n \leq j : |l_{mn} - \xi| \geq \varepsilon\}$$

has double natural density zero for all $\varepsilon > 0$.

Definition 3 [16]. *Let $\mathcal{Z}$ be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on $\mathcal{Z}$ is a function $\|.,.\| : \mathcal{Z}\times\mathcal{Z} \to \mathbb{R}$ which satisfies the following conditions:*

(1) $\|x,y\| = 0$ *if and only if x and y are linearly dependent in $\mathcal{Z}$;*
(2) $\|x,y\| = \|y,x\|$ *for all x and y in $\mathcal{Z}$;*
(3) $\|\alpha x,y\| = |\alpha|\,\|x,y\|$ *for all α in $\mathbb{R}$ and for all x and y in $\mathcal{Z}$;*
(4) $\|x+y,z\| \leq \|x,z\| + \|y,z\|$ *for all x, y, and z in $\mathcal{Z}$.*

Example 2. [34] Let $\mathcal{Z} = \mathbb{R}^2$. Define $\|\cdot,\cdot\|$ on $\mathbb{R}^2$ by $\|x,y\| = |x_1y_2 - x_2y_1|$, where $x = (x_1,x_2)$ and $y = (y_1,y_2) \in \mathbb{R}^2$. Then, $(\mathcal{Z}, \|\cdot,\cdot\|)$ is a 2-normed space.

Definition 4 [35]. *A double sequence* $\{l_{mn}\}$ *in a* 2*-normed space* $(\mathcal{Z}, \|.,.\|)$ *is called statistically convergent to* $\xi \in \mathcal{Z}$ *if, for all* $\varepsilon > 0$ *and all nonzero* $z \in \mathcal{Z}$, *the set*

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|l_{mn} - \xi, z\| \geq \varepsilon\}$$

has double natural density zero; *i.e.,*

$$\lim_{i,j} \frac{1}{ij} \left|\{(m,n), m \leq i, n \leq j : \|l_{mn} - \xi, z\| \geq \varepsilon\}\right| = 0.$$

Definition 5 [35]. *A double sequence* $\{l_{mn}\}$ *in a* 2*-normed space* $(\mathcal{Z}, \|.,.\|)$ *is called a statistically Cauchy double sequence if, for all* $\varepsilon > 0$ *and all* $z \in \mathcal{Z}$, *there exist* $n_0, m_0 \in \mathbb{N}$ *such that, for all* $m, p \geq n_0$ *and* $n, q \geq m_0$, *the set*

$$\{(m,n), m \leq i, n \leq j : \|l_{mn} - l_{pq}, z\| \geq \varepsilon\}$$

has double natural density zero.

Definition 6 [30]. *A binary operation* $\boxdot : [0,1] \times [0,1] \to [0,1]$ *is called a continuous* t*-norm if the following conditions hold*:

(1) $\boxdot$ *is associative and commutative*;
(2) $\boxdot$ *is continuous*;
(3) $x \boxdot 1 = x$ *for all* $x \in [0,1]$;
(4) $x \boxdot y \leq z \boxdot w$ *whenever* $x \leq z$ *and* $y \leq w$ *for all* $x, y, z, w \in [0,1]$.

Definition 7 [30]. *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous* t*-conorm if the following conditions are satisfied*:

(1) $*$ *is associative and commutative*;
(2) $*$ *is continuous*;
(3) $x * 0 = x$ *for all* $x \in [0,1]$;
(4) $x * y \leq z * w$ *whenever* $x \leq z$ *and* $y \leq w$ *for all* $x, y, z, w \in [0,1]$.

Example 3. [20] Here are examples of t-norms:

(1) $x \boxdot y = \min\{x, y\}$;
(2) $x \boxdot y = x.y$;
(3) $x \boxdot y = \max\{x + y - 1, 0\}$. This t-norm is known as Lukasiewicz t-norm.

Example 4. [20] Here are examples of t-conorms:

(1) $x * y = \max\{x, y\}$;
(2) $x * y = x + y - x.y$;
(3) $x * y = \min\{x + y, 1\}$. This is known as the Lukasiewicz t-conorm.

Lemma 1 [33]. *If* $\boxdot$ *is a continuous* t*-norm,* $*$ *is a continuous* t*-conorm, and* $r_i \in (0,1)$ *for* $1 \leq i \leq 7$, *then the following statements hold*:

(1) *if* $r_1 > r_2$, *then there are* $r_3, r_4 \in (0,1)$ *such that* $r_1 \boxdot r_3 \geq r_2$ *and* $r_1 \geq r_2 * r_4$;
(2) *if* $r_5 \in (0,1)$, *then there are* $r_6, r_7 \in (0,1)$ *such that* $r_6 \boxdot r_6 \geq r_5$ *and* $r_5 \geq r_7 * r_7$.

Now we recall the notion of neutrosophic 2-normed space.

Definition 8 [27]. *Let $\mathcal{Y}$ be a vector space, and let*

$$\mathcal{N}_2 = \{< (e,f), \Theta(e,f), \vartheta(e,f), \psi(e,f) >: (e,f) \in \mathcal{Y} \times \mathcal{Y}\}$$

be a 2-normed space such that

$$\mathcal{N}_2 : \mathcal{Y} \times \mathcal{Y} \times \mathbb{R}^+ \to [0,1].$$

Suppose that $\boxdot$ and $*$ are continuous t-norm and t-conorm, respectively. Then, the quadruple $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ is called a neutrosophic 2-normed space (N2-NS) if the following conditions hold for all $e, f, g \in \mathcal{Z}$, $\eta, \zeta > 0$, and $\beta \neq 0$:

(1) $0 \leq \Theta(e,f;\eta) \leq 1$, $0 \leq \vartheta(e,f;\eta) \leq 1$, *and* $0 \leq \psi(e,f;\eta) \leq 1$ *for every* $\eta > 0$;
(2) $\Theta(e,f;\eta) + \vartheta(e,f;\eta) + \psi(e,f;\eta) \leq 3$;
(3) $\Theta(e,f;\eta) = 1$ *iff e and f are linearly dependent*;
(4) $\Theta(\beta e,f;\eta) = \Theta(e,f;\eta/|\beta|)$ *for all* $\beta \neq 0$;
(5) $\Theta(e,f;\eta) \boxdot \Theta(e,g;\zeta) \leq \Theta(e,f+g;\eta+\zeta)$;
(6) $\Theta(e,f;\cdot) : (0,\infty) \to [0,1]$ *is a continuous nonincreasing function*;
(7) $\lim_{\eta\to\infty} \Theta(e,f;\eta) = 1$;
(8) $\Theta(e,f;\eta) = \Theta(f,e;\eta)$;
(9) $\vartheta(e,f;\eta) = 0$ *iff e and f are linearly dependent*;
(10) $\vartheta(\beta e,f;\eta) = \vartheta(e,f;\eta/|\beta|)$ *for all* $\beta \neq 0$;
(11) $\vartheta(e,f;\eta) * \vartheta(e,g;\zeta) \geq \vartheta(e,f+g;\eta+\zeta)$;
(12) $\vartheta(e,f;\cdot) : (0,\infty) \to [0,1]$ *is a continuous nonincreasing function*;
(13) $\lim_{\eta\to\infty} \vartheta(e,f;\eta) = 0$;
(14) $\vartheta(e,f;\eta) = \vartheta(f,e;\eta)$;
(15) $\psi(e,f;\eta) = 0$ *iff e and f are linearly dependent*;
(16) $\psi(\beta e,f;\eta) = \psi(e,f;\eta/|\beta|)$ *for each* $\beta \neq 0$;
(17) $\psi(e,f;\eta) * \psi(e,g;\zeta) \geq \psi(e,f+g;\eta+\zeta)$;
(18) $\psi(e,f;\cdot) : (0,\infty) \to [0,1]$ *is a continuous nonincreasing function*;
(19) $\lim_{\eta\to\infty} \psi(e,f;\eta) = 0$;
(20) $\psi(e,f;\eta) = \psi(f,e;\eta)$;
(21) *If* $\eta \leq 0$, $\Theta(e,f;\eta) = 0$, $\vartheta(e,f;\eta) = 1$, *and* $\psi(e,f;\eta) = 1$.

In this case, $\mathcal{N}_2 = (\Theta, \vartheta, \psi)$ is called neutrosophic 2-norm on $\mathcal{Y}$.

Definition 9 [27]. *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in an N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$. Choose $\varepsilon \in (0,1)$ and $\eta > 0$. Then, $\{l_n\}_{n\in\mathbb{N}}$ is called convergent if there exist $n_0 \in \mathbb{N}$ and $l_0 \in \mathcal{Y}$ such that*

$$\Theta(l_n - l_0, z;\eta) > 1 - \varepsilon, \quad \vartheta(l_n - l_0, z;\eta) < \varepsilon, \quad \psi(l_n - l_0, z;\eta) < \varepsilon$$

for all $n \geq n_0$ and $z \in \mathcal{Z}$; i.e.,

$$\lim_{n\to\infty} \Theta(l_n - l_0, z;\eta) = 1, \quad \lim_{n\to\infty} \vartheta(l_n - l_0, z;\eta) = 0, \quad \lim_{n\to\infty} \psi(l_n - l_0, z;\eta) = 0.$$

In this case, we write

$$\mathcal{N}_2 - \lim_{n\to\infty} l_n = l_0 \quad or \quad l_n \xrightarrow{\mathcal{N}_2} l_0$$

and l_0 is called an $\mathcal{N}_2$-limit of $\{l_n\}_{n\in\mathbb{N}}$.

Definition 10 [27]. *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in an N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$. Choose $\varepsilon \in (0,1)$ and $\eta > 0$. Then, $\{l_k\}_{k\in\mathbb{N}}$ is called statistically convergent to ξ if the natural density of the set*

$$\mathcal{A}(\varepsilon,\eta) = \big\{k \leq n : \Theta(l_k - \xi, z;\eta) \leq 1 - \varepsilon \text{ or } \vartheta(l_k - \xi, z;\eta) \geq \varepsilon \text{ and } \psi(l_k - \xi, z;\eta) \geq \varepsilon\big\}$$

is zero for all $z \in \mathcal{Z}$, i.e., $\delta(\mathcal{A}(\varepsilon,\eta)) = 0$.

Definition 11 [27]. *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in an N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$. Choose $\varepsilon \in (0,1)$ and $\eta > 0$. Then, $\{l_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence if there exists $m_0 \in \mathbb{N}$ such that*

$$\Theta(l_n - l_m, z; \eta) > 1 - \varepsilon, \quad \vartheta(l_n - l_m, z; \eta) < \varepsilon, \quad \psi(l_n - l_m, z; \eta) < \varepsilon$$

for all $n, m \geq m_0$ and $z \in \mathcal{Z}$.

Definition 12 [27]. *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in an N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$, $\varepsilon > 0$, and $\eta > 0$. Then, $\{l_k\}_{k\in\mathbb{N}}$ is called a statistical Cauchy sequence if there exists $n_0 \in \mathbb{N}$ such that*

$$\lim_n \frac{1}{n}\big|\{k \leq n : \Theta(l_k - l_{n_0}, z; \eta) \leq 1 - \varepsilon \text{ or } \vartheta(l_k - l_{n_0}, z; \eta) \geq \varepsilon \text{ and } \psi(l_k - l_{n_0}, z; \eta) \geq \varepsilon\}\big| = 0$$

for every $z \in \mathcal{Z}$ or, equivalently, the natural density of the set

$$\mathcal{A}(\varepsilon, \eta) = \{k \leq n : \Theta(l_k - l_{n_0}, z; \eta) \leq 1 - \varepsilon \text{ or } \vartheta(l_k - l_{n_0}, z; \eta) \geq \varepsilon \text{ and } \psi(l_k - l_{n_0}, z; \eta) \geq \varepsilon\}$$

is zero; i.e., $\delta(\mathcal{A}(\varepsilon, \eta)) = 0$.

3. Main results

Throughout this section, $\mathcal{Z}$ and $\delta_2(\mathcal{A})$ stand for neutrosophic 2-normed space and double natural density of the set $\mathcal{A}$ respectively unless otherwise stated. First, We define the following:

Definition 13. *A double sequence $\{l_{mn}\}$ in an N2-NS $\mathcal{Z}$ is said to be convergent to $\xi \in \mathcal{Z}$ with respect to $\mathcal{N}_2$ if, for all $\sigma \in (0,1)$ and $u > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\Theta(l_{mn} - \xi, z; u) > 1 - \sigma, \quad \vartheta(l_{mn} - \xi, z; u) < \sigma, \quad \psi(l_{mn} - \xi, z; u) < \sigma$$

for all $m, n \geq n_0$ and nonzero $z \in \mathcal{Z}$; i.e.,

$$\lim_{m,n\to\infty} \Theta(l_{mn} - \xi, z; u) = 1, \qquad \lim_{m,n\to\infty} \vartheta(l_{mn} - \xi, z; u) = 0, \qquad \lim_{m,n\to\infty} \psi(l_{mn} - \xi, z; u) = 0.$$

In this case, we write

$$\mathcal{N}_2 - \lim_{m,n\to\infty} l_{mn} = \xi \quad \text{or} \quad l_{mn} \xrightarrow{\mathcal{N}_2} \xi.$$

Definition 14. *A double sequence $\{l_{mn}\}$ in an N2-NS $\mathcal{Z}$ is said to be statistically convergent to $\xi \in \mathcal{Z}$ with respect to $\mathcal{N}_2$ if, for all $\sigma \in (0,1)$, $u > 0$, and nonzero $z \in \mathcal{Z}$,*

$$\delta_2\big(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \Theta(l_{mn} - \xi, z; u) \leq 1 - \sigma \text{ or } \vartheta(l_{mn} - \xi, z; u) \geq \sigma \text{ and } \psi(l_{mn} - \xi, z; u) \geq \sigma\}\big) = 0$$

or, equivalently,

$$\lim_{i,j} \frac{1}{ij}\big|\{m \leq i, n \leq j : \Theta(l_{mn} - \xi, z; u) \leq 1 - \sigma \text{ or } \vartheta(l_{mn} - \xi, z; u) \geq \sigma \text{ and } \psi(l_{mn} - \xi, z; u) \geq \sigma\}\big| = 0.$$

In this case, we write

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi \quad \text{or} \quad l_{mn} \xrightarrow{st_2(\mathcal{N}_2)} \xi$$

and ξ is called an $st_2(\mathcal{N}_2)$-limit of $\{l_{mn}\}$.

Lemma 2. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. Then, for all $\sigma \in (0,1)$, $u > 0$, and nonzero $z \in \mathcal{Z}$, the following statements are equivalent:*

(1) $st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi$;
(2) $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u) \leq 1-\sigma\}) = \delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(l_{mn}-\xi, z; u)\geq\sigma\}) = \delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(l_{mn}-\xi, z; u) \geq \sigma\}) = 0$;
(3) $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u) > 1-\sigma,\ \vartheta(l_{mn}-\xi, z; u) < \sigma,\ \psi(l_{mn}-\xi, z; u) < \sigma\}) = 1$;
(4) $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u)>1-\sigma\}) = \delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(l_{mn}-\xi, z; u) < \sigma\}) = \delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(l_{mn}-\xi, z; u) < \sigma\}) = 1$;
(5) $st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} \Theta(l_{mn}-\xi, z; u) = 1$, $st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} \vartheta(l_{mn}-\xi, z; u) = 0$, *and* $st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} \psi(l_{mn}-\xi, z; u) = 0$.

Theorem 1. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. If*

$$\mathcal{N}_2 - \lim_{m,n\to\infty} l_{mn} = \xi,$$

then

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi.$$

P r o o f. Let

$$\mathcal{N}_2 - \lim_{m,n\to\infty} l_{mn} = \xi.$$

Then, for all $\sigma \in (0,1)$ and $u > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\Theta(l_{mn}-\xi, z; u) > 1-\sigma, \quad \vartheta(l_{mn}-\xi, z; u) < \sigma, \quad \text{and} \quad \psi(l_{mn}-\xi, z; u) < \sigma$$

for all $m,n \geq n_0$ and nonzero $z \in \mathcal{Z}$. So, the set

$$\big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u) \leq 1-\sigma \text{ or } \vartheta(l_{mn}-\xi, z; u) \geq \sigma \text{ and } \psi(l_{mn}-\xi, z; u) \geq \sigma\big\}$$

has at most finitely many terms. Since double natural density of a finite set is zero,

$$\delta_2\big(\big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u) \leq 1-\sigma \text{ or } \vartheta(l_{mn}-\xi, z; u) \geq \sigma \text{ and } \psi(l_{mn}-\xi, z; u) \geq \sigma\big\}\big) = 0.$$

Therefore,

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi.$$

This completes the proof. □

But in the general case, the converse to Theorem 1 does not have to be true, as shown in the following example.

Example 5. Let $\mathcal{Y} = \mathbb{R}^2$ with $\|x,y\| = |x_1y_2 - x_2y_1|$, where $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$. Define a continuous t-norm $\boxdot$ and a continuous t-conorm $*$ as $a \boxdot b = ab$ and $a * b = \min\{a+b, 1\}$ for $a,b \in [0,1]$, respectively. Take $\sigma \in (0,1)$, $x,y \in \mathcal{Y}$, and $u > 0$ such that $u > \|x,y\|$. Consider

$$\Theta(x,y;u) = \frac{u}{u+\|x,y\|}, \quad \vartheta(x,y;u) = \frac{\|x,y\|}{u+\|x,y\|}, \quad \psi(x,y;u) = \frac{\|x,y\|}{u}.$$

Then, $\mathcal{N}_2 = (\Theta, \vartheta, \psi)$ is a neutrosophic 2-norm on $\mathcal{Y}$ and the quadruple $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ becomes a neutrosophic 2-normed space. Define a double sequence $\{l_{mn}\} \in \mathcal{Z}$ by

$$l_{mn} = \begin{cases} (mn, 0), & m = s^2,\ n = t^2,\ s,t \in \mathbb{N}; \\ (0,0), & \text{otherwise.} \end{cases}$$

Then, for nonzero $z \in \mathcal{Z}$, we have

$$\begin{aligned}\mathcal{K}_{s,t}(\sigma,u) &= \big\{m \le s, n \le t : \Theta(l_{mn}, z; u) \le 1-\sigma \text{ or } \vartheta(l_{mn}, z; u) \ge \sigma \text{ and } \psi(l_{mn}, z; u) \ge \sigma\big\} \\ &= \Big\{m \le s, n \le t : \frac{u}{u + \|l_{mn}, z\|} \le 1-\sigma \text{ or } \frac{\|l_{mn}, z\|}{u + \|l_{mn}, z\|} \ge \sigma \text{ and } \frac{\|l_{mn}, z\|}{u} \ge \sigma\Big\} \\ &= \Big\{m \le s, n \le t : \|l_{mn}, z\| \ge \frac{u\sigma}{1-\sigma} \text{ or } \|l_{mn}, z\| \ge u\sigma\Big\} \\ &= \big\{m \le s, n \le t : l_{mn} = (mn, 0)\big\} \\ &= \big\{m \le s, n \le t : m = s^2,\ n = t^2,\ s,t \in \mathbb{N}\big\}\end{aligned}$$

and

$$\frac{1}{st}|\mathcal{K}_{s,t}(\sigma,u)| \le \frac{1}{st}\big|\big\{m \le s, n \le t : m = s^2,\ n = t^2,\ s,t \in \mathbb{N}\big\}\big| \le \frac{\sqrt{s}\sqrt{t}}{st} \to 0 \quad \text{as} \quad s,t \to \infty;$$

i.e.,

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = 0.$$

But $\{l_{mn}\}$ is not convergent with respect to $\mathcal{N}_2$.

Theorem 2. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. If $\{l_{mn}\}$ is statistically convergent with respect to $\mathcal{N}_2$, then an $st_2(\mathcal{N}_2)$-limit of $\{l_{mn}\}$ is unique.*

P r o o f. Suppose that

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi_1, \quad st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi_2,$$

where $\xi_1 \ne \xi_2$. Given $\sigma \in (0,1)$, choose $\lambda \in (0,1)$ such that

$$(1-\lambda) \boxdot (1-\lambda) > 1-\sigma, \quad \lambda * \lambda < \sigma.$$

Now, for all $u > 0$ and $z \in \mathcal{Z}$, we define the sets

$$\begin{aligned}\mathcal{A}_{\Theta 1}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn} - \xi_1, z; u/2) \le 1-\lambda\big\}, \\ \mathcal{A}_{\Theta 2}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn} - \xi_2, z; u/2) \le 1-\lambda\big\}, \\ \mathcal{A}_{\vartheta 1}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(l_{mn} - \xi_1, z; u/2) \ge \lambda\big\}, \\ \mathcal{A}_{\vartheta 2}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(l_{mn} - \xi_2, z; u/2) \ge \lambda\big\}, \\ \mathcal{A}_{\psi 1}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(l_{mn} - \xi_1, z; u/2) \ge \lambda\big\}, \\ \mathcal{A}_{\psi 2}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(l_{mn} - \xi_2, z; u/2) \ge \lambda\big\}.\end{aligned}$$

Since

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi_1, \quad st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi_2,$$

using Lemma 2, we get

$$\delta_2(\mathcal{A}_{\Theta 1}(\lambda,u)) = \delta_2(\mathcal{A}_{\vartheta 1}(\lambda,u)) = \delta_2(\mathcal{A}_{\psi 1}(\lambda,u)) = 0$$

and

$$\delta_2(\mathcal{A}_{\Theta 2}(\lambda,u)) = \delta_2(\mathcal{A}_{\vartheta 2}(\lambda,u)) = \delta_2(\mathcal{A}_{\psi 2}(\lambda,u)) = 0.$$

Now, let

$$\mathcal{A}_{\Theta,\vartheta,\psi}(\lambda,u)=[\mathcal{A}_{\Theta 1}(\lambda,u)\cup\mathcal{A}_{\Theta 2}(\lambda,u)]\cap[\mathcal{A}_{\vartheta 1}(\lambda,u)\cup\mathcal{A}_{\vartheta 2}(\lambda,u)]\cap[\mathcal{A}_{\psi 1}(\lambda,u)\cup\mathcal{A}_{\psi 2}(\lambda,u)].$$

Then, clearly, $\delta_2(\mathcal{A}_{\Theta,\vartheta,\psi}(\lambda,u))=0$; i.e., $\delta_2(\mathcal{A}^c_{\Theta,\vartheta,\psi}(\lambda,u))=1$.

Let $(p,q)\in\mathcal{A}^c_{\Theta,\vartheta,\psi}(\lambda,u)$. Then, the following three cases are possible.

Case i. If $(p,q)\in\mathcal{A}^c_{\Theta 1}(\lambda,u)\cap\mathcal{A}^c_{\Theta 2}(\lambda,u)$, then

$$\Theta(\xi_1-\xi_2,z;u)\geq\Theta(l_{pq}-\xi_1,z;u/2)\boxdot\Theta(l_{pq}-\xi_2,z;u/2)>(1-\lambda)\boxdot(1-\lambda)>1-\sigma.$$

Since $\sigma\in(0,1)$ is arbitrary, we have $\Theta(\xi_1-\xi_2,z;u)=1$, which yields $\xi_1=\xi_2$.

Case ii. If $(p,q)\in\mathcal{A}^c_{\vartheta 1}(\lambda,u)\cap\mathcal{A}^c_{\vartheta 2}(\lambda,u)$, then

$$\vartheta(\xi_1-\xi_2,z;u)\leq\vartheta(l_{pq}-\xi_1,z;u/2)*\vartheta(l_{pq}-\xi_2,z;u/2)<\lambda*\lambda<\sigma.$$

Since $\sigma\in(0,1)$ is arbitrary, we have $\vartheta(\xi_1-\xi_2,z;u)=0$, which yields $\xi_1=\xi_2$.

Case iii. If $(p,q)\in\mathcal{A}^c_{\psi 1}(\lambda,u)\cap\mathcal{A}^c_{\psi 2}(\lambda,u)$, then, similarly to Case *ii*, we get $\xi_1=\xi_2$.

Hence, an $st_2(\mathcal{N}_2)$-limit of $\{l_{mn}\}$ is unique. This completes the proof. □

Theorem 3. *Let $\mathcal{Y}$ be a real vector space, and let $\{l_{mn}\}$ and $\{w_{mn}\}$ be two double sequences in an N2-NS $\mathcal{Z}$. Then, the following statements hold*:

(1) *if $st_2(\mathcal{N}_2)-\lim_{m,n\to\infty}l_{mn}=\xi_1$ and $st_2(\mathcal{N}_2)-\lim_{m,n\to\infty}w_{mn}=\xi_2$, then*

$$st_2(\mathcal{N}_2)-\lim_{m,n\to\infty}l_{mn}+w_{mn}=\xi_1+\xi_2;$$

(2) *if $st_2(\mathcal{N}_2)-\lim_{m,n\to\infty}l_{mn}=\xi_1$ and $c\neq 0$, then $st_2(\mathcal{N}_2)-\lim_{m,n\to\infty}cl_{mn}=c\xi_1$.*

P r o o f. It is easy. So, we omit the details. □

Theorem 4. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. Then,*

$$st_2(\mathcal{N}_2)-\lim_{m,n\to\infty}l_{mn}=\xi$$

if and only if there exists a subset

$$\mathcal{K}=\{m_1<m_2<\cdots<m_p<\cdots;n_1<n_2<\cdots<n_q<\cdots\}\subset\mathbb{N}\times\mathbb{N}$$

such that $\delta_2(\mathcal{K})=1$ and $\mathcal{N}_2-\lim_{p,q\to\infty}l_{m_pn_q}=\xi$.

P r o o f. First, suppose that $st_2(\mathcal{N}_2)-\lim_{m,n\to\infty}l_{mn}=\xi$. Now, for all $u>0$, $k\in\mathbb{N}$, and nonzero $z\in\mathcal{Z}$, define

$$\mathcal{A}_{\mathcal{N}_2}(k,u)=\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:\Theta(l_{mn}-\xi,z;u)>1-\frac{1}{k},\ \vartheta(l_{mn}-\xi,z;u)<\frac{1}{k},\ \psi(l_{mn}-\xi,z;u)<\frac{1}{k}\Big\},\quad(3.1)$$

and

$$\mathcal{B}_{\mathcal{N}_2}(k,u)=\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:\Theta(l_{mn}-\xi,z;u)\leq 1-\frac{1}{k}\text{ or }\vartheta(l_{mn}-\xi,z;u)\geq\frac{1}{k}\text{ and }\psi(l_{mn}-\xi,z;u)\geq\frac{1}{k}\Big\}.$$

Then, clearly, $\mathcal{A}_{\mathcal{N}_2}(k+1,u) \subset \mathcal{A}_{\mathcal{N}_2}(k,u)$ and, by our assumption, we have $\delta_2(\mathcal{B}_{\mathcal{N}_2}(k,u)) = 0$.

Also, from (3.1), we get $\delta_2(\mathcal{A}_{\mathcal{N}_2}(k,u)) = 1$. Now, let us show that, for $(m,n) \in \mathcal{A}_{\mathcal{N}_2}(k,u)$,

$$\mathcal{N}_2 - \lim_{m,n\to\infty} l_{mn} = \xi.$$

Suppose that $\{l_{mn}\}_{(m,n)\in\mathcal{A}_{\mathcal{N}_2}(k,u)}$ is not convergent with respect to $\mathcal{N}_2$. Then, for some $\sigma \in (0,1)$, we have

$$\Theta(l_{mn}-\xi,z;u) \le 1-\sigma, \quad \vartheta(l_{mn}-\xi,z;u) \ge \sigma, \quad \psi(l_{mn}-\xi,z;u) \ge \sigma$$

except for at most finite number of terms $(m,n) \in \mathcal{A}_{\mathcal{N}_2}(k,u)$ and nonzero $z \in \mathcal{Z}$.

Define

$$\mathcal{C}_{\mathcal{N}_2}(\sigma,u) = \big\{(m,n)\in\mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi,z;u) > 1-\sigma \text{ and } \vartheta(l_{mn}-\xi,z;u) < \sigma,\ \psi(l_{mn}-\xi,z;u) < \sigma\big\},$$

where $\sigma > 1/k$. Clearly, $\delta_2(\mathcal{C}_{\mathcal{N}_2}(\sigma,u)) = 0$. Since $\sigma > 1/k$, we have $\mathcal{A}_{\mathcal{N}_2}(k,u) \subset \mathcal{C}_{\mathcal{N}_2}(\sigma,u)$ and, hence, $\delta_2(\mathcal{A}_{\mathcal{N}_2}(k,u)) = 0$, which contradicts $\delta_2(\mathcal{A}_{\mathcal{N}_2}(k,u)) = 1$. Therefore, for $(m,n) \in \mathcal{A}_{\mathcal{N}_2}(k,u)$, we have

$$\mathcal{N}_2 - \lim_{m,n\to\infty} l_{mn} = \xi.$$

Conversely, suppose that there exists a subset

$$\mathcal{K} = \{m_1 < m_2 < \cdots < m_p < \cdots ; n_1 < n_2 < \cdots < n_q < \cdots\} \subset \mathbb{N}\times\mathbb{N}$$

such that

$$\delta_2(\mathcal{K}) = 1, \quad \mathcal{N}_2 - \lim_{p,q\to\infty} l_{m_p n_q} = \xi.$$

Then, for all $\sigma \in (0,1)$ and $u > 0$, there exists $p_0 \in \mathbb{N}$ such that

$$\Theta(l_{m_p n_q}-\xi,z;u) > 1-\sigma, \quad \vartheta(l_{m_p n_q}-\xi,z;u) < \sigma, \quad \psi(l_{m_p n_q}-\xi,z;u) < \sigma$$

for all $p,q \ge p_0$ and nonzero $z \in \mathcal{Z}$. Therefore,

$$\big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi,z;u) \le 1-\sigma \text{ or } \vartheta(l_{mn}-\xi,z;u) \ge \sigma \text{ and } \psi(l_{mn}-\xi,z;u) \ge \sigma\big\}$$
$$\subset \mathbb{N}\times\mathbb{N} \setminus \{m_{p_0+1} < m_{p_0+2}, \ldots ; n_{p_0+1} < n_{p_0+2}, \ldots\}.$$

Hence,

$$\delta_2\big(\big\{(m,n)\in\mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi,z;u) \le 1-\sigma \text{ or } \vartheta(l_{mn}-\xi,z;u) \ge \sigma \text{ and } \psi(l_{mn}-\xi,z;u) \ge \sigma\big\}\big) = 0;$$

i.e., $st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi$. □

Definition 15. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$, $\sigma \in (0,1)$, and let $u > 0$. Then, $\{l_{mn}\}$ is called statistically Cauchy with respect to $\mathcal{N}_2$ if there exist $m_0 = m_0(\sigma)$ and $n_0 = n_0(\sigma) \in \mathbb{N}$ such that*

$$\delta_2\big(\big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-l_{m_0n_0},z;u) \le 1-\sigma \text{ or } \vartheta(l_{mn}-l_{m_0n_0},z;u) \ge \sigma$$
$$\text{and } \psi(l_{mn}-l_{m_0n_0},z;u) \ge \sigma\big\}\big) = 0$$

for nonzero $z \in \mathcal{Z}$.

Theorem 5. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. If*

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi,$$

then $\{l_{mn}\}$ is statistically Cauchy with respect to $\mathcal{N}_2$.

P r o o f. Let

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi$$

and $\sigma \in (0,1)$ be given. Choose $\lambda \in (0,1)$ such that

$$(1-\lambda) \boxdot (1-\lambda) > 1-\sigma, \quad \lambda * \lambda < \sigma.$$

Then, for $\lambda \in (0,1)$, $u > 0$, and nonzero $z \in \mathcal{Z}$, we have $\delta_2(\mathcal{A}_{\mathcal{N}_2}(\lambda, u)) = 0$, where

$$\mathcal{A}_{\mathcal{N}_2}(\lambda, u) = \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u/2) \le 1-\lambda \text{ or } \vartheta(l_{mn}-\xi, z; u/2) \ge \lambda \\ \text{and } \psi(l_{mn}-\xi, z; u/2) \ge \lambda\big\}.$$

Then, $\delta_2(\mathbb{N}\times\mathbb{N} \setminus \mathcal{A}_{\mathcal{N}_2}(\lambda, u)) = 1$. Let $(m_0, n_0) \in \mathcal{A}^c_{\mathcal{N}_2}(\sigma, u)$. So,

$$\Theta(l_{m_0n_0}-\xi, z; u/2) > 1-\lambda,\ \vartheta(l_{m_0n_0}-\xi, z; u/2) < \lambda \text{ and } \psi(l_{m_0n_0}-\xi, z; u/2) < \lambda.$$

Now, we define

$$\mathcal{B}_{\mathcal{N}_2}(\sigma, u) = \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-l_{m_0n_0}, z; u) \le 1-\sigma \text{ or } \vartheta(l_{mn}-l_{m_0n_0}, z; u) \ge \sigma \\ \text{and } \psi(l_{mn}-l_{m_0n_0}, z; u) \ge \sigma\big\}$$

for every nonzero $z \in \mathcal{Z}$. Let us show that $\mathcal{B}_{\mathcal{N}_2}(\sigma, u) \subset \mathcal{A}_{\mathcal{N}_2}(\lambda, u)$. Let $(p,q) \in \mathcal{B}_{\mathcal{N}_2}(\sigma, u)$. Then, we get

$$\Theta(l_{pq}-l_{m_0n_0}, z; u) \le 1-\sigma,\ \vartheta(l_{pq}-l_{m_0n_0}, z; u) \ge \sigma \text{ and } \psi(l_{pq}-l_{m_0n_0}, z; u) \ge \sigma.$$

Case i. Consider $\Theta(l_{pq}-l_{m_0n_0}, z; u) \le 1-\sigma$. Let us show that

$$\Theta(l_{pq}-\xi, z; u/2) \le 1-\lambda.$$

Suppose that

$$\Theta(l_{pq}-\xi, z; u/2) > 1-\lambda.$$

Then, we have

$$1-\sigma \ge \Theta(l_{pq}-l_{m_0n_0}, z; u) \ge \Theta(l_{pq}-\xi, z; u/2) \boxdot \Theta(l_{m_0n_0}-\xi, z; u/2) > (1-\lambda) \boxdot (1-\lambda) > 1-\sigma,$$

which is impossible. Therefore,

$$\Theta(l_{pq}-\xi, z; u/2) \le 1-\lambda.$$

Case ii. Consider $\vartheta(l_{pq}-l_{m_0n_0}, z; u) \ge \sigma$. Let us show that

$$\vartheta(l_{pq}-\xi, z; u/2) \ge \lambda.$$

Suppose that

$$\vartheta(l_{pq}-\xi, z; u/2) < \lambda.$$

Then, we have

$$\sigma \le \vartheta(l_{pq}-l_{m_0n_0}, z; u) \le \vartheta(l_{pq}-\xi, z; u/2) \boxdot \vartheta(l_{m_0n_0}-\xi, z; u/2) < \lambda * \lambda < \sigma,$$

which is impossible. Therefore, we have

$$\vartheta(l_{pq} - \xi, z; u/2) \geq \lambda.$$

Case iii. If we consider $\psi(l_{pq} - l_{m_0n_0}, z; u) \geq \sigma$, then, similarly to Case *ii*, we can show that

$$\psi(l_{pq} - \xi, z; u/2) \geq \lambda.$$

Therefore, $(p, q) \in \mathcal{A}_{\mathcal{N}_2}(\lambda, u)$. Hence, $\mathcal{B}_{\mathcal{N}_2}(\sigma, u) \subset \mathcal{A}_{\mathcal{N}_2}(\lambda, u)$. Since $\delta_2(\mathcal{A}_{\mathcal{N}_2}(\lambda, u)) = 0$, we have $\delta_2(\mathcal{B}_{\mathcal{N}_2}(\sigma, u)) = 0$. So, $\{l_{mn}\}$ is statistically Cauchy with respect to $\mathcal{N}_2$. □

Theorem 6. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. If $\{l_{mn}\}$ is statistically Cauchy with respect to $\mathcal{N}_2$, then it is statistically convergent with respect to $\mathcal{N}_2$.*

P r o o f. Suppose that $\{l_{mn}\}$ is statistically Cauchy with respect to $\mathcal{N}_2$ but not statistically convergent to any $\xi \in \mathcal{Z}$ with respect to $\mathcal{N}_2$. Then, for $\sigma \in (0, 1)$, $u > 0$, and nonzero $z \in \mathcal{Z}$, there exist $m_0 = m_0(\sigma)$ and $n_0 = n_0(\sigma) \in \mathbb{N}$ such that $\delta_2(\mathcal{K}) = 0$, where

$$\mathcal{K} = \big\{(m, n) \in \mathbb{N} \times \mathbb{N} : \Theta(l_{mn} - l_{m_0n_0}, z; u) \leq 1 - \sigma \text{ or } \vartheta(l_{mn} - l_{m_0n_0}, z; u) \geq \sigma \text{ and } \psi(l_{mn} - l_{m_0n_0}, z; u) \geq \sigma\big\},$$

and $\delta_2(\mathcal{M}) = 0$, where

$$\mathcal{M} = \big\{(m, n) \in \mathbb{N} \times \mathbb{N} : \Theta(l_{mn} - \xi, z; u/2) > 1 - \sigma \text{ or } \vartheta(l_{mn} - \xi, z; u/2) < \sigma \text{ and } \psi(l_{mn} - \xi, z; u/2) < \sigma\big\}.$$

Since

$$\Theta(l_{mn} - l_{m_0n_0}, z; u) \geq 2\Theta(l_{mn} - \xi, z; u/2) > 1 - \sigma$$

and

$$\vartheta(l_{mn} - l_{m_0n_0}, z; u) \leq 2\vartheta(l_{mn} - \xi, z; u/2) < \sigma,$$
$$\psi(l_{mn} - l_{m_0n_0}, z; u) \leq 2\psi(l_{mn} - \xi, z; u/2) < \sigma,$$

if

$$\Theta(l_{mn} - \xi, z; \frac{u}{2}) > \frac{1 - \sigma}{2}$$

and

$$\vartheta(l_{mn} - \xi, z; \frac{u}{2}) < \frac{\sigma}{2}, \quad \psi(l_{mn} - \xi, z; u) < \frac{\sigma}{2},$$

we have

$$\delta_2\big(\big\{(m, n) \in \mathbb{N} \times \mathbb{N} : \Theta(l_{mn} - l_{m_0n_0}, z; u) > 1 - \sigma \text{ and } \vartheta(l_{mn} - l_{m_0n_0}, z; u) < \sigma,\ \psi(l_{mn} - l_{m_0n_0}, z; u) < \sigma\big\}\big) = 0.$$

This gives $\delta_2(\mathcal{K}^c) = 0$ and so $\delta_2(\mathcal{K}) = 1$, a contradiction. Therefore, $\{l_{mn}\}$ is statistically convergent to some ξ. □

Definition 16. *An N2-NS $\mathcal{Z}$ is called statistically complete with respect to $\mathcal{N}_2$ if every statistically Cauchy sequence is statistically convergent with respect to $\mathcal{N}_2$.*

Remark 1. In the light of Theorems 5 and 6, we see that every N2-NS is statistically complete for double sequences.

Conclusion and future developments

In this paper, we have dealt with statistical convergent double sequences in an N2-NS and have shown that every N2-NS is statistically complete. Later on, these results may be the opening of new tools to generalize this notion in various directions such as $\mathcal{I}_2$-statistical and $\mathcal{I}_2$-lacunary statistical convergence with respect to $\mathcal{N}_2$. Also, this idea can be used in convergence-related problems in many branches of science and engineering.

Acknowledgements

We express great gratitude and deep respect to the reviewers and the managing editor for their valuable comments which improved the quality of the paper. The second author is grateful to The Council of Scientific and Industrial Research (CSIR), HRDG, India, for the award of Senior Research Fellow during the preparation of this paper.

REFERENCES

1. Atanassov K. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 1986. Vol. 20, No. 1 P. 87–96. DOI: 10.1016/S0165-0114(86)80034-3
2. Atanassov K. T., Gargov G. Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 1989. Vol. 31, No. 3. P. 343–349. DOI: 10.1016/0165-0114(89)90205-4
3. Bustince H., Burillo P. Vague sets are intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 1996. Vol. 79, No. 3. P. 403–405. DOI: 10.1016/0165-0114(95)00154-9
4. Barros L. C., Bassanezi R. C., Tonelli P. A. Fuzzy modelling in population dynamics. *Ecological modelling*, 2000. Vol. 128, No. 1. P. 27–33. DOI: 10.1016/S0304-3800(99)00223-9
5. Bera T., Mahapatra N. K. On neutrosophic soft linear spaces. *Fuzzy Inf. Eng.,* 2017. Vol. 9, No. 3. P. 299–324. DOI: 10.1016/j.fiae.2017.09.004
6. Bera T., Mahapatra N. K. Neutrosophic soft normed linear space. *Neutrosophic Sets Syst.,* 2018. Vol. 23. P. 52–71.
7. Christopher J. The asymptotic density of some k-dimensional sets. *Amer. Math. Monthly*, 1956. Vol. 63, No. 6. P. 399–401. DOI: 10.2307/2309400
8. Connor J. S. The statistical and strong p-Cesaro convergence of sequences. *Analysis*, 1988. Vol. 8, No. 1–2. P. 47–63. DOI: 10.1524/anly.1988.8.12.47
9. Çakallı H., Savaş E. Statistical convergence of double sequences in topological groups. *J. Comput. Anal. Appl.*, 2010. Vol. 12, No. 1A. P. 421–426.
10. Nuray F., Dündar E., Ulusu U. Wijsman statistical convergence of double sequences of sets. *Iran. J. Math. Sci. Inform.*, 2021. Vol. 16, No. 1. P. 55–64.
11. Debnath S., Debnath S., Choudhury C. On deferred statistical convergence of sequences in neutrosophic normed spaces. *Sahand Commun. Math. Anal.,* 2022. Vol. 19, No. 4. P. 81–96. DOI: 10.22130/scma.2022.544537.1031
12. Fast H. Sur la convergence statistique. *Colloq. Math.,* 1951. Vol. 2, No. 3–4. P. 241–244. (in French)
13. Fridy J. A. On statistical convergence. *Analysis*, 1985. Vol. 5, No. 4. P. 301–313. DOI: 10.1524/anly.1985.5.4.301
14. Fridy J. A., Orhan C. Lacunary statistical convergence. *Pacific J. Math.,* 1993. Vol. 160, No. 1. P. 43–51. DOI: 10.2140/pjm.1993.160.43
15. Fradkov A. L., Evans R. J. Control of chaos: Methods and applications in engineering. *Annu. Rev. Control*, 2005. Vol. 29, No. 1. P. 33–56. DOI: 10.1016/j.arcontrol.2005.01.001
16. Gähler S. 2-normed spaces. *Math. Nachr.,* 1964. Vol. 28, No. 1–2. P. 1–43. DOI: 10.1002/mana.19640280102 (in German)
17. Giles R. A computer program for fuzzy reasoning. *Fuzzy Sets and Systems*, 1980. Vol. 4, No. 3. P. 221–234. DOI: 10.1016/0165-0114(80)90012-3
18. Granados C., Dhital A. Statistical convergence of double sequences in neutrosophic normed spaces. *Neutrosophic Sets Syst.*, 2021. Vol. 42. P. 333–344.

19. Hong L., Sun J.-Q. Bifurcations of fuzzy nonlinear dynamical systems. *Commun. Nonlinear Sci. Numer. Simul.*, 2006. Vol. 11, No. 1. P. 1–12. DOI: 10.1016/j.cnsns.2004.11.001
20. Klement E. P., Mesiar R., Pap E. Triangular norms. Position paper I: basic analytical and algebraic properties. *Fuzzy Sets and Systems*, 2004. Vol. 143, P. 5–26. DOI: 10.1016/j.fss.2003.06.007
21. Kirişci M., Şimşek N. Neutrosophic normed spaces and statistical convergence. *J. Anal.,* 2020. Vol. 28, P. 1059–1073. DOI: 10.1007/s41478-020-00234-0
22. Kişi Ö. Ideal convergence of sequences in neutrosophic normed spaces. *J. Intell. Fuzzy Syst.,* 2021. Vol. 41, No. 3. P. 2581–2590. DOI: 10.3233/JIFS-201568
23. Khan V. A., Khan M. D., Ahmad M. Some new type of lacunary statistically convergent sequences in neutrosophic normed space. *Neutrosophic Sets Syst.*, 2021. Vol. 42, No. 1. P. 241–252. URL: https://digitalrepository.unm.edu/nss_journal/vol42/iss1/15
24. Madore J. Fuzzy physics. *Ann. Physics*, 1992. Vol. 219, No. 1. P. 187–198. DOI: 10.1016/0003-4916(92)90316-E
25. Mursaleen M. λ-statistical convergence. *Math. Slovaca*, 2000. Vol. 50, No. 1. P. 111–115. URL: https://dml.cz/handle/10338.dmlcz/136769
26. Mursaleen M., Edely O. H. H. Statistical convergence of double sequences. *J. Math. Anal. Appl.,* 2003. Vol. 288, No. 1. P. 223–231. DOI: 10.1016/j.jmaa.2003.08.004
27. Murtaza S., Sharma A., Kumar V. Neutrosophic 2-normed spaces and generalized summability. *Neutrosophic Sets Syst.,* 2023. Vol. 55, No. 1. Art. no. 25. P. 415–426. URL: https://digitalrepository.unm.edu/nss_journal/vol55/iss1/25
28. Nuray F., Savaş E. Statistical convergence of sequences of fuzzy numbers. *Math. Slovaca*, 1995. Vol. 45, No. 3. P. 269–273. URL: https://dml.cz/handle/10338.dmlcz/129143
29. Steinhaus H. Sur la convergence ordinaire et la convergence asymptotique. *Colloq. Math.,* 1951. Vol. 2, No. 1. P. 73–74. (in French)
30. Schweizer B., Sklar A. Statistical metric spaces. *Pacific J. Math.,* 1960. Vol. 10, No. 1. P. 313–334. DOI: 10.2140/pjm.1960.10.313
31. Šalát T. On statistically convergent sequences of real numbers. *Math. Slovaca*, 1980. Vol. 30, No. 2. P. 139–150. URL: https://dml.cz/handle/10338.dmlcz/136236
32. Smarandache F. Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. *Int. J. Pure. Appl. Math.,* 2005. Vol. 24, No. 3. P. 287–297. URL: https://www.ijpam.eu/contents/2005-24-3/1/1.pdf
33. Saadati R., Park J. H. On the intuitionistic fuzzy topological spaces. *Chaos Solitons Fractals*, 2006. Vol. 27, No. 2. P. 331–344. DOI: 10.1016/j.chaos.2005.03.019
34. Şahiner A., Gürdal M., Saltan S., Gunawan H. Ideal convergence in 2-normed spaces. *Taiwanese J. Math.,* 2007. Vol. 11, No. 5. P. 1477–1484. URL: https://www.jstor.org/stable/43834569
35. Sarabadan S., Talebi S. Statistical convergence of double sequences in 2-normed spaces. *Int. J. Contemp. Math. Sciences*, 2011. Vol. 6, No. 8. P. 373–380.
36. Turksen I. B. Interval valued fuzzy sets based on normal forms. *Fuzzy Sets and Systems*, 1986. Vol. 20, No. 2. P. 191–210. DOI: 10.1016/0165-0114(86)90077-1
37. Zadeh L. A. Fuzzy sets. *Inform. Control*, 1965. Vol. 8, No. 3. P. 338–353. DOI: 10.1016/S0019-9958(65)90241-X