**URAL MATHEMATICAL JOURNAL, Vol. 9, No. 2, 2023, pp. 4–27**

DOI: 10.15826/umj.2023.2.001

# APPROXIMATION OF DIFFERENTIATION OPERATORS BY BOUNDED LINEAR OPERATORS IN LEBESGUE SPACES ON THE AXIS AND RELATED PROBLEMS IN THE SPACES OF $(p, q)$-MULTIPLIERS AND THEIR PREDUAL SPACES[1]

**Vitalii V. Arestov**

Ural Federal University,
51 Lenin ave., Ekaterinburg, 620000, Russian Federation

vitalii.arestov@urfu.ru

**Abstract:** We consider a variant $E_{n,k}(N; r, r; p, p)$ of the four-parameter Stechkin problem $E_{n,k}(N; r, s; p, q)$ on the best approximation of differentiation operators of order $k$ on the class of $n$ times differentiable functions $(0 < k < n)$ in Lebesgue spaces on the real axis. We discuss the state of research in this problem and related problems in the spaces of multipliers of Lebesgue spaces and their predual spaces. We give two-sided estimates for $E_{n,k}(N; r, r; p, p)$. The paper is based on the author's talk at the S.B.Stechkin's International Workshop-Conference on Function Theory (Kyshtym, Chelyabinsk region, August 1–10, 2023).

**Keywords:** Differentiation operator, Stechkin's problem, Kolmogorov inequality, $(p, q)$-Multiplier, Predual space for the space of $(p, q)$-multipliers.

## 1. Introduction

### 1.1. Some notation

In this paper, we use the standard notation for the classical complex spaces of complex-valued measurable (in particular, continuous) functions of one variable on the real axis. Thus, $L_\gamma = L_\gamma(-\infty, \infty)$, $1 \le \gamma < \infty$, is the Lebesgue space of functions $f$ measurable on the real axis $\mathbb{R} = (-\infty, \infty)$ such that the function $|f|^\gamma$ is integrable over the axis; the space $L_\gamma$ is equipped with the norm

$$\|f\|_\gamma = \|f\|_{L_\gamma} = \left( \int |f(t)|^\gamma dt \right)^{1/\gamma};$$

hereinafter, we omit the integration set in integrals over the axis. The space $L_\infty = L_\infty(-\infty, \infty)$ consists of measurable essentially bounded functions on the axis; the space is equipped with the norm

$$\|f\|_\infty = \|f\|_{L_\infty} = \text{ess sup}\,\big\{|f(t)| \colon t \in (-\infty, \infty)\big\}.$$

The space $L_\infty = L_\infty(-\infty, \infty)$ contains the space $C = C(-\infty, \infty)$ of bounded continuous functions on the axis with the uniform norm

$$\|f\|_C = \sup\{|f(t)| \colon t \in (-\infty, \infty)\}.$$

Let $C_0 = C_0(-\infty, \infty)$ be the subspace of $C = C(-\infty, \infty)$ of functions vanishing at infinity. Denote by $V$ the space of (complex) bounded Borel measures on $(-\infty, \infty)$. We will identify this set with the set of (complex) functions $\mu$ of bounded variation on $(-\infty, \infty)$ such that values of their real

[1]This work was supported by the Russian Science Foundation, project no. 22-21-00526, https://rscf.ru/project/22-21-00526/ .

and imaginary parts at the discontinuity points are between the right-sided and left-sided limits. The norm in the space $V$ is the total variation $\bigvee \mu = \bigvee_{-\infty}^{\infty} \mu$ of a measure (a function) $\mu \in V$.

These spaces and their norms are invariant under the group of translations $\{\tau_h,\ h \in \mathbb{R}\}$ defined by the formula $(\tau_h f)(t) = f(t-h)$, $t \in \mathbb{R}$, as well as under the family of operators $\{\sigma_h,\ h \in \mathbb{R}\}$ given by the formula $(\sigma_h f)(t) = f(h-t)$, $t \in \mathbb{R}$. The operators of these two families are related as follows: $\sigma_h = \tau_h \sigma_0$, where $\sigma_0$ is the operator of changing the sign of a function argument: $(\sigma_0 f)(t) = f(-t)$, $t \in \mathbb{R}$.

We define the direct and inverse Fourier transforms of functions (at least from the space $L = L_1(\mathbb{R})$) by the formulas

$$\widehat{f}\,(t) = \int e^{-2\pi t\eta i} f(\eta)\, d\eta, \quad \widecheck{f}\,(t) = \int e^{2\pi t\eta i} f(\eta)\, d\eta = \widehat{f}\,(-t), \tag{1.1}$$

respectively. Properties of the Fourier transform can be found, for example, in [44, Ch. I, Sects. 1, 2].

Let $\mathscr{S}$ be the space of rapidly decreasing, infinitely differentiable functions on the axis, and let $\mathscr{S}'$ be the corresponding dual space of generalized functions (see, for example, [41, 42, 44]). The value of a functional $\theta \in \mathscr{S}'$ on a function $\phi \in \mathscr{S}$ will be denoted by $\langle \theta, \phi \rangle$. The space $\mathscr{S}'$ contains the set $\mathscr{L} = \mathscr{L}(\mathbb{R})$ of functions $f$ measurable and locally integrable on $\mathbb{R}$, and satisfying the condition

$$\int (1 + |t|)^d |f(t)| dt < \infty$$

with some exponent $d = d(f) \in \mathbb{R}$; functions $f \in \mathscr{L}$ are called slowly growing (classical) functions. A function $f \in \mathscr{L}$ is associated with a functional $f \in \mathscr{S}'$ by the formula

$$\langle f, \phi \rangle = \int f(t)\phi(t) dt, \quad \phi \in \mathscr{S}.$$

The convolution $\theta * \phi$ of an element $\theta \in \mathscr{S}'$ and a function $\phi \in \mathscr{S}$ is the function $y(\eta) = \langle \theta, \sigma_\eta \phi \rangle$. If $\theta \in \mathscr{L}$ is a classical function, then

$$(\theta * \phi)(\eta) = \int \theta(t)\phi(\eta - t)\, dt.$$

The Fourier transform $\widehat{\theta}$ of a functional $\theta \in \mathscr{S}'$ is a functional $\widehat{\theta} \in \mathscr{S}'$ acting by the formula $\langle \widehat{\theta}, \phi \rangle = \langle \theta, \widehat{\phi} \rangle$, $\phi \in \mathscr{S}$. If $\theta \in L_\gamma$, $1 \le \gamma \le 2$, then $\widehat{\theta} \in L_{\gamma'}$, $1/\gamma + 1/\gamma' = 1$; moreover, the Hausdorff–Young inequality $\|\widehat{\theta}\|_{\gamma'} \le \|\theta\|_\gamma$ holds (see, for example, [44, Ch. V, Sect. 1]).

### 1.2. Stechkin's problem on the best approximation of differentiation operators by bounded linear operators in Lebesgue spaces on the real axis

Let $r$, $s$, $p$, and $q$ be parameters satisfying the constraints $1 \le r, s, p, q \le \infty$. Let us agree that, for $r = \infty$, by $L_\infty = L_\infty(-\infty, \infty)$, we mean the space $C_0 = C_0(-\infty, \infty)$ of continuous functions on the axis vanishing at infinity. For $p = \infty$, by $L_\infty = L_\infty(-\infty, \infty)$, we mean the classical space of essentially bounded functions on the axis. For $s = \infty$ and $q = \infty$, by $L_\infty = L_\infty(-\infty, \infty)$, we mean the space $C = C(-\infty, \infty)$ of bounded continuous functions on the axis or even the space $C_0 = C_0(-\infty, \infty)$ of continuous functions vanishing at infinity depending on the situation; these situations will be stipulated.

For an integer $n \ge 1$, we define the space $W^n_{r,p}$ of functions $f \in L_r$ that are $n-1$ times continuously differentiable on the axis, their derivatives $f^{(n-1)}$ of order $n-1$ are locally absolutely continuous, and $f^{(n)} \in L_p$. In the space $W^n_{r,p}$, consider the class

$$Q^n_{r,p} = \bigl\{ f \in W^n_{r,p} \colon \|f^{(n)}\|_p \le 1 \bigr\}.$$

Denote by $\mathfrak{B}(L_r, L_s)$ the set of all bounded linear operators from $L_r$ to $L_s$, and let $\mathfrak{B}(N; L_r, L_s)$ for $N > 0$ be the set of operators $T \in \mathfrak{B}(L_r, L_s)$ with the norm $\|T\|_{L_r \to L_s} \le N$. Let $0 \le k < n$ be an integer and $k > 0$ if $r = s$. For an operator $T \in \mathfrak{B}(L_r, L_s)$, define

$$U(T) = \sup \big\{ \|f^{(k)} - Tf\|_q \colon f \in Q^n_{r,p} \big\}.$$

If the difference $f^{(k)} - Tf$ does not belong to the space $L_q$, then we assume that $\|f^{(k)} - Tf\|_{L_q} = \infty$. For $N > 0$, the quantity

$$E(N) = E_{n,k}(N) = E_{n,k}(N; r, s; p, q) = \inf \big\{ U(T) \colon T \in \mathfrak{B}(N; L_r, L_s) \big\} \tag{1.2}$$

is the best approximation (in the space $L_q$) of the differentiation operator of order $k$ on the class $Q^n_{r,p}$ by the set of bounded linear operators $\mathfrak{B}(N; L_r, L_s)$. Stechkin's problem is to study quantity (1.2) and an extremal operator on which the infimum is attained in (1.2); we will call it problem (1.2), and sometimes the problem $E_{n,k}(N; r, s; p, q)$.

Problem (1.2) is a specific version of Stechkin's problem on the best approximation of an unbounded linear operator by bounded linear operators on a class of elements of a Banach space, which arose in his paper [46]. Problem (1.2) and its specific cases were studied by many mathematicians: S.B. Stechkin, L.V. Taikov, Yu.N. Subbotin, V.N. Gabushin, V.I. Berdyshev; V.M.Tikhomirov and his colleagues A.P. Buslaev and G.G. Magaril-Il'yaev; V.F. Babenko and his colleagues and students; V.V. Arestov, R.R. Akopyan, V.G. Timofeev, M.A. Filatova, E.E. Berdysheva, and a lot others; see, for example, the review papers [9, 15, 16] and the bibliography therein. Some specific results will be described in what follows.

Note some facts. Necessary and sufficient conditions for the finiteness of quantity (1.2) are known, see [26] ($s = q$) and [4] ($s \ne q$). Roughly speaking, these conditions are

$$s \ge r, \quad q \ge p. \tag{1.3}$$

More precisely, if conditions (1.3) are satisfied, then there exists $N_0 > 0$ such that $E(N) < \infty$ for $N \ge N_0$. If the problem parameters satisfy the constraints

$$k - \frac{1}{s} + \frac{1}{r} > 0, \quad n - k + \frac{1}{q} - \frac{1}{p} > 0, \tag{1.4}$$

then conditions (1.3) are necessary and sufficient for the quantity $E(N)$ to be finite for any $N > 0$. For a discussion of conditions (1.4), see [4].

Under conditions (1.3) and (1.4), we have the formula

$$E(N) = E(1)N^{-\gamma}, \quad \gamma = (n - k + 1/q - 1/p)/(k + 1/r - 1/s) > 0; \tag{1.5}$$

See [46] for the case $q = p = s = r = \infty$; in the general case, formula (1.5) is justified similarly, see, for example, [2].

The present paper considers problem (1.2) in the case $s = r$ and $q = p$, i.e., the variant $E_{n,k}(N; r, r; p, p)$. We review the results obtained so far in this version of the problem and related problems in multiplier spaces of Lebesgue spaces and predual spaces of multiplier spaces. In the last section of the paper, we give two-sided estimates for the value $E_{n,k}(N; r, r; p, p)$ in this variant of the problem.

### 1.3. Connection with the Kolmogorov inequality

Stechkin's problem (1.2) is related to several other extremal problems of function theory. Among them are the exact Kolmogorov inequalities for differentiable functions on the axis

$$\|f^{(k)}\|_{L_q} \le G \|f\|^{\alpha}_{L_r} \|f^{(n)}\|^{\beta}_{L_p}, \quad f \in W^n_{r,p}, \tag{1.6}$$

$$\alpha = (n - k - 1/p + 1/q)/(n - 1/p + 1/r), \quad \beta = 1 - \alpha.$$

Such inequalities were studied by G.H. Hardy, J.E. Littlewood, E. Landau, J. Hadamard, B. Szőkefalvi-Nagy, A.N. Kolmogorov, S.B. Stechkin, L.V. Taikov, V.N. Gabushin, V.I. Berdyshev, N.P. Kuptsov, A.P. Buslaev, G.G. Magaryl-Il'yaev, V.M. Tikhomirov, V.F. Babenko, etc. (see the bibliography in [9, 16, 17, 48]). V.N. Gabushyn found necessary and sufficient conditions for the existence of inequality (1.6), more precisely, for the finiteness of the constant $G = G(n, k; r, p, q)$ in (1.6). Namely, he proved [24] (see also [27, 28]) that $G < \infty$ if and only if

$$\frac{n-k}{r} + \frac{k}{p} \geq \frac{n}{q}. \tag{1.7}$$

S.B. Stechkin made an important observation that, in the classical case $s = q$, the value (1.2) and the best constant in (1.6) are related by the inequality

$$E_{n,k}(N; r, q; p, q) \geq \beta\alpha^{\alpha/\beta} G^{1/\beta} N^{-\alpha/\beta}, \quad N > 0. \tag{1.8}$$

To avoid discussing specific degenerate values of the parameters, we will assume that conditions (1.3) and (1.4) hold. Inequality (1.8) is a specific version of a more general statement and more general considerations by S.B. Stechkin contained in [46, Sect. 2].

As follows from (1.3) and (1.7), the conditions for the finiteness of the value $E(N) = E_{n,k}(N; r, q; p, q)$ of Stechkin's problem and the best constant $G = G(n, k; r, p, q)$ in (1.6) are different. Consequently, there are cases when $E(N) = \infty$ and $G < \infty$; in this situation, inequality (1.8) is strict. In the case $E(N) < \infty$, depending on the values of the parameters, both possibilities are realized: inequality (1.8) can turn into equality, and inequality (1.8) can be strict. A more informative discussion of this issue can be found in [9, Sect. 4].

Inequality (1.8) is an important tool for studying both the three-parameter version of problem (1.2) (the case $s = q$) and inequality (1.6). Indeed, an arbitrary specific function $f^* \in W^n_{r,p}$ estimates the best constant $G$ in inequality (1.6) from below. This, due to (1.8), gives a lower estimate for the quantity $E_{n,k}(N; r, q; p, q)$. A specific operator $T^* \in \mathfrak{B}(L_r, L_q)$ gives an upper estimate for the quantity $E_{n,k}(N; r, q; p, q)$: $E_{n,k}(N; r, q; p, q) \leq U(T^*)$ for $N = \|T^*\|$; in this case, it is not necessary to have the exact value of $U(T^*)$ but only again an upper estimate. If we managed to choose a function $f^*$ and an operator $T^*$ so that the obtained upper and lower estimates for the quantity $E(N)$ coincide, then we have a solution to both the problems. More precisely, we have exact values of $E(\|T^*\|)$ and the best constant $G$ in (1.6). Moreover, the operator $T^*$ is extremal in Stechkin's problem, and the function $f^*$ is extremal in the Kolmogorov inequality. Along this path, a solution to both problems was found in several new cases; see [9, Sect. 4] and the references therein.

The considerations just outlined are not universal in the study of problem (1.2). Firstly, inequality (1.8) can be strict and, therefore, in this case, it is impossible to obtain an exact lower estimate for $E_{n,k}(N; r, q; p, q)$. Secondly, in the four-parameter case $s \neq q$, there is no analog of inequality (1.8), at least in Lebesgue spaces.

In the study of Stechkin's problem, the property of the translation invariance of problem (1.2) occurs useful. The norms of spaces, the class $Q^n_{r,p}$, and the approximated differentiation operator $D^k = d^k/dt^k$ are invariant under the translation group $\{\tau_h\}$; precisely in this sense, we say that problem (1.2) is translation invariant. Due to this property, in problem (1.2), we can restrict ourselves to approximating operators $T$ that are also translation invariant; details can be found in [4–6, 8, 9]. This property makes it possible to solve Stechkin's problem in some cases (in particular, for $s \neq q$) and, which is no less essential, expands the environment of the problem. It is these issues that most of this paper is devoted to.

The property of invariance of approximating operators in Stechkin's problem and related problems in spaces of periodic functions was obtained and applied in the study of these problems by B.E. Klotz [33, 34].

## 2. Translation invariance of Stechkin's problem

In this section, we present some properties of spaces of bounded linear operators in Lebesgue spaces on the axis that are translation invariant; in particular, we describe their predual spaces.

### 2.1. The space of translation invariant bounded operators

For $1 \le p, q \le \infty$, denote by $\mathfrak{T}_{p,q} = \mathfrak{T}_{p,q}(\mathbb{R})$ the set of bounded linear operators from $L_p = L_p(\mathbb{R})$ to $L_q = L_q(\mathbb{R})$ that are invariant under (any) translation, i.e., such that $\tau_h T = T\tau_h$ on $L_p$ for all $h \in \mathbb{R}$. Extensive research has been devoted to the properties of invariant bounded operators (see [32, 37, 44] and the references therein). It is known (see, for example, [32, Theorem 1.1]) that if $p > q$, then, for $p < \infty$, the set $\mathfrak{T}_{p,q}$ consists only of the operator $T \equiv 0$, and, for $p = \infty$, the restriction of an operator $T \in \mathfrak{T}_{\infty,q}$ to the set $(L_\infty)_0$ of functions from $L_\infty$ having zero limit at infinity is the zero operator. In this regard, when discussing the properties of bounded invariant operators in what follows, we will assume that $1 \le p \le q \le \infty$.

In a joint paper [23], Figà-Talamanca and Gaudry (1967) proved that, for $1 \le p \le q < \infty$, the space $\mathfrak{T}_{p,q}(G)$ of bounded linear operators from $L_p(G)$ to $L_q(G)$ on a locally compact Abelian group $G$ invariant under translation (more precisely, under the group operation) is the conjugate space for a function space $A_{p,q}(G)$ constructively described by them. More precisely, in [23], function spaces $A_{p,q}(G)$ were constructed such that the space $\mathfrak{T}_{p,q}(G)$ of invariant operators is isometrically isomorphic to the dual space $A^*_{p,q}(G)$, in short, $\mathfrak{T}_{p,q}(G) = A^*_{p,q}(G)$. Two years earlier (in 1965), Figa-Talamanca [22] obtained a similar result for the case $1 < q = p < \infty$.

Let $X$ and $Y$ be a pair of normed linear spaces such that $Y$ is the conjugate space of $X$, i.e., $X^* = Y$. In this case, we say that $X$ is the predual space of $Y$. In this terminology, the results of [22] and [23] mean that the spaces $A_{p,q}(G)$ (for $1 \le p \le q < \infty$) are predual of the spaces $\mathfrak{T}_{p,q}(G)$.

The results of [22] and [23] are valid, in particular, for the spaces $\mathfrak{T}_{p,q}(\mathbb{R})$ of bounded linear operators from the space $L_p(\mathbb{R})$ to the space $L_q(\mathbb{R})$ invariant under the group of translations $\tau_h$, $h \in \mathbb{R}$. So, for $1 \le p \le q < \infty$, the spaces $\mathfrak{T}_{p,q}(\mathbb{R})$ are conjugate spaces of the spaces $A_{p,q} = A_{p,q}(\mathbb{R})$ constructed in [22] and [23]; i.e., $A_{p,q}$ are their predual.

In the author's papers (see [12, 14] and the references therein), a function space $F_{p,q} = F_{p,q}(\mathbb{R}) \subset L_r(\mathbb{R})$ was constructed which is the predual space of the space $\mathfrak{T}_{p,q} = \mathfrak{T}_{p,q}(\mathbb{R})$ of translation invariant bounded linear operators from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$. It is described in terms different from [22, 23], however, (for $1 \le p \le q < \infty$) it coincides, more precisely, is isometrically isomorphic to the space $A_{p,q}(\mathbb{R})$ of Figá-Talamanca and Gaudry [23]. The space $F_{p,q}$ will be described and used in what follows.

#### 2.1.1. The space of $(p,q)$-multipliers

Let us discuss some properties of bounded linear operators from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ that are invariant under (any) translation.

It is known (see [32, Theorem 1.2] or [44, Ch. I, Theorem 3.16]) that, if $q \ge p$, then an operator $T \in \mathfrak{T}_{p,q}$ on $\mathscr{S}$ has the form of the convolution with an element $\theta = \theta_T \in \mathscr{S}'$:

$$T\phi = \theta * \phi, \quad \phi \in \mathscr{S}.$$

The set $M_{p,q} = \{\theta_T \colon T \in \mathfrak{T}_{p,q}\} \subset \mathscr{S}'$ is a Banach space with respect to the norm

$$\|\theta_T\|_{M_{p,q}} = \|T\|_{L_p \to L_q}.$$

Elements $\theta \in M_{p,q}$, $1 \le p \le q \le \infty$ are often called $(p,q)$-multipliers.

In what follows, we always assume that $1 \le p \le q \le \infty$. Denote by $\rho$ a parameter chosen from the condition

$$1/p - 1/q = 1 - 1/\rho; \tag{2.1}$$

we have $1 \le \rho \le \infty$. It is known that if $\theta \in L_\rho$ and $x \in L_p$, then $\theta * x \in L_q$ and the Young inequality holds (see, for example, [44, Ch. V, Sect. 1]):

$$\|\theta * x\|_q \le \|\theta\|_\rho \|x\|_p. \tag{2.2}$$

This fact and inequality (2.2) imply the embedding

$$L_\rho \subset M_{p,q}, \quad \frac{1}{\rho} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$$

with the inequality $\|\theta\|_{M_{p,q}} \le \|\theta\|_\rho$, $\theta \in L_\rho$, for the norms of the elements.

Let us mention further known properties of the spaces $M_{p,q}$ (see, for example, [32, Sect. 1.2], [44, Ch. V, Sect. 1]). For two pairs of conjugate exponents $(p,q)$ and $(q',p')$, the equality

$$M_{p,q} = M_{q',p'}$$

holds together with the equality of the norms of the elements: $\|\theta\|_{M_{p,q}} = \|\theta\|_{M_{q',p'}}$, $\theta \in M_{p,q}$. From this and the Riesz–Thorin interpolation theorem (see, for example, [21, Ch. VI, Sect. 10, Theorem 11] or [44, Ch. V, Sect. 1, Theorem 1.16]), it follows that if

$$\frac{1}{\alpha} = \frac{1-t}{p} + \frac{t}{q'}, \quad \frac{1}{\beta} = \frac{1-t}{q} + \frac{t}{p'}, \quad 0 \le t \le 1,$$

then we have the embedding

$$M_{p,q} \subset M_{\alpha,\beta}$$

and the inequality

$$\|\theta\|_{M_{\alpha,\beta}} \le \|\theta\|_{M_{p,q}}, \quad \theta \in M_{p,q}.$$

A constructive description of multipliers is known only in several cases. The structure of the spaces $M_{2,2}$ and $M_{p,\infty} = M_{1,p'}$ is known; namely (see, for example, [32, Sect. 1.2] and [44, Ch. 1, Sect. 3]), the following equalities are valid (together with the equalities of the norms of the elements):

$$M_{2,2} = \widehat{L}_\infty = \{\widehat{\theta} \colon \theta \in L_\infty\},$$

$$M_{p,\infty} = M_{1,p'} = L_{p'} \quad \text{for} \quad 1 \le p < \infty,$$

$$M_{\infty,\infty} = M_{1,1} = V;$$

here, $V = V(\mathbb{R})$ is the space of (complex) bounded Borel measures on $\mathbb{R}$.

### 2.1.2. The predual space of the space of $(p, q)$-multipliers

This section describes function spaces $F_{p,q}$ constructed by the author in [14] and some of their properties. These spaces are predual of the spaces of multipliers $M_{p,q}$: $F^*_{p,q} = M_{p,q}$. The spaces $F_{p,q}$ are described in different terms compared to $A_{p,q}$ in [23], although, in fact, they are isometrically isomorphic [14, Theorem 3.2]. Here, as before, $1 \le p \le q \le \infty$. Let $\gamma$ be a parameter defined by the relation

$$1/\gamma = 1/p - 1/q; \tag{2.3}$$

for $\gamma = \infty$ (i.e., for $q = p$), we assume that $L_\gamma = C_0$. Comparing (2.3) with (2.1), we conclude that $\gamma = \rho'$.

On the set $\mathscr{S}$, we define the functional

$$\|\phi\|_{p,q} = \sup\{|\langle\theta, \phi\rangle| : \theta \in M_{p,q},\ \|\theta\|_{M_{p,q}} \le 1\}, \quad \phi \in \mathscr{S}. \tag{2.4}$$

Functional (2.4) on the set $\mathscr{S}$ is finite and is a norm [14, Lemma 2.1].

Let $F_{p,q} = F_{p,q}(\mathbb{R})$ be the completion of the space $\mathscr{S}$ with respect to the norm (2.4). For all $1 \le p \le q \le \infty$, the space $F_{p,q}$ is a function space; moreover, it is embedded in the space $L_\gamma$ [14, Lemma 2.3]:

$$F_{p,q} \subset L_\gamma \quad \text{and} \quad \|f\|_\gamma \le \|f\|_{F_{p,q}}, \quad f \in F_{p,q}. \tag{2.5}$$

Hereinafter, we use the notation $\|f\|_{p,q}$ for the norms $\|f\|_{F_{p,q}}$ of functions $f \in F_{p,q}$.

For the convenience of reference, we formulate as a separate lemma the following statement from [14, Lemma 2.5].

**Lemma 1.** *For specific values of the parameters, the space $F_{p,q}$ has the following properties.*

(1) *For $q = \infty$,*

$$\begin{gathered} F_{p,\infty} = F_{1,p'} = L_p, \quad 1 \le p < \infty, \\ F_{\infty,\infty} = F_{1,1} = C_0. \end{gathered} \tag{2.6}$$

(2) *For $q = p = 2$,*

$$F_{2,2} = \check{L} = \{f \in C_0 : \widehat{f} \in L\}, \quad \|f\|_{2,2} = \|\widehat{f}\,\|_L, \quad f \in F_{2,2}. \tag{2.7}$$

(3) *Let $q = p$ and $\overline{p} = \max\{p, p'\}$. The spaces $F_{p,p}$ do not decrease in $\overline{p}$; more exactly, if $2 \le \overline{p}_1 \le \overline{p}_2 \le \infty$, then*

$$F_{p_1,p_1} \subset F_{p_2,p_2} \quad \textit{and} \quad \|f\|_{p_2,p_2} \le \|f\|_{p_1,p_1}, \quad f \in F_{p_1,p_1}; \tag{2.8}$$

*in particular, for all $1 \le p \le \infty$,*

$$F_{p,p} \subset C_0 \quad \textit{and} \quad \|f\|_{p,p} \ge \|f\|_{C_0}, \quad f \in F_{p,p},$$

$$F_{2,2} \subset F_{p,p} \quad \textit{and} \quad \|f\|_{p,p} \le \|f\|_{2,2} = \|\widehat{f}\,\|_L, \quad f \in F_{2,2}.$$

The spaces $F_{p,q}$ that are predual spaces of the spaces of $(p, q)$-multipliers will sometimes be briefly called the *predual spaces*.

### 2.2. Two extremal problems related to Stechkin's problem (1.2) in the spaces of multipliers and their predual spaces

Let $r$, $s$, $p$, and $q$ be parameters satisfying the constraints $1 \le r \le s \le \infty$ and $1 \le p \le q \le \infty$. For integer $n \ge 1$, we define the space $\mathcal{W}^n_{r,s;p,q}$ of functions $f \in F_{r,s}$ that are $n-1$ times continuously differentiable on the axis, their derivatives $f^{(n-1)}$ of order $n-1$ are locally absolutely continuous, and $f^{(n)} \in F_{p,q}$. As a consequence of (2.5), we have the embedding $\mathcal{W}^n_{r,s;p,q} \subset W^n_{\gamma_1,\gamma_2}$, where $1/\gamma_1 = 1/r - 1/s$ and $1/\gamma_2 = 1/p - 1/q$.

In the space $\mathcal{W}^n_{r,s;p,q}$, consider the class

$$\mathcal{Q} = \mathcal{Q}^n_{r,s;p,q} = \big\{ f \in \mathcal{W}^n_{r,s;p,q} \colon \|f^{(n)}\|_{p,q} \le 1 \big\}.$$

On this class, consider a variant of Stechkin's problem on the best approximation of the functional $f^{(k)}(0)$ by the ball $M_{r,s}(N)$ of radius $N > 0$ in the space of multipliers $M_{r,s}$:

$$e(N) = e_{n,k}(N) = e_{n,k}(N; r, s; p, q) = \inf \big\{ u(\theta) \colon \theta \in M_{r,s},\ \|\theta\|_{r,s} \le N \big\}, \tag{2.9}$$

where

$$u(\theta) = u_{n,k}(\theta) = \sup \big\{ |f^{(k)}(0) - \langle \theta, f \rangle| \colon f \in \mathcal{Q}^n_{r,s;p,q} \big\}$$

is the deviation of a functional $\theta \in M_{r,s}$ from the functional $f^{(k)}(0)$ on the class $\mathcal{Q}$.

Problem (2.9) is associated with a multiplicative inequality of Kolmogorov type, but in the predual spaces:

$$\|f^{(k)}\|_C \le B_{n,k} \|f\|^{\alpha}_{r,s} \|f^{(n)}\|^{\beta}_{p,q}, \quad f \in \mathcal{W}^n_{r,s;p,q}, \tag{2.10}$$

$$\alpha = \frac{n - k + 1/q - 1/p}{n + 1/q - 1/p + 1/r - 1/s}, \quad \beta = 1 - \alpha = \frac{k + 1/r - 1/s}{n + 1/q - 1/p + 1/r - 1/s};$$

we assume that here $B_{n,k} = B_{n,k}(r, s; p, q)$ is the best (the smallest possible) constant (independent of the function $f$).

The following statement is contained in the author's paper [8, Theorem 3]; however, this result was preceded by several years of research by the author, see [4–6, 8] and [13, 14].

**Theorem 1.** *If $s \ge r \ge 1$, $q \ge p > 1$, and conditions (1.4) hold, then the following equality holds for any $N > 0$ for the values of problems (1.2) and (2.9) and the best constant $B$ in (2.10):*

$$E_{n,k}(N) = e_{n,k}(N) = \beta \alpha^{\alpha/\beta} B^{1/\beta}_{n,k} N^{-\alpha/\beta}. \tag{2.11}$$

*In addition, there is an extremal multiplier in problem (2.9); the convolution with this multiplier is an extremal operator of Stechkin's problem (1.2).*

## 3. Stechkin's problem and related problems in the case $s = r$, $q = p$

In this section, we will discuss Stechkin's problem (1.2) and the corresponding problems (2.9) and (2.10) with the following relationship between the parameters:

$$1 \le s = r \le \infty, \quad 1 \le q = p \le \infty. \tag{3.1}$$

These restrictions and restrictions (1.4) imply that $k > 0$, so from now on $0 < k < n$. Let us agree further in all situations instead of the set of parameters $r, r; p, p$ write $r; p$; so instead of $\mathcal{W}^n_{r,r;p,p}$ the notation $\mathcal{W}^n_{r;p}$ will be used.

In several cases when (3.1) holds, the exact solutions to all three problems are known; a review of the corresponding results will be given here.

For the convenience of further references, we repeat the definitions of the problem and of the quantities in problem (1.2) under restrictions (3.1):

$$E(N) = E_{n,k}(N) = E_{n,k}(N; r; p) = \inf \{U(T) \colon T \in \mathfrak{B}(N; L_r, L_r)\}, \tag{3.2}$$

$$U(T) = \sup \{\|f^{(k)} - Tf\|_p \colon f \in Q^n_{r,p}\}. \tag{3.3}$$

In this case, inequality (2.10) has the form

$$\|f^{(k)}\|_C \le B_{n,k} \|f\|^{\alpha}_{r,r} \|f^{(n)}\|^{\beta}_{p,p}, \quad f \in \mathcal{W}^n_{r;p}, \tag{3.4}$$

$$\alpha = \frac{n-k}{n}, \quad \beta = 1 - \alpha = \frac{k}{n}; \tag{3.5}$$

here, $B_{n,k} = B_{n,k}(r; p)$ is the best (the least possible) constant (independent of the function $f$). Note that indices (3.5) in inequality (3.4) are independent of the parameters $r$ and $p$.

Restrictions (3.1) contain in particular the two sets of parameters $s = r = q = p = \infty$ and $s = r = q = p = 2$, which the study of Stechkin's problem (1.2) began with. As we will see below, these two cases are, in a sense, "extreme" in set (3.1). These two cases are discussed in the subsequent two sections.

### 3.1. The classical variant of Stechkin's problem

Problem (1.2) was first studied by Stechkin in the uniform norm on the axis and semi-axis, see [46] and an earlier paper [45].

We will denote by $E_{n,k}(N; C)$, along with $E_{n,k}(N; \infty; \infty)$, problem (1.2) and the value of this problem in the uniform norm on the axis; more exactly, for

$$s = r = q = p = \infty.$$

As already noted above in Section 1.3 (see inequality (1.8)), Stechkin found out that the problem $E_{n,k}(N; C)$ is related to the exact inequality

$$\|f^{(k)}\|_C \le C_{n,k} \|f\|_C^{(n-k)/n} \, \|f\|^{k/n}_{L_\infty}, \quad f \in W^n_{\infty,\infty}, \tag{3.6}$$

between the norms of derivatives of differentiable functions. Namely, Stechkin showed [46] that the smallest constant $C_{n,k}$ in (3.6) gives an estimate from below of the value $E_{n,k}(N; C)$ (see (1.8)). It turned out later that this estimate is in fact an equality:

$$E_{n,k}(N; C) = k \left(\frac{C_{n,k}}{n}\right)^{n/k} \left(\frac{N}{n-k}\right)^{-(n-k)/k}, \quad N > 0. \tag{3.7}$$

This fact is a consequence of Domar's result [20] and of a more general result by Gabushin [25] on the best approximation of unbounded functionals by bounded ones.

Inequality (3.6) with a certain finite constant was obtained and used by Hardy and Littlewood in 1912 [30]. The exact inequality (3.6), i.e., the inequality with the best constant was first obtained in 1914 by Hadamard [29] for $n = 2$ and $k = 1$; and by Shilov in 1937 [18] for $n = 3, 4$ for all $1 \le k < n$ and for $n = 5$ and $k = 2$. In 1939, Kolmogorov found [35] the exact constant in inequality (3.6) for all $1 \le k < n$ using an elegant comparison theorem. Kolmogorov's result is very striking and important in this topic; in this regard, inequality (3.6) and more general inequalities (1.6) on the axis and semi-axis are often called *Kolmogorov inequalities.*

The Favard–Akhiezer–Krein function

$$f_n(t) = \frac{4}{\pi}\sum_{\ell=0}^{\infty}\frac{\sin\left((2\ell+1)t - n\pi/2\right)}{(2\ell+1)^{n+1}} \tag{3.8}$$

is extremal in inequality (3.6) [35]. For the properties of this function, see, for example, [36, Ch. 5, Sect. 5.4]. The uniform norm of function (3.8) has the following value:

$$K_n = \|f_n\|_C = \frac{4}{\pi}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell(n+1)}}{(2\ell+1)^{n+1}}.$$

For all $1 \le k \le n$, we have the relation $f_n^{(k)} = f_{n-k}$; in particular, $f_n^{(n)}(t) = f_0(t) = \operatorname{sign}\sin t$. The extremal function (3.8) in inequality (3.6) and its properties listed imply the following formula for the best constant in (3.6):

$$C_{n,k} = K_{n-k}\,(K_n)^{-(n-k)/n}; \tag{3.9}$$

for this value, the estimates $1 < C_{n,k} < \pi/2$ hold [35, (3)].

Stechkin proved [45, 46] that the following classical (difference) operators $T_h^{n,k}$ are extremal in the problem $E_{n,k}(N;C)$ for $n = 2$ and 3 and $1 \le k < n$:

$$(T_{2,1}^h f)(t) = (T_{3,1}^h f)(t) = \frac{f(t+h) - f(t-h)}{2h}, \quad N = h^{-1}, \tag{3.10}$$

$$(T_{3,2}^h f)(t) = \frac{f(t+h) - 2f(t) + f(t-h)}{h^2}, \quad N = \frac{4}{h^2}.$$

For $n = 4$ and 5, the solution to this case of problem (1.2) was found (1967) by Arestov [1], and for an arbitrary $n \ge 6$ by Buslaev [19]. For $n \ge 4$, the extremal operators are infinite difference operators with uniform nodes. More precisely, for example, for $k = 1$, the extremal operator has the form

$$T_{n,1}f(t) = h^{-1}\sum_{\ell=0}^{\infty}\alpha_\ell\big(f(t+(2\ell+1)h) - f(t-(2\ell+1)h)\big).$$

The sequence $\{\alpha_\ell\}_{\ell\ge 0}$ is the sum of several geometric progressions. To prove the results, we used the lower estimate (3.7) and the exact Kolmogorov inequality (3.6).

According to the results of Stechkin [46], Arestov [1], and Buslaev [19], in the classical version of Stechkin's problem $E_{n,k}(N;C)$, there is an extremal operator $T_{n,k}^* = T_{n,k}^*(N)$, which is a finite difference operator for $n = 2$ and 3 and infinite difference with a uniform step for $n \ge 4$. The norm of this operator in the space $C$ and the deviation value (3.3) have the following extremal values:

$$\|T_{n,k}^*\|_{C\to C} = N; \quad U_{n,k}(T_{n,k}^*; C) = E_{n,k}(N;C).$$

The operator $T_{n,k}^*$ is bounded linear in the spaces $L_r$ for all $1 \le r < \infty$, and

$$\|T_{n,k}^*\|_{L_r\to L_r} \le N.$$

Let us discuss the corresponding inequality (3.4). According to (2.6), the space $\mathcal{W}_{\infty;\infty}^n = \mathcal{W}_{\infty,\infty;\infty,\infty}^n$ consists of functions $f \in C_0$ continuously differentiable $n$ times on the axis, for which $f^{(n)} \in C_0$. Inequality (3.4) in this case coincides with inequality (3.6) on a narrower space $\mathcal{W}_{\infty;\infty}^n$; inequality (3.6) with constant (3.9) remains exact on $\mathcal{W}_{\infty;\infty}^n$.

## 3.2. Approximation of the differentiation operator in the space $L_2$ and related problems

### 3.2.1. Approximation of the differentiation operator in the space $L_2$

A version of the Stechkin problem on the best approximation of the differentiation operator in the space $L_2(-\infty,\infty)$ (i.e., problem (3.2) for $s=r=q=p=2$) was solved by Subbotin and Taikov [47] back in 1968. They proved the following formula for the best approximation value $E_{n,k}(N;L_2)$:

$$E_{n,k}(N_{n,k}(h);L_2)=\frac{k}{n}h^{n-k},\quad N_{n,k}(h)=\frac{n-k}{n}h^{-k},\quad h>0. \tag{3.11}$$

The extremal operator they constructed will be discussed below. The proof of (3.11) used Stechkin's lower estimate (1.8). The corresponding exact inequality (1.6) in this case has the form

$$\|f^{(k)}\|_{L_2}<\|f\|_{L_2}^{(n-k)/n}\|f^{(n)}\|_{L_2}^{k/n},\quad f\in W^n_{2,2},\quad f\not\equiv 0. \tag{3.12}$$

A proof of inequality (3.12) for $n=2$ and $k=1$ see in [31, Ch. VII, Theorem 261]; the general case is proved similarly.

To prove (3.11), Subbotin and Taikov [47] constructed an extremal operator $T^h_{n,k}$, $h>0$. This operator is a convolution:

$$\widehat{T^h_{n,k}f}=\lambda\cdot\widehat{f},\quad f\in L^2,$$

in which the multiplier $\lambda=\lambda_h$ is defined by the formulas

$$\begin{gathered}\lambda(\eta)=i^k\left((2\pi\eta)^k-\frac{k}{n}h^{n-k}(2\pi\eta)^n\operatorname{sign}\eta^{n-k}\right),\quad |\eta|\le\frac{1}{2\pi h}\left(\frac{n}{k}\right)^{1/(n-k)},\\ \lambda(\eta)=0,\quad |\eta|>\frac{1}{2\pi h}\left(\frac{n}{k}\right)^{1/(n-k)}.\end{gathered} \tag{3.13}$$

Note that function (3.13) differs from the multiplier of [47] by a change of a variable; this is because the definition of the Fourier transform adopted here differs from that used in [47] by a factor of $-2\pi$ in the exponent.

### 3.2.2. The space $\mathcal{W}^n_{2;2}$

Before considering inequality (2.10) and problem (2.9) in the case $s=r=q=p=2$, we discuss the properties of functions from the space $\mathcal{W}^n_{2;2}$.

**Lemma 2.** *The space $\mathcal{W}^n_{2;2}$ consists of functions $f\in C_0$ that can be represented in the form*

$$f(t)=\check{x}(t)=\int e^{2\pi t\eta i}x(\eta)\,d\eta, \tag{3.14}$$

*where the function $x=\widehat{f}$ belongs to $L$ and has the property*

$$y(\eta)=(2\pi\eta i)^n x(\eta)\in L. \tag{3.15}$$

*Moreover,*

$$f^{(n)}(t)=\check{y}(t)=\int e^{2\pi t\eta i}(2\pi\eta i)^n x(\eta)\,d\eta.$$

P r o o f. The space $\mathcal{W}^n_{2;2}$ is formed by functions $f \in F_{2,2}$ such that $f^{(n)} \in F_{2,2}$. The derivative $f^{(n)}$ is understood in the sense of the theory of generalized functions, see, for example, [42, 44]. Namely, for a pair of functions $f, g \in \mathscr{L} = \mathscr{L}(\mathbb{R})$, it is assumed that $g = f^{(n)}$ if the following equality holds for all functions $\phi \in \mathscr{S}$:

$$\int g(t)\phi(t)dt = (-1)^n \int f(t)\phi^{(n)}(t)dt. \tag{3.16}$$

According to (2.7) and (1.1), a function $f \in F_{2,2}$ has the form (3.14). Its derivative $g = f^{(n)}$ has a similar form:

$$g\,(t) = \int e^{2\pi t\eta i} y(\eta)\, d\eta, \quad y \in L. \tag{3.17}$$

Substituting representations (3.14) and (3.17) into (3.16), we obtain

$$\int \phi(t) \int e^{2\pi t\eta i} y(\eta)\, d\eta dt = (-1)^n \int \phi^{(n)}(t) \int e^{2\pi t\eta i} x(\eta)\, d\eta dt.$$

We may change the orders of integration on both sides of this relation:

$$\int y(\eta) \int e^{2\pi t\eta i} \phi(t)\, dt\, d\eta = (-1)^n \int x(\eta) \int e^{2\pi t\eta i} \phi^{(n)}(t)\, dt\, d\eta. \tag{3.18}$$

Let us introduce the notation

$$\psi(\eta) = \check{\phi}(\eta) = \int e^{2\pi t\eta i} \phi(t)\, dt. \tag{3.19}$$

Together with the function $\phi$, the function $\psi$ also belongs to the space $\mathscr{S}$. Relation (3.19) implies that

$$\phi(\eta) = \widehat{\psi}(\eta) = \int e^{-2\pi t\eta i} \psi(t)\, dt. \tag{3.20}$$

Differentiate relation (3.20) $n$ times:

$$\phi^{(n)}(\eta) = \int e^{-2\pi t\eta i} (-2\pi t i)^n \psi(t)\, dt.$$

Hence, we conclude that

$$\widecheck{\phi^{(n)}}(\eta) = \int e^{2\pi t\eta i} \phi^{(n)}(t)\, dt = (-2\pi\eta i)^n \psi(\eta). \tag{3.21}$$

Substituting (3.21) and (3.19) into (3.18), we obtain

$$\int y(\eta)\psi(\eta)\, d\eta = (-1)^n \int x(\eta)(-2\pi\eta i)^n \psi(\eta)\, d\eta$$

and

$$\int \left(y(\eta) - (2\pi\eta i)^n x(\eta)\right) \psi(\eta)\, d\eta = 0, \quad \psi \in \mathscr{S}.$$

The Fourier transform, and therefore the inverse Fourier transform (3.19), is a bijection of $\mathscr{S}$ onto itself, and therefore $\psi$ in the last relation is an arbitrary function from $\mathscr{S}$. Hence,

$$y(\eta) - (2\pi\eta i)^n x(\eta) = 0, \qquad \text{a.e. on the axis.}$$

Property (3.15) is justified. Lemma 2 is proved. □

Consider now the corresponding inequality (3.4). It is convenient to study it in terms of Fourier transforms of functions $f \in \mathcal{W}^n_{2;2}$. Let us introduce the notation

$$Y^n = \widehat{\mathcal{W}^n_{2;2}} = \{x = \widehat{f} : f \in \mathcal{W}^n_{2;2}\} = \{x \in L\colon (2\pi ti)^n x \in L\}.$$

In terms of functions from the space $Y^n$, inequality (3.4) takes the following form in this case:

$$\|\check{x}^{(k)}\|_C \le B_{n,k}\,\|x\|_L^{(n-k)/n}\,\|(2\pi ti)^n x\|_L^{k/n}, \quad x \in Y^n. \tag{3.22}$$

Obviously, for every function $x \in Y^n$, the function $|x|$ also belongs to the space $Y^n$, and the function

$$\check{x}^{(k)}(t) = \int e^{2\pi t\eta i}(2\pi\eta i)^k x(\eta)\,d\eta$$

satisfies the relations

$$\|\check{x}^{(k)}\|_C \le \|\,\widecheck{|x|}^{(k)}\|_C = |\,\widecheck{|x|}^{(k)}(0)\,| = \|(2\pi ti)^k x\|_L.$$

Therefore, inequality (3.22) is equivalent to the inequality

$$\|(2\pi ti)^k x\|_L \le B_{n,k}\,\|x\|_L^{(n-k)/n}\,\|(2\pi ti)^n x\|_L^{k/n}, \quad x \in Y^n, \tag{3.23}$$

(with the same value of the best constant $B_{n,k}$).

### 3.2.3. Stechkin's problem in the space of multipliers $M_{2,2}$ and the corresponding inequality in the predual space

Consider now the corresponding variant of Stechkin's problem (2.9) on the best approximation of the functional

$$\check{x}^{(k)}(0) = \int (2\pi\eta i)^k x(\eta)d\eta$$

by the space of multipliers $M_{2,2}$. The class $Q^n_{2;2} \subset \mathcal{W}^n_{2;2}$ is correspond in $Y^n$ to the class of functions

$$\Theta^n_2 = \widehat{Q^n_{2;2}} = \big\{x \in L\colon (2\pi ti)^n x \in L,\ \|(2\pi ti)^n x\|_L \le 1\big\}.$$

As a result, we have the problem

$$e_{n,k}(N) = e_{n,k}(N;2;2) = \inf\big\{u(\theta)\colon \theta \in L_\infty,\ \|\theta\|_{L_\infty} \le N\big\}, \tag{3.24}$$

where

$$u(\lambda) = u_{n,k}(\lambda) = \sup\Big\{\Big|\int (2\pi\eta i)^k x(\eta)d\eta - \int \lambda(\eta)x(\eta)\,d\eta\Big|\colon x \in \Theta^n_2\Big\}.$$

The best upper estimate for value (3.24) is given by multiplier (3.13). The relevant properties of this multiplier are summarized in the following lemma; all of them are available in [47].

**Lemma 3.** *The following two statements are valid for function* (3.13).

(1) *Function* (3.13) *is continuous and bounded on the axis, and*

$$\|\lambda\|_{C(-\infty,\infty)} = \big|\lambda(\pm(2\pi h)^{-1})\,\big| = h^{-k}\frac{n-k}{n}. \tag{3.25}$$

(2) *The function*

$$\Delta(\eta) = \frac{(2\pi\eta i)^k - \lambda(\eta)}{(2\pi\eta i)^n} \tag{3.26}$$

*belongs to the space* $L_\infty(-\infty,\infty)$, *and*

$$\|\Delta\|_{L_\infty(-\infty,\infty)} = \frac{k}{n}h^{n-k}. \tag{3.27}$$

P r o o f. The continuity, boundedness, and property (3.25) for function (3.13) are rather evident.

Let us now study function (3.26). For

$$0 < |\eta| \le \frac{1}{2\pi h}\left(\frac{n}{k}\right)^{1/(n-k)},$$

we have

$$\Delta(\eta) = i^k \frac{k}{n}\frac{h^{n-k}(2\pi\eta)^n \mathrm{sign}\,\eta^{n-k}}{(2\pi\eta i)^n} = \frac{k}{n}h^{n-k}\mathrm{sign}\,\eta^{n-k} i^{k-n}.$$

In the case when

$$|\eta| \ge \frac{1}{2\pi h}\left(\frac{n}{k}\right)^{1/(n-k)},$$

we have

$$\Delta(\eta) = \frac{(2\pi\eta i)^k}{(2\pi\eta i)^n} = \frac{1}{(2\pi\eta i)^{n-k}};$$

hence,

$$|\Delta(\eta)| \le \frac{k}{n}h^{n-k}, \quad |\eta| \ge \frac{1}{2\pi h}\left(\frac{n}{k}\right)^{1/(n-k)}.$$

This implies property (3.27) of function (3.26). Lemma 3 is proved. □

The following statement is contained in equality (3.11), inequality (3.12), Lemma 2, and Theorem 1. However, its proof will be given here. This proof largely repeats that of statement (3.11) in [47].

**Theorem 2.** *The following statements are valid for value* (3.24) *and the best constant* $B_{n,k}$ *in inequality* (3.23) *for* $0 < k < n$.

(1) *For all* $h > 0$,

$$e_{n,k}(N_{n,k}(h)) = \frac{k}{n}h^{n-k}, \quad N_{n,k}(h) = h^{-k}\frac{n-k}{n}, \quad h > 0; \tag{3.28}$$

*and functional* (3.13) *is extremal.*

(2) *The best constant in inequality* (3.23) *is one*:

$$B_{n,k} = 1. \tag{3.29}$$

P r o o f. (1) First, we obtain an upper estimate for the value $e_{n,k}(N)$. To do this, we use multiplier (3.13). Relations (3.25) and (3.27) imply the following upper estimate for $e_{n,k}(N)$:

$$e_{n,k}(N_{n,k}(h)) \le \frac{k}{n}h^{n-k}, \quad N_{n,k}(h) = h^{-k}\frac{n-k}{n}, \quad h > 0. \tag{3.30}$$

(2) Let us now obtain a lower estimate for the best constant $B_{n,k}$ in inequality (3.23). We start with the function

$$f(t) = e^{2\pi ti} = \int e^{2\pi t\eta i}d\mu(\eta);$$

here $\mu$ is the measure on the axis, which can be written as $d\mu(\eta) = \delta(\eta-1)\,d\eta$, where $\delta$ is the Dirac $\delta$-function. For $\rho > 0$, we define a function $x_\rho$ on the axis by the relation

$$x_\rho(\eta) = \begin{cases} \dfrac{1}{\rho}, & \eta \in [1, 1+\rho]\,; \\ 0, & \eta \notin [1, 1+\rho]\,. \end{cases}$$

For this function, we have $\|x_\rho\|_L = 1$ and

$$\|(2\pi ti)^k x_\rho\|_L \to (2\pi)^k, \quad \|(2\pi ti)^n x_\rho\|_L \to (2\pi)^n \quad \text{as } \rho \to +0.$$

Substitute the function $x_\rho$ into inequality (3.23) and let $\rho \to +0$. As a result,

$$B_{n,k} \geq \frac{\|(2\pi ti)^k x_\rho\|_L}{\|x_\rho\|_L^{(n-k)/n} \|(2\pi ti)^n x_\rho\|_L^{k/n}} \to 1.$$

Thus, the following lower estimate holds for the best constant $B_{n,k}$ in inequality (3.23):

$$B_{n,k} \geq 1. \tag{3.31}$$

(3) Statement (2.11) and estimates (3.30) and (3.31) imply equalities (3.28) and (3.29). Theorem 2 is proved. □

Inequality (3.23) is an inequality of the Carlson type; the studies of V.I. Levin, F.I. Andrianov, and others were devoted to such inequalities in the middle of the last century, see [38], [31, Levin V.I., Stechkin S.B. Additions to the Russian edition], [3], and the references therein. For statements like Theorem 2 related to Carlson's inequalities, see [3].

In the previous two Sections 3.1 and 3.2, Stechkin's lower estimate (1.8) for the value of the best approximation of the differentiation operator in terms of the best constant in the corresponding Kolmogorov inequality was applied in the study of Stechkin's problem. At the time of studying Stechkin's problem, the exact constant in the corresponding inequalities (3.6) and (3.12) was known; moreover, inequality (1.8) gave an exact estimate for the value of the best approximation. In the next two Sections 3.3 and 3.4, Stechkin's problem will be discussed in situations where there is no corresponding inequality (1.8). A lower estimate for the best approximation will be based on the considerations of the translation invariance of Stechkin's problem; more precisely, the statements of Theorem 1 will be used.

### 3.3. Approximation in the uniform norm on the axis by operators bounded in the space $L_r$: the case $1 \leq s = r \leq \infty$, $p = q = \infty$

Here, we discuss Stechkin's problem (3.2) for values of the parameters

$$1 \leq s = r \leq \infty, \quad p = q = \infty. \tag{3.32}$$

For real $r$, $1 \leq r \leq \infty$, and integer $n \geq 1$, the space $W^n_{r,\infty}$ consists of functions $f \in L_r$ that are $n-1$ times continuously differentiable on the axis, their derivatives $f^{(n-1)}$ of order $n-1$ are locally absolutely continuous, and $f^{(n)} \in L_\infty$. In the space $W^n_{r,\infty}$, consider the class

$$Q^n_{r,\infty} = \{f \in W^n_{r,\infty} \colon \|f^{(n)}\|_{L_\infty} \leq 1\}.$$

Denote by $\mathfrak{B}(L_r)$ the set of all bounded linear operators in the space $L_r$. Let $\mathfrak{B}(N; L_r)$ for $N > 0$ be the set of operators $T \in \mathfrak{B}(L_r)$ with the norm $\|T\|_{L_r \to L_r} \leq N$. In this section, for $r = \infty$, we mean by $L_\infty$ the space $C = C(-\infty, \infty)$.

We are interested in the best approximation (in the space $C$) of the differentiation operator $D^k = d^k/dt^k$ on the class $Q^n_{r,\infty}$ by the set of bounded linear operators $\mathfrak{B}(N; L_r)$:

$$E_{n,k}(N) = E_{n,k}(N; r; \infty) = \inf\{U(T) \colon T \in \mathfrak{B}(N; L_r)\}, \quad N > 0, \tag{3.33}$$
$$U(T) = U_{n,k}(T; r; \infty) = \sup\{\|f^{(k)} - Tf\|_C \colon f \in Q^n_{r,\infty}\}.$$

The problem is to calculate value (3.33) and an extremal operator on which the infimum in (3.33) is attained.

### 3.3.1. Case $n \geq 3$, $1 \leq r \leq \infty$

Recall that

$$K_n = \|f_n\|_C = \frac{4}{\pi} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell(n+1)}}{(2\ell+1)^{n+1}} \tag{3.34}$$

is the uniform norm of the Favard–Akhiezer–Krein function (3.8). Define

$$\overline{K}_n = \frac{4}{\pi} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^{n+1}}. \tag{3.35}$$

This is, in a sense, the "norm" of the same function $f_n$ in the space $F_{2,2}$. Comparing (3.35) with (3.34), we see that $K_n \leq \overline{K}_n$; more exactly,

$$K_n = \overline{K}_n \text{ if } n \text{ is odd}; \quad K_n < \overline{K}_n \text{ if } n \text{ is even.}$$

The following two statements are valid [13] for the problem $E_{n,k}(N; L_r)$.

**Theorem 3.** *The following two-sided estimates for the value of problem* (3.33) *hold for all* $n \geq 2$, $1 \leq k < n$, *and* $1 \leq r \leq \infty$:

$$k \left(\frac{K_{n-k}}{n}\right)^{n/k} \left(\frac{N\,\overline{K}_n}{n-k}\right)^{-(n-k)/k} \leq E_{n,k}(N; r; \infty) \leq k \left(\frac{K_{n-k}}{n}\right)^{n/k} \left(\frac{N\,K_n}{n-k}\right)^{-(n-k)/k}. \tag{3.36}$$

**Theorem 4.** *The following statements hold in problem* (3.33) *for odd* $n \geq 3$ *and arbitrary* $k$, $1 \leq k < n$.

(1) *The following formula holds for value* (3.33) *independently of* $r$, $1 \leq r \leq \infty$:

$$E_{n,k}(N; r; \infty) = E_{n,k}(N, C) = k \left(\frac{K_{n-k}}{n}\right)^{n/k} \left(\frac{N\,K_n}{n-k}\right)^{-(n-k)/k}.$$

(2) *An operator* $T^*_{n,k} = T^*_{n,k}(N)$ *that is extremal in the problem* $E_{n,k}(N, C)$ *is also extremal in the problem* $E_{n,k}(N, L_r)$ *for all* $r$, $1 \leq r < \infty$.

### 3.3.2. Case $n = 2$, $r = 2$

For even $n \geq 2$ and $1 < r < \infty$, the statements of Theorem 4, generally speaking, no longer hold. The author's paper [11] provides a solution to problem (3.33) for

$$n = 2 \quad (k = 1); \quad r = s = 2, \quad p = q = \infty. \tag{3.37}$$

In this case, the first inequality in (3.36) is exact. More precisely, the following statement is true.

**Theorem 5.** *The following formula holds for values of the parameters* (3.37) *for all* $h > 0$:

$$E_{2,1}(N_{2,1}(h); 2; \infty) = \frac{\pi h}{4}, \quad N_{2,1}(h) = \frac{\pi^2}{2h} \left(4 \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^3}\right)^{-1}. \tag{3.38}$$

*An extremal operator in* (3.38) *is the singular convolution operator on the space* $L_2$ *defined by the formula*

$$(\Theta_h f)(t) = A(h) \int_0^{\pi h} (f(t+u) - f(t-u))\, y\left(uh^{-1}\right)\, du,$$

*where*

$$y(u) = \frac{\pi - u}{4 \sin u}, \quad u \in (0, \pi); \quad A(h) = h^{-2} \left(4 \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^3}\right)^{-1}.$$

For comparison, consider the result of Stechkin [46] for $n=2$, $k=1$, and $(q=p=)s=r=\infty$:

$$E_{2,1}(N_{2,1}(h);\infty;\infty)=\frac{\pi h}{4},\quad N_{2,1}(h)=h^{-1}.$$

An extremal operator is the difference operator (3.10):

$$(T^h_{2,1}f)(t)=\frac{f(t+h)-f(t-h)}{2h}.$$

### 3.3.3. Inequalities for values of the parameters (3.32) in predual spaces

In the case $1\le s=r\le\infty$ and $p=q=\infty$ under consideration, inequality (2.10) has the form

$$\|f^{(k)}\|_C\le B_{n,k}(r;\infty)\|f\|_{r,r}^{(n-k)/n}\big(\|f^{(n)}\|_\infty\big)^{k/n},\quad f\in\mathcal{W}^n_{r;\infty}. \tag{3.39}$$

For $s=r=\infty$, this is the classical variant (3.6) of the inequality between the uniform norms of derivatives studied by Kolmogorov. In the case $s=r=2$, inequality (3.39) takes the form

$$\|f^{(k)}\|_C\le B_{n,k}(2;\infty)\|\widehat{f}\|_1^{(n-k)/n}\big(\|f^{(n)}\|_\infty\big)^{k/n},\quad f\in\mathcal{W}^n_{2;\infty}. \tag{3.40}$$

The following inequality holds [11, 13] for the best constants in (3.39) and, in particular, in (3.40):

$$B_{n,k}(r;\infty)\le B_{n,k}(\infty;\infty)=C_{n,k},\quad 1\le r\le\infty; \tag{3.41}$$

recall that $C_{n,k}$ was defined in (3.9). For odd $n\ge 3$, we have the equality $B_{n,k}(r;\infty)=B_{n,k}(\infty)$, $1\le r\le\infty$, and the Favard–Akhiezer–Krein function $f_n$ (3.8) is extremal for all $r$.

For even $n\ge 2$, this is, generally speaking, no longer the case. At least for $n=2$ $(k=1)$ and $r=2$, the best constant in inequality (3.40) has the following value [11]:

$$B_{2,1}(2;\infty)=\frac{\pi}{2}\left(\frac{4}{\pi}\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)^3}\right)^{-1/2}, \tag{3.42}$$

and the Favard–Akhiezer–Krein function $f_2$ is extremal again. The following estimates hold for constant (3.42):

$$\sqrt{\frac{\pi}{2}}<B_{2,1}(2;\infty)<\sqrt{2} \tag{3.43}$$

(see details in [11]). According to Hadamard's result [29], the best constant in inequality (3.6) for $n=2$ and $k=1$ is $C_{2,1}=\sqrt{2}$. Consequently, the second inequality (3.43) means that $B_{2,1}(2;\infty)<B_{2,1}(\infty;\infty)=C_{2,1}$, so that inequality (3.41) is strict in this case.

Inequalities of type (3.40) containing the norms of intermediate and highest derivatives and the norm of the Fourier transform of functions, with norm parameters different from (3.40), also arose in the studies by Magaril-Il'yaev and Osipenko of extremal problems of recovering functions from information about their spectrum [39, 40].

## 3.4. Case $1\le s=r\le\infty$, $p=q=2$

Here we will discuss Stechkin's problem (3.2) studied in [5, 7, 10] for the parameter values

$$1\le s=r\le\infty,\quad p=q=2. \tag{3.44}$$

For real $r$, $1\le r\le\infty$, and integer $n\ge 1$, the space $W^n_{r,2}$ consists of functions $f\in L_r$ that are $n-1$ times continuously differentiable on the axis, their derivatives $f^{(n-1)}$ of order $n-1$ are locally absolutely continuous, and $f^{(n)} \in L_2$. Here, for $r = \infty$, we mean by $L_\infty$ the space $C = C(-\infty, \infty)$. In the space $W^n_{r,2}$, consider the class $Q^n_{r,2} = \{f \in W^n_{r,2} \colon \|f^{(n)}\|_{L_2} \le 1\}$. As was said above, $\mathfrak{B}(L_r)$ denotes the set of all bounded linear operators in the space $L_r$, and $\mathfrak{B}(N; L_r)$ for $N > 0$ is the set of operators $T \in \mathfrak{B}(L_r)$ with the norm $\|T\|_{L_r \to L_r} \le N$.

We are interested in the best approximation in the space $L_2$ of the differentiation operator $D^k$ on the class $Q^n_{r,2}$ by the set of bounded linear operators $\mathfrak{B}(N; L_r)$:

$$E_{n,k}(N) = E_{n,k}(N; r; 2) = \inf\left\{U_{n,k}(T; r; 2) \colon T \in \mathfrak{B}(N; L_r)\right\}, \quad N > 0,$$

$$U(T) = U_{n,k}(T; r; 2) = \sup\left\{\|f^{(k)} - Tf\|_{L_2} \colon f \in Q^n_{r,2}\right\}.$$

### 3.4.1. Case $n \ge 3$, $1 \le r \le \infty$ [5]

**Theorem 6.** *For $n \ge 3$, $1 \le k < n$, $1 \le r \le \infty$, and all $h > 0$,*

$$E_{n,k}(N_{n,k}(h); r; 2) = \frac{k}{n} h^{n-k}, \tag{3.45}$$

*where*

$$N_{n,k}(h) = \frac{n-k}{n} h^{-k}. \tag{3.46}$$

For $r = 2$, statement (3.45)+(3.46) is statement (3.11) of Subbotin and Taikov [47]. To justify (3.45)+(3.46), the author used in [5] an operator that differs from the one in [47]; for more detailed discussion see Section 3.4.3.

### 3.4.2. Cases $n = 2$, $k = 1$, $r = \infty$ [7, 10], and $r = 2$ [47]

**Theorem 7.** *For all $h > 0$,*

$$E_{2,1}(N_{2,1}(h); r; 2) = \frac{1}{2} h,$$

*where*

$$N_{2,1}(h) = \frac{1}{2} h^{-1}$$

*for $r = 2$ and*

$$N_{2,1}(h) = \frac{16}{h\pi^3} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^3}$$

*for $r = \infty$.*

### 3.4.3. Extremal operators

Case $n \ge 3$ and $k = 1$. Let us describe the construction of an extremal operator [5]. Let $\eta$ be the $2\pi$-periodic odd function defined on $[0, \pi]$ by the relations

$$\eta(t) = \begin{cases} t - \dfrac{1}{n}\left(\dfrac{2}{\pi}\right)^{n-1} t^n, & t \in \left[0, \dfrac{\pi}{2}\right], \\ \eta(\pi - t), & t \in \left[\dfrac{\pi}{2}, \pi\right]. \end{cases}$$

The Fourier series of this function has the form

$$\eta(t) = \sum_{l=0}^{\infty} c_l \sin(2l+1)t, \quad c_\ell = \frac{4}{\pi} \int_0^{\pi/2} \eta(t) \sin(2l+1)t dt.$$

The coefficients of this expansion for $n \ge 3$ have the following signs (see [5, proof of Theorem 4.1]):

$$(-1)^l c_l \ge 0, \quad l \ge 0. \tag{3.47}$$

For a number $h > 0$, we set $\nu = \nu(h) = \pi h/2$ and define an operator $T_{n,1}$ by the formula

$$(T_{n,1}f)(t) = \frac{1}{2\nu(h)} \sum_{l=0}^{\infty} c_l \big\{ f(t + (2l+1)\nu) - f(t - (2l+1)\nu) \big\}.$$

It is clear that $T_{n,1}$ is a bounded linear operator in the space $L_r$ for all $1 \le r \le \infty$ and

$$\|T_{n,1}\|_{L_r \to L_r} = \frac{1}{\nu} \sum_{l=0}^{\infty} |c_l| = \frac{1}{\nu} \eta\left(\frac{\pi}{2}\right) = \frac{n-1}{nh}.$$

For this operator,

$$U_{n,k}(T_{n,1}; r; 2) = \frac{k}{n} h^{n-k}. \tag{3.48}$$

It is this operator that is extremal in (3.45) for $n \ge 3$, $k = 1$, and all $1 \le r \le \infty$. It is different from the operator constructed by Subbotin and Taikov [47] for $r = 2$.

Case $n = 2$, $k = 1$, and $r = \infty$. For $n = 2$, the property of signs (3.47) is violated. More precisely, we have

$$\eta(t) = \sum_{l=0}^{\infty} c_l \sin(2l+1)t, \quad c_l = \frac{8}{\pi^2} \frac{1}{(2l+1)^3}.$$

The operator $T_{2,1}$ defined by the formula

$$(T_{2,1}f)(t) = \frac{1}{2\nu(h)} \sum_{l=0}^{\infty} c_l \big\{ f(t + (2l+1)\nu(h)) - f(t - (2l+1)\nu(h)) \big\},$$

where $\nu = \nu(h) = \pi h/2$, is a bounded linear operator in $C$ and

$$\|T_{2,1}\|_{C \to C} = \frac{1}{\nu} \sum_{l=0}^{\infty} c_l = \frac{16}{\pi^3 h} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^3} = N_{2,1}(h).$$

The norm of the operator $T_{2,1}$ has a different expression in comparison with (3.46). The same formula (3.48) holds for the value of the deviation.

Paper [47] by Subbotin and Taikov contains the case $n = 2$, $k = 1$, and $r = 2$ as a special case.

### 3.4.4. Inequalities for cases (3.44) in predual spaces

Let us discuss now inequality (2.10) for the set of parameters (3.44) in the space

$$\mathcal{W}^n_{r;2} = \big\{ f \in F_{r,r} \colon \widehat{f^{(n)}} \in L \big\} = \big\{ f \in F_{r,r} \colon f^{(n)} = \check{z},\ z \in L \big\}, \tag{3.49}$$

and, in particular, in the space

$$\mathcal{W}^n_{\infty;2} = \big\{ f \in C_0 \colon \widehat{f^{(n)}} \in L \big\} = \big\{ f \in C_0 \colon f^{(n)} = \check{z},\ z \in L \big\}. \tag{3.50}$$

**Theorem 8** [5]. *The following inequality holds for functions of space* (3.49) *for* $1 \le r \le \infty$, $n \ge 3$, *and* $1 \le k < n$:

$$\|f^{(k)}\|_C \le B_{n,k}(r;2)\|f\|_{r,r}^{(n-k)/n}\|\widehat{f^{(n)}}\|_L^{k/n}, \quad f \in \mathcal{W}^n_{r;2}, \tag{3.51}$$

*with the smallest possible constant*

$$B_{n,k}(r;2) = 1. \tag{3.52}$$

*For all* $n \ge 3$ *and* $1 \le k < n$, *an "ideal" extremal function is* sin.

For $n = 2$ ($k = 1$), inequality (3.51) with constant (3.52) holds for $r = 2$ (see Theorem 9 below) and does not hold for $r = \infty$. The value of the constant $B_{2,1}(r;2)$ for other values of $r$ is currently unknown. The following statement highlights the case $r = \infty$ of Theorem 8 and adds information about inequality (3.51) in the case $n = 2$ and $r = 2$.

**Theorem 9** [7]. *The following inequality holds for functions of space* (3.50) *for* $n \ge 2$, $1 \le k < n$, *and* $r = \infty$:

$$\|f^{(k)}\|_C \le B_{n,k}(\infty;2)\|f\|_C^{(n-k)/n}\|\widehat{f^{(n)}}\|_L^{k/n}, \quad f \in \mathcal{W}^n_{\infty;2},$$

*with the smallest possible constants*

$$B_{2,1}(\infty;2) = \left\{\frac{32}{\pi^3}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^3}\right\}^{1/2} > 1, \quad n = 2, \quad k = 1;$$

$$B_{n,k}(\infty;2) = 1, \quad n \ge 3, \quad 1 \le k < n.$$

*For* $n \ge 3$, *an "ideal" extremal function is* sin. *For* $n = 2$, *it is the entire function*

$$f(t) = \frac{1}{2}\int_0^\pi \frac{\pi - u}{\sin u}\sin 2\pi t u\, du.$$

## 4. Two-sided estimates for the value of Stechkin's problem (3.2)

For parameters $1 \le r, p \le \infty$, define $\overline{r} = \max\{r, r'\}$ and $\overline{p} = \max\{p, p'\}$. In statements of this section, we assume the following condition on two pairs of parameters $r_1, r_2$ and $p_1, p_2$:

$$\overline{r}_1 \le \overline{r}_2, \quad \overline{p}_1 \le \overline{p}_2. \tag{4.1}$$

**Theorem 10.** *The following two statements hold for the value* $E_{n,k}(N;r;p) = E_{n,k}(N;r,r;p,p)$ *for* $1 \le r \le \infty$, $1 < p \le \infty$, *and* $0 < k < n$.

(1) *For all* $N > 0$, *the value* $E_{n,k}(N;r;p)$ *of Stechkin's problem* (3.2) *does not decrease in the parameters* $\overline{r}$ *and* $\overline{p}$; *more exactly, if two pairs of parameters* $r_1, r_2$ *and* $p_1, p_2$ *satisfy conditions* (4.1), *then the following inequality holds*:

$$E_{n,k}(N;r_1;p_1) \le E_{n,k}(N;r_2;p_2).$$

(2) *For all* $N > 0$, *the following (exact) two-sided estimates hold for the values of Stechkin's problems* (3.2) *and* (2.9):

$$\beta\alpha^{\alpha/\beta}N^{-\alpha/\beta} \le E_{n,k}(N;r;p) = e_{n,k}(N;r;p) \le \beta\alpha^{\alpha/\beta}\left(C_{n,k}\right)^{1/\beta}N^{-\alpha/\beta},$$

*where*

$$\alpha = \frac{n-k}{n}, \quad \beta = \frac{k}{n}.$$

### 4.1. Auxiliary statement

**Lemma 4.** *The following two statements hold for the best constant $B_{n,k} = B_{n,k}(r;p)$ in inequality* (3.4) *for $1 \le r \le \infty$, $1 \le p \le \infty$, and $0 < k < n$.*

(1) *If two pairs of parameters $r_1, r_2$ and $p_1, p_2$ satisfy conditions* (4.1)*, then the following inequality holds for the best constant in inequality* (3.4):

$$B_{n,k}(r_1;p_1) \le B_{n,k}(r_2;p_2). \tag{4.2}$$

(2) *The following (exact) two-sided estimates hold*:

$$1 \le B_{n,k}(r;p) \le C_{n,k} \left( < \frac{\pi}{2} \right). \tag{4.3}$$

P r o o f. The constant in inequality (3.4) can be represented in the form

$$B_{n,k}(r;p) = \sup \left\{ \frac{\|f^{(k)}\|_C}{\|f\|_{r,r}^{(n-k)/n} \|f^{(n)}\|_{p,p}^{k/n}} : f \in \mathcal{W}^n_{r;p},\ f \not\equiv 0 \right\}. \tag{4.4}$$

According to statement (2.8) of Lemma 1, under conditions (4.1), we have the embeddings

$$F_{r_1,r_1} \subset F_{r_2,r_2} \quad \text{and} \quad \|f\|_{r_2,r_2} \le \|f\|_{r_1,r_1}, \quad f \in F_{r_1,r_1},$$

$$F_{p_1,p_1} \subset F_{p_2,p_2} \quad \text{and} \quad \|g\|_{p_2,p_2} \le \|g\|_{p_1,p_1}, \quad g \in F_{p_1,p_1},$$

and hence the embeddings

$$\mathcal{W}^n_{r_1;p_1} \subset \mathcal{W}^n_{r_2;p_2}; \tag{4.5}$$

moreover, the following inequalities hold on $\mathcal{W}^n_{r_1;p_1}$:

$$\|f\|_{r_2,r_2} \le \|f\|_{r_1,r_1}, \quad \|f^{(n)}\|_{p_2,p_2} \le \|f^{(n)}\|_{p_1,p_1}, \quad f \in \mathcal{W}^n_{r_1;p_1}. \tag{4.6}$$

Representation (4.4), embedding (4.5), and inequality (4.6) imply property (4.2).

In particular, we have the inequalities

$$B_{n,k}(2;2) \le B_{n,k}(r;p) \le B_{n,k}(\infty;\infty).$$

According to the result (3.29) of Lemma 2, $B_{n,k}(2;2) = 1$. In the case $r = p = \infty$, the value $B_{n,k}(\infty;\infty)$ coincides with the best constant (3.9) in the Kolmogorov inequality (3.6): $B_{n,k}(\infty;\infty) = C_{n,k}$. Thus, statement (4.3) is verified. Lemma 4 is proved.

### 4.2. The proof of Theorem 10

Both statements of Theorem 10 follow from the corresponding statement of Lemma 4 and statement (2.11) of Theorem 1. Theorem 10 is proved.

## 5. Conclusions

As can be seen from the results described above, even in the case (3.1), the topics considered here are far from exhausted. One of the main reasons for the difficulties in studying Stechkin's problem is that the description of $(p,q)$-multipliers and the value of the norm of multipliers are known only in several exceptional cases (see Section 2.1.1).

For example, Stechkin's problem and the corresponding inequalities between the norms of derivatives in the case of equal exponents

$$1 \le s = r = q = p \le \infty$$

are of interest. Denote by $E_{n,k}(N)_p$ Stechkin's problem and its value for this case. This case is embedded in the assumptions and conclusions of Theorem 10. According to Theorem 10, the value $E_{n,k}(N)_p$ does not decrease in the parameter $\overline{p} = \{p, p'\}$ and the following estimates hold:

$$E_{n,k}(N)_2 \le E_{n,k}(N)_p \le E_{n,k}(N)_\infty .$$

The solution to Stechkin's problem $E_{n,k}(N)_p$ is known only in the cases $p = \infty$, 2, and 1. Of course, one of the reasons for this is that the description of the multipliers of the Lebesgue spaces $L^p(-\infty, \infty)$ is known only for these values of the parameter $p$.

Let $B_{n,k}(p)$ be the best constant in the corresponding inequality

$$\|f^{(k)}\|_p \le B_{n,k}(p)\, \|f\|_p^{(n-k)/n} \|f^{(n)}\|_p^{k/n}, \quad f \in W^n_{p,p},$$

between the $p$-norms of the derivatives. The following estimates are known for the best constant in this inequality:

$$B_{n,k}(2) = 1 \le B_{n,k}(p) \le B_{n,k}(\infty) = C_{n,k} < \frac{\pi}{2}. \tag{5.7}$$

The second inequality in (5.7) is Stein's result [43]. To justify the first inequality, one should substitute the (appropriately smoothed) function sin into (5.7). No results regarding monotonicity of the value $B_{n,k}(p)$ in $p$ are unknown to the author.

## Acknowledgements

The author expresses his deep gratitude to the reviewer, who thoroughly read the paper and made several helpful comments.

## REFERENCES

1. Arestov V. V. On the best approximation of differentiation operators. *Math. Notes*, 1967. Vol. 1, No. 2. P. 100–103. DOI: 10.1007/BF01268057
2. Arestov V. V. On the best uniform approximation of differentiation operators. *Math. Notes*, 1969. Vol. 5, No. 3. P. 167–173. DOI: 10.1007/BF01388621
3. Arestov V. V. Approximation of linear operators, and related extremal problems. *Proc. Steklov Inst. Math.*, 1977. Vol. 138. P. 31–44.
4. Arestov V. V. Approximation of operators that are invariant under a shift. *Proc. Steklov Inst. Math.*, 1977. Vol. 138. P. 45–74.
5. Arestov V. V. Approximation of operators of convolution type by linear bounded operators. *Proc. Steklov Inst. Math.*, 1981. Vol. 145. P. 1–18.
6. Arestov V. V. Approximation of invariant operators. *Math. Notes*, 1983. Vol. 34, No. 1. P. 489–499. DOI: 10.1007/BF01160861
7. Arestov V. V. On the best approximation of the differentiation operator. In: *Priblizhenie funkcii polinomami i splainami* [Approximation of Functions by Polynomials and Splines]: collection of papers. Sverdlovsk, 1985. P. 3–14. (in Russian)
8. Arestov V. V. Best approximation of translation invariant unbounded operators by bounded linear operators. *Proc. Steklov Inst. Math.*, 1994. Vol. 198. P. 1–16.
9. Arestov V. V. Approximation of unbounded operators by bounded operators and related extremal problems. *Russian Math. Surveys*, 1996. Vol. 51, No. 6. P. 1093–1126. DOI: 10.1070/rm1996v051n06abeh003001

10. Arestov V. V. On the best approximation of the differentiation operator. *Ural Math. J.*, 2015. Vol. 1, No. 1. P. 20–29. DOI: 10.15826/umj.2015.1.002
11. Arestov V. V. Best uniform approximation of the differentiation operator by operators bounded in the space $L_2$. *Proc. Steklov Inst. Math.*, 2020. Vol. 308, Suppl. 1. P. S9–S30. DOI: 10.1134/S0081543820020029
12. Arestov V. V. On the conjugacy of the space of multipliers. *Trudy Inst. Mat. Mekh. UrO RAN*, 2019. Vol. 25, No. 4. P. 5–14. DOI: 10.21538/0134-4889-2019-25-4-5-14 (in Russian)
13. Arestov V. V. Uniform approximation of differentiation operators by bounded linear operators in the space $L_r$. *Anal. Math.*, 2020. Vol. 46. P. 425–445. DOI: 10.1007/s10476-020-0040-z
14. Arestov V. V. Predual spaces for the space of $(p, q)$-multipliers and their application in Stechkin's problem on approximation of differentiation operators. *Anal. Math.*, 2023. Vol. 49, No. 1. P. 43–65. DOI: 10.1007/s10476-022-0184-0
15. Arestov V. V., Akopyan R. R. Stechkin's problem on the best approximation of an unbounded operator by bounded ones and related problems. *Trudy Inst. Mat. Mekh. UrO RAN*, 2020. Vol. 26, No. 4. P. 7–31. DOI: 10.21538/0134-4889-2020-26-4-7-31 (in Russian)
16. Arestov V. V., Gabushin V. N. Best approximation of unbounded operators by bounded operators. *Russian Math.* (*Iz. VUZ*), 1995. Vol. 39, No. 11. P. 38–63.
17. Babenko V. F., Korneichuk N. P., Kofanov V. A., Pichugov S. A. *Neravenstva dlya proizvodnykh i ikh prilozheniya* [Inequalities for Derivatives and Their Applications]. Kiev: Naukova Dumka, 2003. 591 p. (in Russian)
18. Bosse Yu. G. (Shilov G. E.) On inequalities between derivatives. In: *Collection of Works of Student Research Societies of Moscow State University*. Moscow: Izd. MGU, 1937. Vol. 1. P. 68–72. (in Russian)
19. Buslaev A. P. Approximation of a differentiation operator. *Math. Notes*, 1981. Vol. 29, No. 5. P. 372–378. DOI: 10.1007/BF01158361
20. Domar Y. An extremal problem related to Kolmogoroff's inequality for bounded functions. *Arkiv för Mat.*, 1968. Vol. 7, No. 5. P. 433–441.
21. Dunford N., Schwartz J. T. *Linear Operators. Part 1: General Theory.* New York: Interscience, 1958.
22. Figà-Talamanca A. Translation invariant operators in $L_p$. *Duke. Math. J.*, 1965. Vol. 32, No. 3. P. 495–502. DOI: 10.1215/S0012-7094-65-03250-3
23. Figà-Talamanca A., Gaudry G. I. Density and representation theorems for multipliers of type $(p, q)$. *J. Australian Math. Soc.*, 1967. Vol. 7, No. 1. P. 1–6. DOI: 10.1017/S1446788700005012
24. Gabushin V. N. Inequalities for the norms of a function and its derivatives in metric $L_p$. *Math. Notes*, 1967. Vol. 1, No. 3. P. 194–198. DOI: 10.1007/BF01098882
25. Gabushin V. N. Best approximations of functionals on certain sets. *Math. Notes Acad. Sci. USSR*, 1970. Vol. 8, No. 5. P. 780–785. DOI: 10.1007/BF01146932
26. Gabushin V. N. The best approximation of the differentiation operator in the metric of $L_p$. *Math. Notes*, 1972. Vol. 12, No. 5. P. 756–760. DOI: 10.1007/BF01099059
27. Gabushin V. N. Inequalities between derivatives in $L_p$-metrics for $0 < p \le \infty$. *Math. USSR-Izv.*, 1976. Vol. 10, No. 4. P. 823–844.
28. Gabushin V. N. Inequalities for derivatives of solutions of ordinary differential equations in the metric of $L_p$ $(0 < p \le \infty)$. *Differ. Equ.*, 1988. Vol. 24, No. 10. P. 1088–1094.
29. Hadamard J. Sur le module maximum d'une fonction et de ses dérivées. *Soc. math. France, Comptes rendus des Séances*, 1914. Vol. 41. P. 68–72. (in French)
30. Hardy G. H., Littlewood J. E. Contribution to the arithmetic theory of series. *Proc. London Math. Soc.* 2, 1912. Vol. 11. P. 411–478.
31. Hardy G. H., Littlewood J. E., Pólya G. *Inequalities.* Cambridge: Cambridge Univ. Press, 1934.
32. Hörmander L. Estimates for translation invariant operators in $L^p$ spaces. *Acta Math.*, 1960. Vol. 104, No. 1–2. P. 93–140. DOI: 10.1007/BF02547187
33. Klots B. E. Linear deviations of the classes $\widetilde{W}_p^\alpha$ and approximations in spaces of multipliers. *Math. Notes*, 1975. Vol. 18, No. 1. P. 640–646. DOI: 10.1007/BF01461146
34. Klots B. E. Approximation of differentiable functions by functions of large smoothness. *Math. Notes*, 1977. Vol. 21, No. 1. P. 12–19. DOI: 10.1007/BF02317028

35. Kolmogorov A. N. On inequalities between upper estimates of consecutive derivatives of an arbitrary function defined on an infinite interval. In: *Selected Works: Mathematics and Mechanics.* Moscow: Nauka, 1985. P. 252–263. (in Russian)
36. Korneichuk N. P. *Ekstremal'nye zadachi teorii funkcii* [Extremal Problems of Approximation Theory]. Moscow: Nauka, 1976. 320 p. (in Russian)
37. Larsen R. *An Introduction to the Theory of Multipliers.* Berlin etc.: Springer, 1971. 282 p.
38. Levin V. I. Exact constants in inequalities of the Carlson type. *Doklady Akad. Nauk. SSSR*, 1948. Vol. 59. P. 635–638. (in Russian)
39. Magaril-Il'yaev G. G., Osipenko K. Yu. Optimal recovery of functions and their derivatives from inaccurate information about the spectrum and inequalities for derivatives. *Funct. Anal. Appl.*, 2003. Vol. 37. P. 203–214. DOI: 10.1023/A:1026084617039
40. Magaril-Il'yaev G. G., Osipenko K. Yu. Optimal recovery of values of functions and their derivatives from inaccurate data on the Fourier transform. *Sbornik Math.*, 2004. Vol. 195, No. 10. P. 1461–1476. DOI: 10.1070/SM2004v195n10ABEH000853
41. Rudin W. *Functional Analysis.* New York: McGraw-Hill, 1973. 397 p.
42. Shilov G. E. *Matematicheskii analiz. Vtoroi spetsial'nyi kurs* [Mathematical Analysis: The Second Special Course]. Moscow: Nauka, 1965. 328 p. (in Russian)
43. Stein E. M. Functions of exponential type. *Ann. Math.* (2), 1957. Vol. 65. P. 582–592.
44. Stein E. M., Weiss G. *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton: Princeton Univ. Press, 1971. 297 p.
45. Stechkin S. B. Inequalities between norms of derivatives of an arbitrary function. *Acta Sci. Math.*, 1965. Vol. 26, No. 3–4. P. 225–230.
46. Stechkin S. B. Best approximation of linear operators. *Math. Notes*, 1967. Vol. 1, No. 2. P. 91–99. DOI: 10.1007/BF01268056
47. Subbotin Yu. N., Taikov L. V. Best approximation of a differentiation operator in $L_2$-space. *Math. Notes*, 1968. Vol. 3, No. 2. P. 100–105. DOI: 10.1007/BF01094328
48. Tikhomirov V. M., Magaril-Il'yaev G. G. Inequalities for derivatives. In: *Kolmogorov A. N. Izbrannye trudy. Matematika i mekhanika* [Selected Works. Mathematics and Mechanics]: collection of papers. Moscow: Nauka, 1985. P. 387–390. (in Russian)