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COMPUTING THE REACHABLE SET BOUNDARY FOR AN ABSTRACT CONTROL SYSTEM: REVISITED

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Abstract: A control system can be treated as a mapping that maps a control to a trajectory (output) of the system. From this point of view, the reachable set, which consists of the ends of all trajectories at a given time, can be considered an image of the set of admissible controls into the state space under a nonlinear mapping. The paper discusses some properties of such abstract reachable sets. The principal attention is paid to the description of the set boundary.

Keywords: Reachable set, Nonlinear mapping, Control system, Extremal problem, Maximum principle.

1. Introduction

The paper explores the issue of describing the boundary of the reachable set of a nonlinear control system. A reachable set consists of all state vectors that can be reached along trajectories generated by admissible controls. For a system with geometric (point-wise) constraints, it is known that control steering the trajectory to the boundary of the set satisfies Pontryagin's maximum principle [13, 16]. Many algorithms for computing reachable sets are established based on solving optimal control problems and (or) use of the maximum principle [2, 5, 12, 14, 17]. For systems with integral constraints, some properties of reachable sets and algorithms for their construction are given in [6, 7, 15].

For integral quadratic constraints, it was shown in [8, 10] that any admissible control leading to the reachable set boundary provides a local extremum in some optimal control problem. Therefore, this control satisfies the maximum principle. This result was generalized in [11] for several mixed integral constraints in which the integrands depend on both control and state variables. In [9] (see, also [1]), we proposed to consider the reachability problem in terms of nonlinear mappings of Banach spaces. With this approach, the reachable set is treated as the image of the set of all admissible controls under the action of a nonlinear mapping. In the present paper, we extend the results of [9] to a broader class of abstract control systems. These systems are determined by differentiable maps of Banach spaces with different types of constraints on controls. The paper weakens the conditions of [9], which makes it possible to consider the problem with constraints specified by nonsmooth functionals. The use of nonsmooth analysis constructions allowed us to consider problems with multiple constraints within the framework of a unified scheme.

2. Single constraint control systems

Let us consider the system

$$\dot{x}(t) = f_1(t, x(t)) + f_2(t, x(t))u(t), \quad x(t_0) = x_0, \quad u(\cdot) \in U, \tag{2.1}$$

on a time interval $[t_0, t_1]$. Here, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$, and U is a given set in the space $\mathbb{L}_p$, $p > 1$.

Functions $f_2 : \mathbb{R}^{n+1} \to \mathbb{R}^{n\times r}$ are considered to have continuous Fréchet derivatives in x and satisfying the conditions:

$$\|f_1(t,x)\| \le l_1(t)(1+\|x\|), \quad \|f_2(t,x)\|_{n\times r} \le l_2(t), \quad t_0 \le t \le t_1, \quad x \in \mathbb{R}^n.$$

Here, $l_1(\cdot) \in \mathbb{L}_1$ and $l_2(\cdot) \in \mathbb{L}_2$, where $\mathbb{L}_1$ and $\mathbb{L}_2$ denote the spaces of summable and square summable functions, respectively.

For any $u(\cdot) \in \mathbb{L}_1$, there is a unique absolutely continuous solution $x(t, u(\cdot))$ to system (2.1) such that $x(t_0) = x_0$.

A reachable set $G(t_1)$ of system (2.1) at time t_1 under the constraint $u(\cdot) \in U \subset \mathbb{L}_1$ is defined as follows:

$$G(t_1) = \big\{y \in \mathbb{R}^n : y = x(t_1, u(\cdot)),\ u(\cdot) \in U\big\}.$$

This definition of a reachable set fits into the framework of the following abstract construction. Let X and Y be real Banach spaces, and let $U \subset X$ be a given set. We will call a map $F : U \to Y$ an abstract control system. Here, $u \in U$ is called a control and the set U is called a constraint. The reachable set G of this system is

$$G = \big\{y \in Y : y = F(u),\ u \in U\big\}.$$

Thus, $G = F(U)$ is an image of the set U under the mapping F.

Further, we set

$$U = \big\{u \in X : \varphi(u) \le \mu\big\},$$

so U is a level set of a continuous function $\varphi : X \to \mathbb{R}$; $\mu > 0$ is a given number. In control problems for system (2.1), one can take $X = \mathbb{L}_p$, $p > 1$, including $p = \infty$, as the space X and $Y = \mathbb{R}^n$.

The mapping F in this case is determined as

$$F(u) = F(u(\cdot)) = x(t_1, u(\cdot)). \tag{2.2}$$

With standard requirements on system (2.1) (see, for example, [10]), $F(u(\cdot))$ is a single-valued mapping having a continuous Fréchet derivative $F'(u(\cdot)) : \mathbb{L}_2 \to \mathbb{R}^n$:

$$F'_u(u(\cdot))\Delta u(\cdot) = \Delta x(t_1).$$

Here, $\Delta x(t)$ is a solution to system (2.1) linearized around $(x(t, u(\cdot)), u(t))$,

$$\begin{gathered} \dot{\Delta x}(t) = A(t)\Delta x(t) + B(t)\Delta u(t), \quad \Delta x(t_0) = 0, \\ A(t) = \frac{\partial f_1}{\partial x}(t, x(t)) + \frac{\partial}{\partial x}\big[f_2(t, x(t))u(t)\big], \quad B(t) = f_2(t, x(t)), \end{gathered} \tag{2.3}$$

corresponding to the control $\Delta u(t)$. If system (2.3) is controllable on $[t_0, t_1]$, then $\operatorname{Im} F'(u(\cdot)) = \mathbb{R}^n$.

Let us consider the geometric constraints on controls that are standard for control theory:

$$u(t) \in \Omega, \quad \text{a.e.} \quad t \in [t_0, t_1].$$

In many cases, the set Ω can be represented as

$$\Omega = \{v \in \mathbb{R}^r : \|Qv\| \le 1\},$$

where Q is a matrix and $\|\cdot\|$ is some norm in $\mathbb{R}^m$. It is clear that we can take here $X = \mathbb{L}_\infty$ and

$$\varphi(u(\cdot)) = \operatorname*{ess\,sup}_{t_0 \le t \le t_1} \|Qu(t)\|.$$

Such a functional is obviously continuous in the space $\mathbb{L}_\infty$.

Another example of control constraints is an integral constraint. In this case, $X = \mathbb{L}_p$, $p > 1$, and

$$\varphi(u(\cdot)) = \int_{t_0}^{t_1} \|u(t)\|^p \, dt.$$

We call the joint constraints on both control and state variables of the form

$$\varphi(u(\cdot)) := \int_{t_0}^{t_1} \big(Q(t, x(t)) + u^\top(t) R(t, x(t)) u(t)\big) \, dt \le \mu, \quad u(\cdot) \in \mathbb{L}_2,$$

the isoperimetric constraints.

Let $B_X(x, r)$ and $B_Y(y, r)$ be the balls of radius r centered at $x \in X$ and $y \in Y$, respectively. Further analysis is based on a well-known Lyusternik's theorem.

Theorem 1 [4, Theorem 2]. *Let a mapping F from a Banach space X to a Banach space Y be continuously Fréchet differentiable at a point $\hat{u}$ and such that $\operatorname{Im} F'(\hat{u}) = Y$. Then there are a neighborhood V of the point $\hat{u}$ and a number $s > 0$ such that, for any $B_X(u, r) \subset V$,*

$$B_Y(F(u), sr) \subset F(B_X(u, r)).$$

The condition $\operatorname{Im} F'(\hat{u}) = Y$ is called the Lyusternik (regularity) condition. If this condition is met, F is said to be regular at the point $\hat{u}$.

Using this theorem we get the following statement.

Theorem 2. *Let W be some neighborhood of the set U, let $F : W \to Y$ be a mapping continuously Fréchet differentiable at a point $\hat{u} \in U$, and let $\operatorname{Im} F'(\hat{u}) = Y$. To $\hat{x} = F(\hat{u}) \in \partial G$, it is necessary that $\hat{u}$ be a local extremum in the problem*

$$\varphi(u) \to \min, \quad F(u) = \hat{x}, \tag{2.4}$$

and $\varphi(\hat{u}) = \mu$.

P r o o f. The proof is by contradiction. Assume that $\varphi(\hat{u}) < \mu$. Since $\varphi(u)$ is continuous at the point $\hat{u}$, there is a neighborhood V_1 of $\hat{u}$ such that $\varphi(u) < \mu \ \forall u \in V_1$. Let us choose a neighborhood V and a number s whose existence follows from Theorem 1. Then, for any ball $B_X(\hat{u}, r) \in V \bigcap V_1$, we have

$$B_X(\hat{u}, r) \subset U,$$
$$B_Y(\hat{x}, sr) = B_Y(F(\hat{u}), sr) \subset F(B_X(\hat{u}, r)) \subset F(U) = G,$$

which contradicts the condition $\hat{x} \in \partial G$. Hence, $\varphi(\hat{u}) = \mu$.

Let us again choose V and s from Theorem 1. Assume that $\hat{u}$ is not a local minimum in (2.4). Then there is $\bar{u} \in V$ such that $F(\bar{u}) = \hat{x}$ and $\varphi(\bar{u}) < \varphi(\hat{u}) = \mu$. Let us choose $r > 0$ such that $B_X(\bar{u}, r) \subset V$. Then, by Theorem 1,

$$B_Y(\hat{x}, sr) = B_Y(F(\bar{u}), sr) \subset F(B_X(\bar{u}, r)) \subset F(U) = G$$

contrary to the condition $\hat{x} \in \partial G$. This completes the proof. □

Let us write down the necessary extremum condition for problem (2.4), assuming that $\varphi(u)$ is continuously differentiable at $\hat{u}$. Since the constraint $F(u) = \hat{x}$ is regular at the point $\hat{u}$, there is a Lagrange multiplier $y^* \in Y^*$ such that

$$\varphi'(\hat{u}) + F'^*(\hat{u})y^* = 0. \tag{2.5}$$

Here, $F'^*(\hat{u})$ denotes the operator conjugate to the continuous linear operator $F'(\hat{u})$.

If $\varphi'(\hat{u}) \neq 0$, then equality (2.5) implies that $y^* \neq 0$. If we divide both sides of equality (2.5) by $\|y^*\|$, then it takes the form

$$F'^*(\hat{u})y^* + \lambda\varphi'(\hat{u}) = 0, \tag{2.6}$$

where $\|y^*\| = 1$ and $\lambda > 0$. Since $\varphi(\hat{u}) - \mu = 0$, we also have the equality

$$\lambda(\varphi(\hat{u}) - \mu) = 0. \tag{2.7}$$

It is easy to see that relations (2.6) and (2.7) also give the necessary optimality conditions for the problem

$$\langle y^*, F(u)\rangle \to \min, \quad \varphi(u) \leq \mu, \tag{2.8}$$

where $\langle \cdot, \cdot \rangle$ denotes a bilinear form establishing the duality of the spaces Y and Y^*. Here, equality (2.6) means that the derivative of the Lagrange function

$$L(u, \lambda) = \langle y^*, F(u)\rangle + \lambda(\varphi(u) - \mu)$$

in u is equal to zero, and equality (2.7) is a complementary slackness condition. Thus, the following statement is true.

Theorem 3. *Assume that $F(\hat{u}) = \hat{x} \in \partial G$, $u \in U$, $F(u)$ is regular, and $\varphi(u)$ is continuously differential at the point $\hat{u}$ and $\varphi(\hat{u}) \neq 0$. Then, there is $y^* \in Y^*$, $\|y^*\| = 1$, such that $\hat{u}$ satisfies the necessary extremum conditions* (2.6) *and* (2.7) *in problem* (2.8).

As it is easy to see, problem (2.8) can be rewritten in the equivalent form

$$\langle z^*, y\rangle \to \max, \quad y \in G.$$

where $z^* = -y^*$. The latter is the problem of calculating the support function of G. Recall that a support function $\psi_G(z^*)$ is defined on Y^* by the equality

$$\psi_G(z^*) = \sup_{y \in G}\langle z^*, y\rangle.$$

The point at which the supremum is reached is called the support point. Since the reachable set G in the nonlinear case is not necessarily convex, the boundary point $\hat{x}$ is not necessarily a support point. But it meets the necessary optimality conditions as if it would be a support point.

Next, we will consider the case when φ is not continuously differentiable but is Lipschitz continuous at the point $\hat{u}$. For simplicity, we will assume also that $Y = \mathbb{R}^n$.

Denote by $\partial_C f(u)$ the Clarke subdifferential of a function f at a point u. If f is Lipschitz continuous in some neighborhood of u, then $\partial_C f(u) \neq \varnothing$ is a convex weakly* compact set [3].

Let L be a Lagrange function

$$L(u, \lambda, y^*) = \lambda\varphi(u) + \langle y^*, F(u) - \hat{x}\rangle,$$

where $\lambda \geq 0$ and $y^* \in Y^* = \mathbb{R}^n$ are Lagrange multipliers.

Assume that $\hat{u}$ is a local solution to problem (2.4) and $\varphi(u)$ is Lipschitz continuous at the point $\hat{u}$. Then, there exist $\lambda \geq 0$ and $y^* \in \mathbb{R}^n$, $\lambda + \|y^*\| \neq 0$, such that

$$0 \in \partial_C L(\hat{u}, \lambda, y^*) = \lambda \partial_C \varphi(\hat{u}) + F'^*(\hat{u}) y^*, \tag{2.9}$$

where $\partial_C L$ is taken with respect to u (see, for example, [3, Theorem 6.1.1]). Let us show that $\lambda > 0$. Indeed, if $\lambda = 0$, then $\|y^*\| \neq 0$ and $F'^*(\hat{u})y^* = 0$. This contradicts the regularity of F at the point $\hat{u}$.

Without loss of generality, we set $\lambda = 1$. Suppose that $0 \notin \partial_C \varphi(\hat{u})$. Then $F'^*(\hat{u})y^* \neq 0$ and condition (2.9) takes the form

$$-F'^*(\hat{u}) y^* \in \partial_C \varphi(\hat{u}). \tag{2.10}$$

Let us show that this inclusion is a necessary extremum condition in problem (2.8). Let

$$L(u, \alpha, \beta) = \alpha \langle y^*, F(u) \rangle + \beta(\varphi(u) - \mu)$$

be the Lagrange function for problem (2.8). If $\hat{u}$ is a local minimum point in problem (2.8), then there are $\alpha \geq 0$ and $\beta \geq 0$, $\alpha + \beta \neq 0$, such that

$$0 \in \partial_C L(\hat{u}, \alpha, \beta). \tag{2.11}$$

Note that if $0 \notin \partial_C \varphi(\hat{u})$, then $\alpha > 0$ and $\beta > 0$. Indeed, if $\alpha = 0$, then $\beta > 0$ and $0 \in \partial_C \varphi(\hat{u})$. If $\beta = 0$, then $\alpha F'^*(\hat{u}) y^* = 0$ and $\alpha > 0$, which is impossible due to the regularity condition. Divide both sides of inclusion (2.11) by β and take $\alpha y^*/\beta$ as a new vector y^*. Then inclusion (2.11) takes the form (2.10).

As a result, we get the following statement.

Theorem 4. *Assume that $F(\hat{u}) = \hat{x} \in \partial G$, $\hat{u} \in U$, $F(u)$ is regular, and $\varphi(u)$ is Lipschitz continuous at the point $\hat{u}$ and $0 \notin \partial_C \varphi(\hat{u})$. Then there is $y^* \in Y^*$, $\|y^*\| = 1$, such that $\hat{u}$ satisfies the necessary extremum condition* (2.10) *in problem* (2.8).

Remark 1. If $\varphi(u)$ is convex, then $\partial_C \varphi(u) = \partial \varphi(u)$ is a subdifferential of a convex function. The condition $0 \notin \partial_C \varphi(\hat{u})$ in this case is equivalent to Slater's condition: there is $\bar{u}$ such that $\varphi(\bar{u}) < \varphi(\hat{u})$.

Remark 2. If a mapping F is defined by formula (2.2) and $\varphi(u(\cdot))$ is an integral quadratic in u functional, then Theorem 2 implies the necessary extremum conditions [10] in the form of Pontryagin's maximum principle.

Note that, under integral quadratic constraints, the relations of the maximum principle follow directly from the extremum conditions (2.10). Below we present its proof. Assume that $X = \mathbb{L}_2$, $Y = \mathbb{R}^n$, the mapping F is defined by formula (2.2), and $\varphi(u(\cdot)) = 1/2 \langle u(\cdot), u(\cdot) \rangle$ is an integral quadratic functional. In this case, $\partial \varphi(u(\cdot)) = \{\varphi'(u(\cdot))\} = \{u(\cdot)\}$ and the equality $\varphi(\hat{u}(\cdot)) = \mu$ implies that $\varphi'(\hat{u}(\cdot)) \neq 0$. Therefore, (2.10) takes the following equivalent form:

$$F'^*(\hat{u}) z^* = \hat{u}, \quad z^* = -y^*, \quad z^* \neq 0.$$

Recall that $F'(u) = F'(u(\cdot))$ is defined by the equality $F'(u(\cdot))\Delta u(\cdot) = \Delta x(t_1)$, where $x(t)$ is the solution of (2.3). Let us represent this solution in the integral form

$$\Delta x(t_1) = \int_{t_0}^{t_1} X(t_1, \tau) B(\tau) \Delta u(\tau)\, d\tau,$$

where $X(t,\tau)$ is the Cauchy matrix. For any $z^* \in \mathbb{R}^n$, we have

$$\big(z^*, F'(u(\cdot))\Delta u(\cdot)\big) = \big\langle F'^*(u(\cdot))z^*, \Delta u(\cdot)\big\rangle = z^{*\top}\int_{t_0}^{t_1} X(t_1,\tau)B(\tau)\Delta u(\tau)d\tau$$
$$= \int_{t_0}^{t_1} p^\top(\tau)B(\tau)\Delta u(\tau)d\tau,$$

where $p(\tau) = X^\top(t_1,\tau)z^*$ satisfies the adjoint equation

$$\dot p(t) = -A^\top(t)p(t), \quad p(t_1) = z^*.$$

Thus, we have

$$F'^*(u(\cdot))z^* = B^\top(\cdot)\,p(\cdot) = \hat u(\cdot),$$

which implies that

$$\hat u(t) = B^\top(t)p(t), \quad t_0 \le t \le t_1.$$

Finally, we obtain a system of relations of the maximum principle for the boundary control $\hat u(t)$ (see [10])

$$\dot x(t) = f_1(t,x(t)) + f_2(t,x(t))B(t)p(t), \quad x(t_0) = x_0, \tag{2.12}$$
$$\dot p(t) = -A^\top(t)p(t), \quad p(t) \ne 0, \quad \hat u(t) = B(t)p(t), \tag{2.13}$$
$$A(t) = \frac{\partial f_1}{\partial x}(t,x(t)) + \frac{\partial}{\partial x}[f_2(t,x(t))\hat u(t)], \quad B(t) = f_2(t,x(t)).$$

Now suppose that the constraints have the form

$$\gamma(u(t)) \le \mu, \quad \text{a.e. in } [t_0,t_1],$$

where $\gamma(u)$ is a convex function in $\mathbb{R}^r$ (for example, a norm in $\mathbb{R}^r$). In this case, we can take $X = \mathbb{L}_\infty$ and

$$\varphi(u(\cdot)) = \operatorname*{ess\,sup}_{t_0\le t\le t_1} \gamma(u(t)).$$

Such a functional is obviously convex and continuous in the space X. Assume that there is $\bar u \in \mathbb{R}^r$ such that $\gamma(\bar u) < \mu$. As before, we believe that $Y = \mathbb{R}^n$. Since $\varphi(u(\cdot))$ is convex, we can substitute $\partial_C\varphi(\hat u(\cdot))$ by a subdifferential of the convex function $\partial\varphi(\hat u(\cdot))$.

If $F(\hat u(\cdot)) \in \partial G$, then $\varphi(\hat u(\cdot)) = \mu$ and hence $0 \notin \partial\varphi(\hat u(\cdot))$. Thus,

$$F'^*(\hat u(\cdot))z^* \in \partial\varphi(\hat u(\cdot))$$

for some $z^* \in \mathbb{R}^n$, $z^* \ne 0$. Here, the point $F'^*(\hat u(\cdot))z^*$ belongs to the space $\mathbb{L}_\infty^*$. Similar to the previous case, it can be proven that $F'^*(\hat u(\cdot))z^* = B^\top(\cdot)p(\cdot)$, where $p(t) \ne 0$ is a solution to the adjoint system.

From the properties of $\partial\varphi(\hat u(\cdot))$, we get

$$\varphi(u(\cdot)) - \varphi(\hat u(\cdot)) \ge \langle F'^*(\hat u(\cdot))z^*, u(\cdot) - \hat u(\cdot)\rangle$$

for every $u(\cdot) \in \mathbb{L}_\infty$. From this inequality, for every $u(\cdot)$ such that $\varphi(u(\cdot)) \le \mu$, we have

$$0 \ge \int_{t_0}^{t_1} p^\top(\tau)B(\tau)(u(\tau) - \hat u(\tau))d\tau. \tag{2.14}$$

Choose a point $\tau \in (t_0, t_1)$ and a vector $v \in \mathbb{R}^r$ such that $\gamma(v) \le \mu$, and sufficiently small $\varepsilon > 0$. Let

$$u(t) = \begin{cases} \hat{u}(t), & t \notin [\tau, \tau+\varepsilon], \\ v, & t \in [\tau, \tau+\varepsilon]. \end{cases}$$

Then, (2.14) implies the inequality

$$\frac{1}{\varepsilon}\int_{\tau}^{\tau+\varepsilon} p^\top(t)B(t)\hat{u}(t)dt \ge \frac{1}{\varepsilon}\int_{\tau}^{\tau+\varepsilon} p^\top(t)B(t)vdt.$$

Passing here to the limit, we get

$$p^\top(\tau)B(\tau)\hat{u}(\tau) \ge p^\top(\tau)B(\tau)v$$

for almost every $\tau \in [t_0, t_1]$ and every v such that $\gamma(v) \le \mu$. So, we have

$$p^\top(\tau)B(\tau)\hat{u}(\tau) = \max_{\gamma(v)\le\mu} p^\top(\tau)B(\tau)v,$$
$$\dot{p}(\tau) = -A(\tau)p(\tau), \quad p(\cdot) \ne 0.$$

Introducing the Hamiltonian

$$H(t,x,p,u) = p^\top(f_1(t,x) + f_2(t,x)u),$$

we can write the last relations in the standard form of the maximum principle:

$$H(\tau, x(\tau), p(\tau), \hat{u}(\tau)) = \max_{\gamma(v)\le\mu} H(\tau, x(\tau), p(\tau), v), \quad \text{a.e.} \quad \tau \in [t_0, t_1], \tag{2.15}$$

$$\dot{p}(\tau) = -A(\tau)p(\tau) = -\frac{\partial H}{\partial x}(\tau, x(\tau), p(\tau), \hat{u}(\tau)), \quad \tau \in [t_0, t_1]. \tag{2.16}$$

3. Multiple constraints on the control

In this section, we consider constraints specified by the inequalities

$$\varphi_i(u) \le \mu_i, \quad i = 1, \dots, k. \tag{3.1}$$

Here, $\varphi_i : X \to \mathbb{R}$ are functionals and μ_i, $i = 1, \dots, k$, are given positive numbers.

One can assume without loss of generality that $\mu_i = 1$, $i = 1, \dots, k$. Then (3.1) can be replaced by the single constraint $\varphi(u) \le 1$ by setting

$$\varphi(u) = m(\varphi_1(u), \dots, \varphi_k(u)), \quad m(x) = m(x_1, \dots, x_k) = \max_{1\le i\le k} x_i.$$

Since $m(x)$ is a continuous function, the functional $\varphi(u)$ is obviously continuous at a point of continuity of all functionals $\varphi_i(u)$. Therefore, for describing the reachable set boundary, we can use Theorem 2, which leads to the following statement.

Corollary 1. *Let W be a neighborhood of the set U, and let $F : W \to Y$ be a mapping continuously Fréchet differentiable at the point $\hat{u} \in U$ such that $\operatorname{Im} F'(\hat{u}) = Y$. Assume that*

$$G = \{F(u) : \varphi_i(u) \le 1,\ i = 1, \dots, k\},$$

where $\varphi_i(u)$ are continuous at the point $\hat{u}$. To $\hat{x} = F(\hat{u}) \in \partial G$, it is necessary that $\hat{u}$ be a local extremum in the problem

$$\varphi(u) = m(\varphi_1(u), \dots, \varphi_k(u)) \to \min, \quad F(u) = \hat{x},$$

and $\varphi(\hat{u}) = 1$.

The derivation of extremum conditions in this problem is more complicated than before because the function $m(x)$ is not differentiable. However, the superposition $\varphi(u) = m(\varphi_1(u), \dots, \varphi_k(u))$ is locally Lipschitz at the point $\hat{u}$ if such are the functions $\varphi_i(u)$. Moreover, if each of the functions $\varphi_i(u)$ is either convex or continuously differentiable at the point $\hat{u}$, then

$$\partial_C \varphi(\hat{u}) = \operatorname{co} \bigcup_{i \in I(\hat{u})} \partial_C \varphi_i(\hat{u}), \tag{3.2}$$

where $I(\hat{u}) = \{i : \varphi_i(\hat{u}) = \varphi(\hat{u})\}$ and $\operatorname{co} A$ denotes a convex hull of A [3].

Let the conditions of Corollary 1 be satisfied. Let initially all functionals φ_i be continuously differentiable at $\hat{u}$. Then $\partial_C \varphi_i(\hat{u}) = \{\varphi_i'(\hat{u})\}$ and, taking into account (3.2), we get

$$\partial_C \varphi_i(\hat{u}) = \Big\{ \sum_{i \in I(\hat{u})} \alpha_i \varphi_i'(\hat{u}) : \sum_{i \in I(\hat{u})} \alpha_i = 1,\ \alpha_i \geq 0 \Big\}$$
$$= \Big\{ \sum_{1 \leq i \leq k} \alpha_i \varphi_i'(\hat{u}) : \sum_{1 \leq i \leq k} \alpha_i = 1,\ \alpha_i \geq 0,\ \alpha_i(\varphi_i(\hat{u}) - 1) = 0,\ i = 1, \dots, k \Big\}.$$

Here, the condition $0 \notin \partial_C \varphi_i(\hat{u})$ takes the form

$$\sum_{1 \leq i \leq k} \alpha_i = 1, \quad \alpha_i \geq 0, \quad \alpha_i(\varphi_i(\hat{u}) - 1) = 0, \quad i = 1, \dots, k \quad \Rightarrow \quad \sum_{1 \leq i \leq k} \alpha_i \varphi_i'(\hat{u}) = 0.$$

In particular, it is satisfied if the vectors $\varphi_i(\hat{u})$ form a positive linear independent set. If this condition is met, we can write down the necessary condition for the inclusion $F(\hat{u}) \in \partial G$ as follows:

$$F'^*(\hat{u}) z^* = \sum_{1 \leq i \leq k} \alpha_i \varphi_i'(\hat{u}), \quad \sum_{1 \leq i \leq k} \alpha_i = 1, \quad \alpha_i \geq 0, \quad \alpha_i(\varphi_i(\hat{u}) - 1) = 0, \quad i = 1, \dots, k.$$

Using the previous scheme, we can also write this condition in the form of Pontryagin's maximum principle [16] (see also [11]).

Let us next consider a system with double control constraints. We will assume that one of the constraints is specified by a convex differentiable functional $\varphi_1(u)$ and the second by a convex functional $\varphi_2(u)$. An example of such a problem is system (2.1) with integral quadratic and geometric constraints. If $\varphi_2(\hat{u}) < \varphi_1(\hat{u})$, then $\partial_C \varphi(\hat{u}) = \{\varphi_1'(\hat{u})\}$; if $\varphi_1(\hat{u}) < \varphi_2(\hat{u})$, then $\partial_C \varphi(\hat{u}) = \{\partial \varphi_2(\hat{u})\}$; and, finally, if $\varphi_1(\hat{u}) = \varphi_2(\hat{u})$, then $\partial_C \varphi(\hat{u}) = \operatorname{co}(\{\varphi_1'(\hat{u})\} \cup \partial \varphi_2(\hat{u}))$.

Lemma 1. *Let $a \in X$, and let $B \subset X$ be a convex set. Then*

$$\operatorname{co}(\{a\} \cup B) = C := \bigcup_{0 \leq \lambda \leq 1} (\lambda a + (1 - \lambda) B).$$

P r o o f. Obviously, $C \subset \operatorname{co}(\{a\} \cup B)$. To prove the lemma, it suffices to prove the convexity of C. Let

$$c_1 = \lambda_1 a + (1 - \lambda_1) b_1, \quad c_2 = \lambda_2 a + (1 - \lambda_2) b_2, \quad b_1, b_2 \in B.$$

Let us choose $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, and show that

$$c_3 = \alpha c_1 + \beta c_2 \in \lambda_3 a + (1 - \lambda_3) B$$

for some $\lambda_3 \in [0, 1]$. To this end, we try to find numbers $\alpha_1, \beta_1 \geq 0$, $\alpha_1 + \beta_1 = 1$, such that

$$\alpha c_1 + \beta c_2 = \alpha(\lambda_1 a + (1 - \lambda_1) b_1) + \beta(\lambda_2 a + (1 - \lambda_2) b_2) = \lambda_3 a + (1 - \lambda_3)(\alpha_1 b_1 + \beta_1 b_2).$$

Equating the coefficients at the vectors a, b_1, and b_2 on both sides of the equality, we obtain

$$\lambda_3 = \alpha\lambda_1 + \beta\lambda_2, \quad \alpha(1-\lambda_1) = \alpha_1(1-\lambda_3), \quad \beta(1-\lambda_2) = \beta_1(1-\lambda_3).$$

This implies the inequality $0 \le \lambda_3 \le 1$. For $0 \le \lambda_3 < 1$, we have

$$\alpha_1 = \frac{\alpha(1-\lambda_1)}{1-\lambda_3}, \quad \beta_1 = \frac{\beta(1-\lambda_2)}{1-\lambda_3};$$

so, $\alpha_1, \beta_1 \ge 0$ and $\alpha_1 + \beta_1 = 1$. If $\lambda_3 = 1$, then either $\alpha\lambda_1 = 1$ or $\beta\lambda_2 = 1$. In both of these cases, we get $c_3 = a$. This completes the proof. □

Let us further assume that Slater's condition is satisfied: there exists $\bar{u}$ such that $\varphi_i(\bar{u}) < 1$, $i = 1, 2$. Then the condition $0 \notin \partial_C\varphi(\hat{u})$ is satisfied. Indeed, suppose on the contrary that $0 \in \partial_C\varphi(\hat{u})$. Then, it follows from Lemma 1 that there is $\lambda \in [0, 1]$ such that

$$0 \in \lambda\varphi_1'(\hat{u}) + (1-\lambda)\partial\varphi_2(\hat{u}) = \partial(\lambda\varphi_1 + (1-\lambda)\varphi_2)(\hat{u}).$$

For the convex function $\lambda\varphi_1 + (1-\lambda)\varphi_2$, the last condition is necessary and sufficient for the minimum at $\hat{u}$. Thus,

$$(\lambda\varphi_1 + (1-\lambda)\varphi_2)(\hat{u}) \le (\lambda\varphi_1 + (1-\lambda)\varphi_2)(\bar{u}),$$

which contradicts Slater's condition.

Let further $X = \mathbb{L}_\infty$ and

$$\varphi_1(u(\cdot)) = c/2\langle u(\cdot), u(\cdot)\rangle = c/2\int_{t_0}^{t_1} u^\top(t)u(t)dt, \quad \varphi_2(u(\cdot)) = \operatorname*{ess\,sup}_{t_0 \le t \le t_1} \gamma(u(t)). \tag{3.3}$$

The constant $c > 0$ is chosen here such that to write down the constraints in the form $\varphi_i(u(\cdot)) \le 1$, $i = 1, 2$. Since $\varphi_1'(u(\cdot)) = cu(\cdot)$, the optimality conditions $F'^*(\hat{u}(\cdot))z^* \in \partial\varphi(\hat{u}(\cdot))$ take the form

$$F'^*(\hat{u}(\cdot))z^* - \lambda c\hat{u}(\cdot) \in (1-\lambda)\partial\varphi_2(\hat{u}(\cdot))$$

for some $\lambda \in [0, 1]$.

For $\lambda = 0$, we get a maximum principle of the form (2.15), (2.16).

For $\lambda = 1$, we get (2.12), (2.13).

Finally, for $0 < \lambda < 1$, we get

$$F'^*(\hat{u}(\cdot))w^* - \sigma c\hat{u}(\cdot) \in \partial\varphi_2(\hat{u}(\cdot)),$$

where $w^* = z^*/(1-\lambda)$ and $\sigma = \lambda/(1-\lambda)$. Introducing the Hamiltonian

$$H(t, x, p, \sigma, u) = -\sigma cu + p^\top(f_1(t, x) + f_2(t, x)u),$$

we can write these relations in the form of maximum principle:

$$H(\tau, x(\tau), p(\tau), \sigma, \hat{u}(\tau)) = \max_{\gamma(v) \le \mu} H(\tau, x(\tau), p(\tau), \sigma, v), \quad \text{a.e.} \quad \tau \in [t_0, t_1],$$

$$\dot{p}(\tau) = -A(\tau)p(\tau) = -\frac{\partial H}{\partial x}(\tau, x(\tau), p(\tau), \sigma, \hat{u}(\tau)), \quad \tau \in [t_0, t_1].$$

Thus, we arrive at the following statement.

Corollary 2. *Let functionals $\varphi_i(u(\cdot)) : \mathbb{L}_\infty \to \mathbb{R}$, $i = 1, 2$, be given by equalities (3.3), and let $F(u(\cdot)) = x(t_1)$, where $x(t)$ is a solution to system (2.1). Let*

$$G = \{F(u(\cdot)) : \varphi_i(u(\cdot)) \le 1, \ i = 1, 2\}.$$

If $F(\hat{u}(\cdot)) \in \partial G$ and system (2.1) linearized around $\hat{u}(\cdot)$ is controllable, then there exist a function $p(\cdot) \neq 0$ and a number $\sigma \ge 0$ such that the relations of maximum principle are satisfied.

4. Conclusion

The paper proposes a unified scheme for studying extremal properties of the reachable set boundary. Within the framework of this approach, the reachable set is treated as the image of the set of admissible controls under a nonlinear mapping of a Banach space. The proposed scheme is based on the results of nonlinear and nonsmooth analysis and is equally applicable to systems with integral and geometric control constraints, including multiple constraints.

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